The analysis of 6-component measurements of a random electromagnetic wave field in a magnetoplasma. I : The direct problem

L.R.O. Storey, François Lefeuvre

CENTRE NATIONAL D'ETUDES DES TELECOMMUNICATIONS

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THE ANALYSIS OF 6-COMPONENT MEASUREMENTS OF A RANDOM
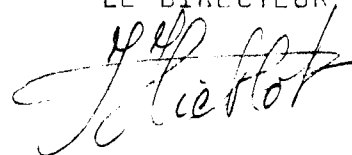ELECTROMAGNETIC WAVE FIELD IN A MAGNETOPLASMA. I - THE DIRECT PROBLEM

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L. R. O. STOREY and F. LEFEUVRE

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THE ANALYSIS OF 6-COMPONENT MEASUREMENTS OF A RANDOM
ELECTROMAGNETIC WAVE FIELD IN A MAGNETOPLASMA.
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ABSTRACT

        This is the first of a series of papers, the general
subject of which is how to interpret a set of simultaneous measu-
rements of the three electric and three magnetic components of a
random electromagnetic wave field in a magnetoplasma. The point at
which the measurements are made is assumed to be stationary with
respect to the plasma. In this first paper, the following problems
are treated : how to define, within the framework of classical
electrodynamics, a distribution function that characterizes the
statistics of a linear random electromagnetic wave field in a
lossless magnetoplasma ; the direct problem of predicting the
statistical properties of measurements of the six components of a
field of this type, when the distribution function is known.

CONTENTS

I. GENERAL INTRODUCTION

This paper presents the first step in the development
of a method for the analysis of experimental data on linear
random (i.e. weakly turbulent) electromagnetic wave fields in
a magnetoplasma. The method is suitable, in particular, for
analysing measurements of the six components of the fields of
certain natural very-low-frequency (VLF) or extremely-low-
frequency (ELF) electromagnetic wave phenomena, in the study
of the interactions between waves and energetic particles in
the earth's magnetosphere.

The wave phenomena in question are those that appear
to be continuous and structureless, such as VLF and ELF
hiss, in contrast to discrete phenomena such as whistlers or
 VLF emissions which have clear-cut structures in the frequency-
time plane (RUSSELL et al. , 1972). Their interactions comprise
the plasma instabilities that produce them, and also the
perturbations that they cause to particles other than those
involved in their production.

In the development of the theory of magnetospheric
wave-particle interactions during recent years, the situations
envisaged have become progressively more and more complex.
The earliest work concerned beams of mono-energetic particles
with a single pitch angle, interacting with plane monochromatic
waves propagated at a fixed angle to the magnetic field. Soon,
however, this situation was seen to be unrealistic. In the
magnetosphere, the interacting particles are spread in energy
and in pitch angle, and must be characterized by their statistical
distribution with respect to these two variables. Accordingly
the next developments in the theory concerned interactions
between the population of energetic electrons or protons,
with a given distribution function, and plane monochromatic
waves. In point of fact, this is a reasonable model for the
interactions involving natural **whistlers** from lightning
strokes. Hence much theoretical work has been done on interactions

between whistlers and energetic particles, in particular on whistler -triggered V L F emissions, and on the similar discrete emissions that can be triggered by artificial signals of fixed frequency radiated from VLF transmitters on the ground.

Equally important, however, are the interactions involving natural VLF noise that is generated entirely within the magnetosphere, doubtless throughout large regions, and which covers a wide range of frequencies. At a given point of observation, which may be either inside or outside the source region, the corresponding waves arrive from different parts of this region by different paths, with different wave-normal directions, and all frequencies are present simultaneously. Accordingly, in some of the most recent work on the pitch-angle diffusion of radiation-belt particles under the influence of natural VLF waves, the latter are recognized as being spread in frequency and in direction of propagation, and hence -like the particles- as needing to be characterized by some kind of distribution function (LYONS et al. , 1972 ; SCHULZ and LANZEROTTI, 1974).

Therefore, in order to provide inputs to the theory and to test its outputs, one needs to be able to determine, from experimental data, distribution functions for the waves as well as for the particles. Methods for measuring particle distribution functions are well developed, yet almost no thought has been given as to how to make the corresponding measurements for waves ; indeed, only recently has the need for such measurements been acknowledged clearly (STOREY, 1971).

The present paper is an initial step towards the development of one possible method for measuring distribution functions for electromagnetic waves in space plasmas : electrostatic waves require different methods, which will not be discussed here.

The first question that arises is the choise of the data on which to base the method, and the answer depends on what practical limitations to data collection are accepted. Here it will be assumed that data are available at any instant from only one point in space. In the case of VLF electroma-

gnetic wave fields observed from a single spacecraft, the
assumption is reasonable because the wavelengths in the
medium are generally much larger than the dimensions of any
practicable antennae. which means that the wave fields are
uniform on the scale of the measuring device. (This is one
respect in which the case of electrostatic waves is different).
While accepting this limitation for present purposes, we should
remember that it could be overcome by collecting data with
a suitably-disposed pair or cluster of spacecraft.

The electromagnetic field data that are available at
a single point in space comprise the three orthogonal components
of the wave electric field, and the three orthogonal components
of the wave magnetic field, in as wide a band of frequencies as
one may choose to observe. They can be measured by suitable
antennae : dipoles for the electric components, loops for
the magnetic components. These methods of measurement, though
simple in principle, are often less so in practice, particularly
for the electric field, but they have been discussed elsewhere
(MOZER, 1973) and will not be considered here. The measurements
will be assumed to be perfect, free both from systematic
errors and from noise of technological origin, and attention
will be confined to the question of how to analyze them.

Up to now, most of the methods for analyzing measurements
of the six components of an electromagnetic wave field in
space have been based on the assumption that the field observed
in any narrow band of frequencies is locally and instanta-
neously like that of a single plane wave (GRARD, 1968 ;
SHAWHAN, 1970 ; MEANS, 1972). In this case, the data can be
interpreted readily to yield all the characteristic parameters
of the wave : direction of propagation, amplitude, polarization
ellipses for the electric and magnetic vectors, wave impedance.
When data on all six components are available, the method of
interpretation does not involve any assumption about the nature
of the medium ; indeed, some of the properties of the medium
can be deduced from those of the wave (STOREY, 1959).
Conversely, if the properties of the medium are known, then
certain items of the information obtained about the wave are
redundant. Specifically, when the frequency of observation
has been chosen and the direction of propagation measured,
then the shapes of the polarization ellipse and the value

of the wave impedance can he calculated from straightforward
propagation theory. By comparing these theoretical predictions
with the corresponding experimental results, the original
assumption of a plane wave can be put to the test. When
studying V L F   waves in the magnetosphere, the polarization
is particularly useful for this purpose because, under very
diverse conditions, its theoretical form is almost independent
of the plasma parameters. When this test is applied to
discrete phenomena such as whistlers, the agreement with the
theory is often satisfactory (MEANS, 1972). On the other
hand, when it is applied to continuous, structureless phenomena
such as VLF and ELF hiss, the results show frequent
discrepancies : the observed shapes of the polarization ellipses
depart from their theoretical ones, and indeed fluctuate
considerably (THORNE et al. , 1973). It follows that, for
such apparently random fields, the plane-wave model is much
too simple, and that some kind of statistical model is required
instead.

STOREY (1971), following a suggestion made privately
by D.A. Gurnett, advocated that a natural noise field of this
type be considered as a continuum of superposed plane waves,
of different frequencies and propagated in different directions,
without any mutual phase coherence. This last condition is
what is meant by the statement at the beginning of the
present paper, to the effect that the field in question is
weakly turbulent : if the observations are being made within
the source region, the condition requires that the field be
so weak that no nonlinear effects occur ; alternatively, even
when such effects occur at the source, the condition can still
be satisfied if the resulting phase coherence between different
frequences is destroyed by dispersion along the path to the
point of observation, the latter being outside the source
region. The properties of such a weakly turbulent, incoherent,
or random wave field can be described statistically by a
function that specifies how the wave energy density is
distributed with respect to the wave-number vector $\vec{k}$, or
alternatively with respect to the angular frequency ω and
the direction of propagation. This concept, which is familiar
to theore ticians of plasma turbulence (BEKEFI, 1966), has
yet to win general acceptance among experimental space
physicists.

In the light of it, the question of how to analyze measurements of the six components of a random electromagnetic wave field in a magnetoplasma reduces to the more specific one of how best to use these measurements to determine the wave distribution function.

A brief discussion of this question has been published previously (STOREY and LEFEUVRE, I974). The present paper is intended to be the first in a series of papers that reply to it in detail. Here we are concerned with the direct problem in the statistics of random electromagnetic wave fields : namely, given a field characterized by a known distribution function, what are the statistical properties of its six components ?

The plan of the paper is as follows : section 2 presents the formal definition of the distribution function that is used to describe a random wave field statistically ; section 3 describes the standard methods that are used for the statistical description of the set of signals received on six antennae that are immersed in the field and measure its six components ; section 4 gives the solution to the direct problem (as defined above) of the relationship between the statistics of the wave field and of the received signals : finally section 5 offers some provisional conclusions.

Many assumptions and approximations are made in order to simplify the problem : all the statistical properties of the wave field are stationary in time and space ; the plasma is infinite, uniform, cold and collisionless ; the point of observation is fixed with respect to the plasma ; wave propagation is described by the magneto-ionic theory in its simplified version in which the forced motion of the ions is ignored. Their validity, and that of the basic concepts of the paper, are discussed in appendix A.

The rationalized MKSA system of units is used throughout.

2. STATISTICAL DESCRIPTION OF THE WAVE FIELD

In the kinetic theory of gases, the distribution function F_s $(t, \vec{r}, \vec{v})$ for the particle species s is defined in such a way that

$$F_s (t, \vec{r}, \vec{v}) \; d^3r \quad d^3v$$

is the ensemble average number of these particles, at the time t, in the volume element d^3r at the point $\vec{r}$ in coordinate space, whose velocity vectorslie in the element d^3v at the point $\vec{v}$ in velocity space.

By analogy, the distribution function F_m ($t, \vec{r}, \vec{k}$) for the wave mode m may be defined on such a way that

$$F_m (t, \vec{r}, \vec{k}) \; d^3r \quad d^3k$$

is the ensemble average amount of wave energy in this mode at the time t and in the volume element d^3r, due to homogeneous plane waves whose wave-number vectors lie in the element d^3k at the point $\vec{k}$ in wave-number space. However, since $\vec{k}$ and $\vec{r}$ are not independent variables, the definition of F_m involves certain conceptual difficulties that do not arise for F_s ; these are discussed in appendix A.

This wave distribution function is essentially the same as that introduced by BEKEFI (1966), in terms, however, of "light particles", which he does not define clearly. If they are what are usually called "plasmons", and each carries a quantum of energy $\hbar \, \omega$ where $\hbar$ is Planck's constant and ω the wave angular frequency corresponding to the vector $\vec{k}$, then Bekefi's distribution function, which is even more closely analogous to that for a species of particles of finite mass, is equal to F_m / $\hbar \, \omega$. Since our treatment is purely classical, we prefer to work directly in terms of the wave energy.

Note that the definition of the function F_m, in terms of the energy in a particular wave mode, involves the implicit assumption that the medium is of the type that supports wave propagation in distinct modes, i.e. that it is anisotropic. Surprisingly, this assumption very much simplifies the problem in hand. The nature of the simplification may be appreciated by considering how one would specify the intensity and state of polarization of a quasi-monochromatic wave field that is random in the restricted sense that it is spread in frequency, but not in respect of its direction of

propagation which is fixed and known. In the case of an
anisotropic medium, it suffices to specify the energy density
in each of the possible wave modes ; thus for a cold magneto-
plasma, which supports two electromagnetic wave modes, only
two parameters are required. On the other hand, in the case
of an isotropic medium such as free space or a non-magnetized
plasma, no less than four parameters are required, for
instance the Stokes parameters (BORN and WOLF, 1964). The
method of analysis developed in the present paper has not
yet been extended to the case of waves in isotropic media.

From here on, several assumptions will be made in
order to simplify the discussion : first, the medium will
be assumed to be uniform ; the wave field will be assumed to
be stationary in time, and the observations to be made at
a fixed point in the medium, so the variables t and $\vec{r}$ can be
omitted ; also, it will be assumed provisionally that only
one wave mode is present, and the subscript m will be dropped
accordingly. These assumptions are discussed in the appendix A.
They enable is to write the wave distribution function simply
as $F(\vec{k})$.

Now when the medium is optically uniaxial, as is the
case for a stationary magnetoplasma, it is more convenient
to express the wave distribution function in terms of the
angular frequency ω and of the direction of propagation
(i.e. the direction of the $\vec{k}$ vector), as specified by the
angle θ between this vector and the optic axis, and by
an azimuthal angle φ (fig. I) ; the angle θ varies
from 0 to π , and φ from 0 to 2 π . In this
representation, the distribution function will be written
F (ω, θ , φ) and the element of wave energy is F (ω, θ , φ)
d^3r d ω dθ dφ . The relation between this and the previous
form of the distribution function is

(1. 1.) $F(\omega, \theta, \phi) = V_g^{-1} k^2 \sin \theta \; F(\vec{k})$

where k = $|\vec{k}|$ and $V_g = |(\partial \omega / \partial k)_\theta|$ is the
modulus of the group velocity (BEKEFI, 1966) ; the vector $\vec{V}_g$

has the direction of the wave normal (or the opposite
direction, if $(\partial\omega/\partial k)_\theta$ is negative), as distinct from the
group-ray velocity which has the direction of the geometric
optics ray (HELLIWELL, 1965).

The representation of the distribution function
for an electromagnetic wave mode in a magnetoplasma in
terms of the variables ω, θ, and ϕ is analogous to that
for a charged particle species, in the presence of a magnetic
field, in terms of the energy E and the pitch-angle α .
There is the difference that, for low-energy particles, the
distribution function has cylindrical symmetry around the
magnetic field, so no azimuthal angle is required, whereas
for waves, although most known source processes give rise to
distributions having such symmetry, this is lost progressively
by refraction as the waves are propagated away from their source
region. Incidentally, it should he remembered that the
distribution functions of the energetic charged particles
in the earth's radiation belts are not cylindrically
symmetrical at energies such that the radii of gyration are
comparable with the linear scale of the gradients of the
magnetic field (i.e. with an earth radius) or with the
atmospheric scale height in the case of particles trapped
on low L - shells (HESS. 1968).

There is one respect in which the representation
F (ω . θ, ϕ) of the wave distribution function is actually
better than the representation F ($\vec{k}$) : namely, it is
unambiguous. For electromagnetic waves in a given mode,
propagated in a given direction, it is possible for a given
value of k to correspond to two (or more) different values
of ω . This point is illustrated in fig. 2, which shows
the real branches of the dispersion curves for waves propa-
gated perpendicularly to the magnetic field (θ = $\pi/2$) ,
the electron gyrofrequency has been taken equal to the
plasma frequency. The abscissa is ω / Π , where Π is the
angular plasma frequency, and the ordinate is (c/Π) k,

where c is the speed of light in free space ; the dashed
line is the free-space dispersion curve $\omega = ck$. Note that
the function ω (k) is ambiguous for the extraordinary (X)
mode, though not for the ordinary (O), whereas the function
k (ω) is single-valued for both modes. In general, for a
given mode, a given value of ω always corresponds to a
unique value of k, and for this reason the representation
F (ω, θ, ϕ) is to be preferred.

In conclusion of this section, and in anticipation of

Another reason is that the experimenter, by means
of narrow-band filters, can select for observation waves
with frequencies ω that lie between limits of his choosing.
Of course, the observed frequencies are equal to the true
wave frequencies only if the point of observation is
stationary with respect to the medium, because any motion
gives rise to Doppler shifts proportional to k ; here this
has been assumed to be the case (see appendix A.). From this
point onwards, therefore, the distribution function for wave
energy will be used exclusively in the representation
F (ω, θ, ϕ).

In conclusion of this section, and in anticipation of
those that follow, it should be stated that the term "energy"
is employed in this paper in two different senses. In the
context of wave propagation theory, it is used in its strict
and customary sense as above. However, when discussing the
analysis of data on the six field components, we shall use
it also in the sense of signal theory, where for instance
if x(t) and y (t) are two real time series, then
$\int_0^T x^2$ (t) dt and $\int_0^T y^2$ (t) dt are their respective energies
in the time interval $0 < t < T$, while $\int_0^T x$ (t) y (t) dt is
their interaction energy. We shall distinguish between these
two senses by using the terms "wave energy" and "signal
energy" respectively ; equally we shall speak of "wave power"
and "signal power".

3. STATISTICAL DESCRIPTION OF THE RECEIVED SIGNALS

3.1. Wide-band signals

When we seek to predict the statistics of the six
components of a random wave field, the first question to ask
is how these statistics should be described. As already
mentioned, the field is pictured as being made up of a
continuum of elementary plane waves of different frequencies
and with random phases, propagated through the medium in
different directions. We suppose that, at a fixed point in
the medium, continuous measurements are made of three
orthogonal electric components and three orthogonal magnetic
components of the total field, using for instance three electric
dipole antennae and three magnetic loop antennae. For the
time being, we suppose that these measurements are made over
a wide frequency band. We take a right-handed set of rectangular
coordinate axes Oxyz, with Oz parallel to the steady magnetic
field, and suppose for convenience that the antennae are
aligned parallel to these axes. Then the experimental data
consist of six wide-band signals, each proportional to one
of the six axial field components.

First let us consider the individual properties of
any one of these signals, separate from the other five. It
is the sum of all the waveforms induced on the corresponding
antenna by all the different elementary plane waves. The
elementary wareform induced by any one of these waves is a
sinusoid, the phase of which bears no special relationship
to that of the sinusoid induced by any other such wave. Thus
the total signal is the sum of an infinite number of elementary
sinusoids, with infinitesimal amplitudes and random phases.
It is well known that a signal of this type is a stationary
Gaussian random process of zero mean value, and that, as such,
its statistical properties are described completely by its
mean auto-covariance function or by its mean signal power
spectrum ("auto-spectrum").

As usual when discussing random processes, it is
convenient to introduce the notion of a statistical ensemble.

We imagine an ensemble of random wave fields. each giving
rise to a specimen of the random signal considered. We
assume that, on going from one member to another of the
ensemble of fields, the elementary plane waves change in
amplitude, frequency, direction of propagation, and phase,
while remaining consistent collectively with the given wave
distribution function. On this assumption, the received signal
is ergodic, so its auto-corariance function and auto-spectrum
can be defined alternatively in terms of time averages or
of ensemble averages.

Now let us consider the six received signals as a
whole. Besides their individual properties, we need to know
what relationships exist between them. These are described
by the cross-covariance functions for all the fifteen possible
pairs of signals, or by the corresponding cross-spectra.

Here we prefer to work with signal auto-spectra and
cross-spectra, rather than to use covariance functions, because
the spectra are related more simply to the wave distribution
function, as we shall show later ; of course, we can get from
a covariance function to the corresponding power spectrum, or
vice versa, by a direct or inverse Fourier transform
respectively (JENKINS and WATTS, 1969).

First, let us define a 6 x 6 <u>spectral matrix</u>, which
groups all the statistical data together in a convenient way.
Let E_x, E_y, E_z be the axial components of the electric field
of the wave at the point of observation, and H_x, H_y, H_z the
components of the wave magnetic field. From these variables,
we define a generalized electric field vector $\vec{E}$ as follows :

(1.2a) $\qquad \mathcal{E}_1 = E_x \qquad \mathcal{E}_2 = E_y \qquad \mathcal{E}_3 = E_z$

(1.2b) $\qquad \mathcal{E}_4 = Z_o H_x \qquad \mathcal{E}_5 = Z_o H_y \qquad \mathcal{E}_6 = Z_o H_z$

where Z_o is the wave impedance of free space. (Note : another
way of defining a generalized electric field vector is
mentioned in appendix B). This step in the argument is just
a convenient way of regrouping the data, and has no physical

significance. Let E_i be any component of this vector, the subscript i running from 1 to 6. We now introduce the 6 x 6 spectral matrix, of which any element $S_{ij}(\omega)$ is either the auto-power spectrum of the field component E_i (if i = j), or the cross-power spectrum of the components E_i and E_j (if i ≠ j).

These are "two-sided" signal power spectra : that is, they have values at negative as well as at positive frequencies. They are the Fourier transforms of the corresponding covariance functions. Since the latter functions are purely real, it follows that $S_{ij}(\omega) = S_{ij}^*(-\omega)$, where the asterisk denotes the complex conjugate ; therefore it suffices to quote the expressions for the spectra at positive frequencies only.

At any one frequency ω, the spectral matrix contains 36 items of statistical information about the field. It might be thought that the number of independent items is less than this, because the matrix is Hermitian : $S_{ij}(\omega) = S_{ji}^*(\omega$ Hence here are only 6 independent auto-spectra and 15 cross-spectra. Note, however, that while the auto-spectra are purely real, since the auto-covariance functions are symmetrical, the cross-spectra are complex, since the cross-covariance functions are asymmetrical in general. Moreover the real and imaginary parts of the cross-spectra are mutually independent, i.e. there is no general relation between them. Hence the number of independent items of information is 36 as stated.

3.2. Narrow-band signals

Now let us consider the case where the 6 data signals are received in a band of width $\Delta\omega$, centred on some frequency ω_0 . We suppose moreover that this band is narrow, by which we mean two things : firstly, that $\Delta\omega << \omega_0$; secondly, that all the 36 signal power spectra $S_{ij}(\omega)$ are essentially constant and equal to $S_{ij}(\omega_0)$ throughout this band.

Then clearly the information contained in the received signals relates only to the values of the 36 quantities $S_{ij}(\omega_o)$; the question is how to extract the relevant information from the data.

Let $X_i(t)$ be the real narrow-band signal obtained by filtering the original wide-band signal $E_i(t)$ through the receiver pass-band. Let $\chi_i(t)$ be the corresponding analytic signal. i.e. the representation of $X_i(t)$ by a complex exponential (HELLSTROM, 1968). We now define the covariance matrix $\{c_{ij}\}$ of the set of 6 narrow-band real signals, with (1.3)

$$c_{ij} = \langle \chi_i(t) \quad \chi_j^{\times}(t) \rangle$$

where the triangular brackets denote the (BORN and WOLF, 1964) ensemble average or "expected value" of the product. If the signals are stationary and ergodic, as has been assumed, then these same quantities are given by the corresponding time averages in the limit of very long times. A practical method for estimating the elements c_{ij} of the covariance matrix in this way has been described by MEANS (1972).

In order to relate these elements to those of the spectral matrix at the frequency ω_o . it is necessary to define the bandwidth $\Delta\omega$ more precisely. If the transfer function of the receiver is $Y(\omega)$, between the point where the signal $E_i(t)$ enters it and that at which $X_i(t)$ emerges, and if, at positive frequencies, this function takes the form of a single peak with its summit at ω_o , then we define

$$(1.4) \qquad \Delta\omega = \frac{\int_0^\infty |Y(\omega)|^2 \, d\omega}{|Y(\omega_o)|^2}$$

In engineering parlance, this quantity is known as the noise bandwidth (BENDAT and PIERSOL, 1971).

With $\Delta\omega$ thus defined, it follows from elementary considerations of signal energy conservation that

$$(1.5) \qquad c_{ij} = 4 \, \Delta\omega \, |Y(\omega_o)|^2 \, S_{ij}(\omega_o)$$

The factor 4 stems from the fact that, at positive frequencies, the amplitudes of the Fourier components of $\chi_i(t)$ are twice those of the corresponding components of $X_i(t)$.

4. DIRECT RELATIONSHIP BETWEEN WAVE AND SIGNAL STATISTICS

The problem is to relate the spectral matrix, which describes the statistics of the 6 received signals, to the distribution function, which describes those of the random wave field. In order to do this, we shall show how each of these entities is made up of contributions from all the various elementary waves ; we begin with the case where these waves all belong to the same magneto-ionic mode.

First let us consider one such wave, which we shall distinguish from its fellows by placing the subscript 1 in front of every symbol that refers to it. The i'th component of its generalized electric field, which will be called $_1\varepsilon_i$ (t, $\vec{r}$), varies sinusoidally as a function of time and of spatial position :

$$(1.6) \qquad _1\varepsilon_i (t, \vec{r}) \quad = \quad Re \{ _1e_i \exp [i (_1\omega t - _1\vec{k}.\vec{r})]\}$$

On the right-hand side of this equation, the symbol Re denotes the extraction of the real part, $_1e_i$ is the complex amplitude, the angular frequency $_1\omega$ is real and positive, and the wave-number vector $_1\vec{k}$ is real because the medium is assumed to be lossless, and also because only homogeneous waves are being considered, as befits a situation in which there are no boundaries.

Now if, in coordinate space, the wave has an electro-magnetic energy density $_1\rho$, then its distribution function, which we shall call $_1f (\vec{k})$, is a 3-dimensional Dirac distribution of strength ρ at the point $_1\vec{k}$ in $\vec{k}$ space :

$$(1.7) \qquad _1f (\vec{k}) \quad = \quad _1\rho \quad \delta(\vec{k} - _1\vec{k})$$

The alternative form of this distribution function is

$$(1.8) \qquad _1f (\omega, \theta.\phi) \quad = \quad _1\rho \quad \delta(\omega - _1\omega) \, \delta(\theta - _1\theta) \, \delta(\phi - _1\phi)$$

where $_1\theta$ and $_1\phi$ are the angles that specify the direction of $_1\vec{k}$. The equivalence of these two expressions is evident, because in both cases the wave energy per unit volume of coordinate space is given by a triple integral over the distribution function :

(1.9) $_1\rho$ = $\int\int\int {}_1f(\vec{k})\, d^3k$ = $\int\int\int {}_1f(\omega, \theta, \phi)\, d\omega\, d\theta\, d\phi$

In spite of superficial appearances, the expressions (1.7)
and (1.8) are consistent with (1.1) ; the algebraic factors
on the right-hand side of (1.1) are incorporated in the
delta-functions on the right-hand side of (1.8).

The elements of the spectral matrix for this same
elementary wave are, for ω positive,

(1.10) $_1s_{ij}(\omega)$ = $\frac{\pi}{2}\; {}_1e_i\; {}_1e_j^{x}\; \delta(\omega - {}_1\omega)$ ($\omega > 0$)

Probably the simplest way to derive this result is to obtain
the expression for the cross-covariance function of the
periodic signals $_1\epsilon_i(t)$ and $_1\epsilon_j(t)$ at a fixed point
$\vec{r}$, and then to take its Fourier transform.

Now, instead of considering just one elementary wave,
let us consider the subset of elementary waves for which the
parameters $_1\omega$, $_1\theta$, and $_1\phi$ lie respectively in the ranges
from ω to $\omega + d\omega$, from θ to $\theta + d\theta$, and from ϕ to $\phi + d\phi$.
Their contribution to the wave energy per unit volume of
coordinate space is $\sum_1 {}_1\rho$, where the sum is taken over
this subset. By definition, the distribution function for
the random field is related to this quantity as follows :

(1.11) $F(\omega, \theta, \phi)\, d\omega\, d\theta\, d\phi$ = $\left\langle \sum_1 {}_1\rho \right\rangle$

Here the brackets on the right-hand side denote the ensemble
average.

Similarly, the elementary waves in this subset make
the following contribution to the signal energy of interaction
between the field components E_i and E_j , in the frequency
range from ω to $\omega + d\omega$:

(1.12) $\sigma_{ij}(\omega)\, d\omega$ = $\frac{\pi}{2}\left\langle \sum_1 {}_1e_i\; {}_1e_j^{x} \right\rangle$ ($\omega > 0$)

At this point we introduce a crucial idea : for given
values of ω , θ , and ϕ, the wave energy density ρ is propor-
tional to the square of the amplitude of any one field component ;
the same is true of the product $_1e_i\; {}_1e_j^{x}$; therefore the
ratio

(1.13) a_{ij} $=$ $\dfrac{e_i \; e_j{}^\times}{\rho}$

is independent of the amplitude, and is essentially the
same for all the elementary waves in the subset defined
above, i.e. it is independent of l. With all the possible
pairs of values of the indices i and j, the identity
(1.13) defines 36 such ratios. The expressions for them,
as functions of ω , θ , and ϕ and of the characteristic
parameters of the plasma, are given by the propagation
theory for the type of wave considered, i.e. by the
magneto-ionic theory in the present instance ; they are
quoted in the appendix B. It is at this point that our
knowledge of the properties of the medium enters into the
argument.

This idea enables us to write

(1.14) σ_{ij} (ω) $d\omega$ $=$ $\dfrac{\pi}{2}$ a_{ij} (ω, θ, ϕ) $\langle \sum\limits_{l} {}_{l}\rho \rangle$ $(\omega > o)$

(1.15) $=$ $\dfrac{\pi}{2}$ a_{ij} (ω, θ, ϕ) F (ω, θ, ϕ) $d\omega \, d\theta \, d\phi$ $(\omega > o$

with (1.15) following from (1.11), and enabling us to cancel
$d\omega$ on both sides.

Finally, in order to obtain the total auto-spectral
(i = j) or cross-spectral (i ≠ j) signal energy in the
frequency range from ω to $\omega + d\omega$, it suffices to sum over
the complete set of elementary waves in this range, with all
possible directions of propagation. In other words, (1.15)
must be integrated over the full ranges of the angles θ and ϕ .
Then, dividing by $d\omega$, we obtain

(1.16) S_{ij} (ω) $=$ $\dfrac{\pi}{2} \int_0^\pi \int_o^{2\pi} a_{ij}$ (ω, θ, ϕ) F (ω, θ, ϕ) $d\theta \, d\phi$

for ω positive. This is the required result, for the special
case where only one of the two magneto-ionic modes are
present.

In the general case where both modes are present
simultaneously, each spectral matrix element is just the
sum of the contributions from the two modes :

(1.17) S_{ij} (ω) $=$ $\dfrac{\pi}{2} \sum\limits_{m} \int_0^\pi \int_0^{2\pi}$ a_{ijm} (ω, θ, ϕ) F_m (ω, θ, ϕ) $d\theta \, d\phi$

where the subscript m denotes the wave mode.

From their definitions, it follows that the spectral matrix S_{ij} , together with the kernels a_{ij} for the two modes, are tensors of rank 2.

5. CONLUSION

Subject to the assumptions made in this paper. equation (1.17) is the solution for the <u>direct problem</u> of determining the statistics of the received signals when those of the wave field are known.

As such, it already provides a weak basis for comparison between theory and experiment in the study of the origin of certain natural random electromagnetic wave fields, such as those of magnetospheric VLF and ELF hiss. If the theory predicts the wave distribution function explicitly, and if an accurate estimate of the spectral matrix is available from experimental data, then the two can be compared by means of this equation.

More often, however, the task of comparing theory and experiment is less simple. The data may be degraded by noise, or the time of observation too short to yield good statistics. The theory may involve one or more unknown parameters, which have to be adjusted in order to make its predictions agree with the data as closely as possible. There may be several competing theories, none of which explains the data perfectly, and then one wants to know whether any of these theories is acceptable, and if so, which is the most plausible. Or again. one may be interested in analyzing some experimental data to find out whatever one can about the wave distribution function. in the absence of any theoretical model. All there are different forms of the <u>inverse problem</u>, which is more difficult and will be treated in subsequent papers.

At the same time, further work is needed on the direct problem, in order to free the theory from the limitations accepted in this paper. For instance, before it can be applied to space-probe data on the random field of Alfvén waves in the solar wind, the theory must be generalized to take account of the motion of the point of observation with respect to the medium, and to make use of data on the wave-induced fluctuations of plasma velocity. The application to random wave fields other than electromagnetic must be considered. The need to study the simple special case of isotropic media

has been noted already, in section 2. Another interesting
special case is that in which the wave field is random in
space but not in time, an example being the field created
when an initially plane monochromatic wave traverses a
spatially irregular but temporally stationary medium. The
study of this case would begin to forge the links between two
domains hitherto independent, namely previous work on the
propagation of deterministic wave fields through random media,
and the present work on the analysis of measurements of random
wave fields in uniform media.

A C K N O W L E D G E M E N T S

R E F E R E N C E S

BEKEFI, G. "Radiation Processes in Plasmas",
Wiley, New York, 1966.

BENDAT, J.S. and A.G. PIERSOL, "Random Data :
Analysis and Measurement Procedures".
Wiley, New York, 1971.

BORN, M. and E. WOLF, "Principles of Optics" (4th. ed.),
Pergamon, London, 1970.

BUDDEN, K.G. "Radio Waves in the Ionosphere",
University Press, Cambridge (G.B.) 1961.

GRARD, R. "Interprétation de mesures de champ electromagnétique
T.B.F. dans la magnétosphère",
Ann. Géophys. 24, 955-971, 1968.

HELLIWELL. R.A. "Whistlers and related Ionospheric
Phenomena". Stanford University Press.
Stanford. 1965.

HELLSTROM. C.W. "Statistical Theory of Signal Detection"
(2 nd.ed.). Pergamon London,1968.

HESS. W.N. "The Radiation Belt and the Magnetosphere".
Blaisdell. Toronto, 1968.

JENKINS. G.M. and D.G. WATTS. "Spectral Analysis and its
Applications". Holden-Day, San Francisco. 1969.

LYONS. L.R., THORNE. R.M. and C.F. KENNEL. "Pitch-angle
diffusion of radiation-belt electrons within
the plasmasphere", J. Geophys. Res. 77,
3455-3474. 1972.

MEANS. J.D. "The use of the three-dimensional covariance
            matrix in analyzing the polarization properties
            of plane waves". J. Geophys.
            Res. 77. 5551 - 5559, 1972.

MOZER. F.S. "Analysis of techniques for measuring DC and AC
            electric fields in the magnetosphere". Space
            Sci. Rev. 14, 272-313. 1973.

RATCLIFFE. J.A. "The Magneto-ionic Theory and its Applications
            to the Ionosphere". University Press,
            Cambridge (G.B.). 1959.

RUSSELL. C.T. Mc. PHERRON, R.L., and P.J. COLEMAN,
            "Fluctuating magnetic fields in the magnetosphere :
            1- ELF and VLF fluctuations". Space Sci.
            Rev. 12. 810-856, 1972.

SCHULZ. M. and L.J. LANZEROTTI. "Particle Diffusion in the
            Radiation Belts". Springer-Verlag, Berlin, 1974.

SHAWHAN. S.D. "The use of multiple receivers to measure
            the wave characteristics of very-low-frequency
            noise in space". Space Sci. Rev. 10, 689-736. 1970.

STOREY, L.R.O. "A method for measuring local electron density
            from an artificial satellite". J. Res. Nat. Bur.
            Stand. 63 D. 325-340, 1959.

STOREY, L.R.O. "Electric field experiments : - alternating
            fields", in "The ESRO geostationary magnetospheric
            satellite". European Space Research Organisation,
            Neuilly-sur-Seine, Report n° SP-60, 267-279, 1971.

STOREY. L.R.O. and F. LEFEUVRE. "Theory for the interpretation
            of measurements of a random electromagnetic wave
            field in space". in "Space Research XIV" (Eds :
            M.J. RYCROFT and R.D. REASENBERG) Akademie-Verlag,
            Berlin, 381-386, 1974.

THORNE. R.M.. SMITH. E.J. , BURTON. R.K. and R.E. HOLZER.
"Plasmaspheric hiss". J. Geophys. Res. $\underline{78}$. 1581-
1596, 1973.

APPENDIX A : Validity of the basic concepts and assumptions.

In section 2, it is assumed that the random wave field is statistically stationary in space and in time, and that only one wave mode is present. These assumptions are less restrictive than might appear at first sight. The conditions under which they may be considered to be satisfied adequately are discussed below.

Approximate spatial uniformity is required only to ensure that the structure of a single wave be very little different from what it would be in an infinite uniform medium, with properties like those of the real medium at the point considered. For this purpose, it is necessary that the medium be uniform on a scale much larger than a wavelength. The residual large-scale non-uniformity limits the meaningful detail that the distribution function can exhibit : if L is the distance over which the medium is substantially uniform, then this function can he specified with a resolution, in any component of $\vec{k}$, of the order of $\Delta k = 2\pi /L$.

Approximate temporal stationarity of the field is needed to provide reasonable data statistics, which means that the received signals must be statistically stationary over time intervals much longer than a wave period. If they are in fact so for a time T, then the distribution function can he measured with a resolution in frequency of the order of $\Delta\omega = 2\pi/T$ (moreover, the point of observation is effectively stationary if all Doppler shifts due to its motion are less than $\Delta\omega$).

Under the conditions enunciated in the previous two paragraphs, it is possible to define a wave distribution function that varies with time t and with spatial position $\vec{r}$, as was mentioned in section 2. For this purpose, the fields must he truncated hy means of a suitable "window" function, of linear dimensions L and duration T, centred on the point-instant ($\vec{r}$, t) considered, before taking their Fourier transforms ; the choice of L and T also limits the meaningful resolution of F in wave number and in frequency respectively.

These same conditions justify the use of geometric optics to determine how a given distribution function evolves in time and in space (BEKEFI, 1966). In the present paper, such questions have been avoided by the assumption of stationarity.

Finally, the assumption that only one wave mode is present is satisfied over wide frequency ranges, in particular in the range from the lower hybrid frequency up to the electron gyro-frequency, in which only the whistler (ordinary) mode is propagated. Even at frequencies where both modes can be propagated, natural source mechanisms are likely to excite one much more strongly than the other. Finally, as we saw in section 4, this assumption can be dropped if necessary without greatly complicating the theory.

APPENDIX B : expressions for the kernels a_{ij} and b_{ij}

The 36 functions a_{ij} (ω,θ,ϕ), which are the kernels of the set of integral equations (1.16), were defined by (1.13) in the course of an analysis based on a right-handed rectangular coordinate system Oxyz, with Oz parallel to the steady magnetic field. For pratical purposes, algebraic expressions are required for these weighting functions in terms of the variables ω, θ and ϕ , and of the characteristic parameters of the medium.

However, since the medium has been assumed to be uniaxial, the required expressions would have been simpler if the analysis had been developed using a "circularly-polarized" coordinate system (STIX, 1962), in which the complex amplitudes of the sinusoidal components of the electric field vector $\vec{e}$ of an elementary plane wave are

(1.18) $e_R = e_x + i e_y$ $e_L = e_x - i e_y$ $e_P = e_z$

and similarly for the magnetic vector $\vec{h}$. The subscripts R, L, and P stand for "right-handed", "left-handed", and "parallel" respectively. We now define a generalized electric field vector $\vec{f}$, the 6 components of which have the complex amplitudes

(1.19a) $f_1 = e_R$ $f_2 = e_L$ $f_3 = e_P$
(1.19b) $f_4 = Z_0 h_R$ $f_5 = Z_0 h_L$ $f_6 = Z_0 h_P$

Then, redeveloping the analysis along the lines of section 4, we are led to define a new set of 36 kernels,

(1.20) $b_{ij} = \dfrac{f_i f_j^{\times}}{\rho}$

for which we shall now quote the expressions.

First, however, we must define the plasma parameters. In this we shall follow STIX (1962), with some minor differences : in particular, we suppose that the plasma contains only electrons and positive ions, and that the latter, of which there may be several species present, are all singly charged. Denoting the different types of charged particle - including the electrons - by a subscript k, let n_k be the number density for each type, m_k the mass

per particle, and q_k the charge per particle. For each type of particle, we now define two characteristic frequencies : the angular plasma frequency Π_k, such that

(1.21) $$\Pi^2_K = \frac{n_k q^2_k}{\varepsilon_0 m_k}$$

where ε_0 is the electric permittivity of free space, and the angular gyrofrequency

(1.22) $$\Omega_k = - \frac{q_k B_0}{m_k}$$

where B_0 is the magnitude of the induction vector of the steady magnetic field. The minus sign has been put into this definition so as to give Ω_k the sign of the natural sense of gyration of the particles around the field (e.g. positive for the electrons, which gyrate in a right-handed sense). Lastly, we define three dimensionless quantities :

(1.23a) $$R = 1 - \sum_k \frac{\Pi^2_k}{\omega(- \Omega_k)}$$

(1.23b) $$L = 1 - \sum_k \frac{\Pi^2_k}{\omega(\omega + \Omega_k)}$$

(123c) $$P = 1 - \sum_k \frac{\Pi^2_k}{\omega 2}$$

These quantities are identical with those that Stix denotes by the same symbols ; in the expressions that follow , they represent the properties of the plasma.

From these we must define several derived quantities, the first of which is the phase refractive index n for electromagnetic waves in either of the two magneto-ionic modes. This quantity is the solution of the equation

(1.24) $$\tan^2\theta = - \frac{P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}$$

which is a quadratic in n^2 with purely real roots (STIX, 1962).

For a given θ and given plasma parameters. (1.24) may have two, one, or no roots with n^2 positive. Only such positive roots, which correspond to freely propagated modes, are to be considered here ; negative roots correspond to evanescent modes. Moreover. for a given positive n^2, only the positive value of n will be considered, the negative value corresponding to a wave propagated in the opposite direction. On the questions of the nomenclature for the two modes, and of their correspondence with the roots, the reader should consult the standard works on the magneto-ionic theory (RATCLIFFE 1959, BUDDEN 1961).

One more dimensionless quantity remains to be defined. It involves the group refractive index

$$(1.25) \qquad n_g = \left[\frac{\partial}{\partial \omega} (\omega n) \right]^{-1}$$

and two other parameters :

$$(1.26) \quad \lambda = 2n \left\{ (P-n^2)\left[1+\frac{(P-L)(R-n^2)^2}{(P-R)(L-n^2)^2}\right] - (R-n^2)\left[1+\frac{(P-L)(R-n^2)}{(P-R)(L-n^2)}\right] \right\} \cos^2\theta$$

$$(1.27) \quad \mu = \frac{n(R-L)^2 (P-n^2)^2}{(P-R) (L-n^2)^2} \sin^2\theta$$

In these terms, the required quantity is

$$(1.28) \qquad \xi = 8 n_g \left[\varepsilon_0 (\lambda + \mu) \right]^{-1}$$

We are now in a position to list the 36 kernels b_{ij}. In point of fact, it suffices to list the 6 kernels with $i=j$, and 15 kernels with $i \neq j$, since the remaining 15 are given by the relation $b_{ji} = b^x_{ij}$, which follows from their definition in (1.20). Here are the 21 expressions :

$$(1.29) \quad b_{11} = \frac{(P-n^2)^2}{P-R} \xi$$

$$(1.30) \quad b_{12} = \frac{(R-n^2)(P-n^2)^2}{(L-n^2) (P-R)} \xi \exp(2i\phi)$$

$$(131) \quad b_{13} = \frac{(R-n^2)(P-n^2)}{P-R} \xi \cot\theta \exp(i\phi)$$

(1.32) $b_{14} = -i\left[n(P-n^2)\xi\cos\theta\right]$

(1.33) $b_{15} = i\left[\dfrac{n(R-n^2)(P-L)(P-n^2)}{(L-n^2)(P-R)}\ \xi\ \cos\theta\ \exp(2i\phi)\right]$

(1.34) $b_{16} = -i\left[\dfrac{n(R-L)(P-n^2)^2}{2(P-R)(L-n^2)}\ \xi\ \sin\theta\ \exp(i\phi)\right]$

(1.35) $b_{22} = \dfrac{(P-n^2)^2(R-n^2)^2}{(P-R)(L-n^2)^2}\ \xi$

(1.36) $b_{23} = \dfrac{(P-n^2)(R-n^2)^2}{(P-R)(L-n^2)}\ \xi\ \cot\theta\ \exp(-i\phi)$

(1.37) $b_{24} = -i\left[\dfrac{n(P-n^2)(R-n^2)}{L-n^2}\ \xi\cos\theta\ \exp(2i\phi)\right]$

(1.38) $b_{25} = i\left[\dfrac{n(P-n^2)(P-L)(R-n^2)^2}{(P-R)(L-n^2)^2}\ \xi\ \cos\theta\right]$

(1.39) $b_{26} = -i\left[\dfrac{n(R-L)\ (R-n^2)(P-n^2)^2}{2(P-R)(L-n^2)^2}\ \xi\sin\theta\ \exp(-i\phi)\right]$

(1.40) $b_{33} = \dfrac{(R-n^2)^2}{P-R}\ \xi\ \cot^2\theta$

(1.41) $b_{34} = -i\left[n(R-n^2)\ \xi\ \cos\theta\cot\theta\exp(-i\phi)\right]$

(1.42) $b_{35} = i\left[\dfrac{n(P-L)(R-n^2)^2}{(P-R)(L-n^2)}\ \xi\cos\theta\cot\theta\exp(i\phi)\right]$

(1.43) $b_{36} = -i\left[\dfrac{n(R-L)(R-n^2)\ (P-n^2)}{2(P-R)\ (L-n^2)}\ \xi\cos\theta\right]$

(1.44) $b_{44} = n^2(P-R)\ \xi\cos^2\theta$

(1.45) $b_{45} = -\left[\dfrac{n^2(P-L)(R-n^2)}{L-n^2}\ \xi\cos^2\theta\exp(2i\phi)\right]$

(1.46) $b_{46} = \dfrac{n^2(R-L)(P-n^2)}{2(L-n^2)}\ \xi\ \sin\theta\ \cos\theta\ \exp(i\phi)$

$$(1.47) \quad b_{55} = - \frac{n^2 (P-L)^2 (R-n^2)^2}{(P-R)(L-n^2)^2} \; \xi \; \cos^2\theta$$

$$(1.48) \quad b_{56} = - \frac{n^2 (P-L)(R-L)(P-n^2)(R-n^2)}{2(P-R)(L-n^2)^2} \; \xi \sin\theta \, \cos\theta \, \exp(-i\phi)$$

$$(1.49) \quad b_{66} = \frac{n^2 (P-n^2)^2 (R-L)^2}{4(P-R)(L-n^2)^2} \; \xi \; \sin^2\theta$$

The derivation of these results. which is straightforward but lengthy, will be published elsewhere. They apply to both magneto-ionic modes ; however, the values of n and of ξ are different for the two modes. so in general the b_{ij} are different also.

These kernels can be used directly for interpreting data on random wave fields, provided that the 6 field components are transformed to the circularly-polarized coordinate system before being used to estimate their spectral matrix. On the other hand. if one prefers to work with data in the rectangular system $Oxyz$, then one requires the kernels a_{ij}. which are given in terms of the b_{ij} by the following expressions :

$$(1.50) \quad a_{11} = \frac{1}{4} (b_{11} + b_{12} + b_{21} + b_{22})$$

$$(151) \quad a_{12} = \frac{i}{4} (b_{11} - b_{12} + b_{21} - b_{22})$$

$$(1.52) \quad a_{13} = \frac{1}{2} (b_{13} + b_{23})$$

$$(1.53) \quad a_{22} = \frac{1}{4} (b_{11} - b_{12} - b_{21} + b_{22})$$

$$(1.54) \quad a_{23} = \frac{i}{2} (b_{23} - b_{13})$$

$$(1.55) \quad a_{33} = b_{33}$$

The 3 other kernels with i and $j \leqslant 3$ can be found by using the relation $a_{ji} = a_{ij}^{x}$. From this set of 9 kernels, the remaining 27 with i and/or $j \geqslant 4$ can be obtained by adding 3 to the first and/or second subscript of every term in each expression. These results follow quite simply from the definitions (1.18) and (1.19).
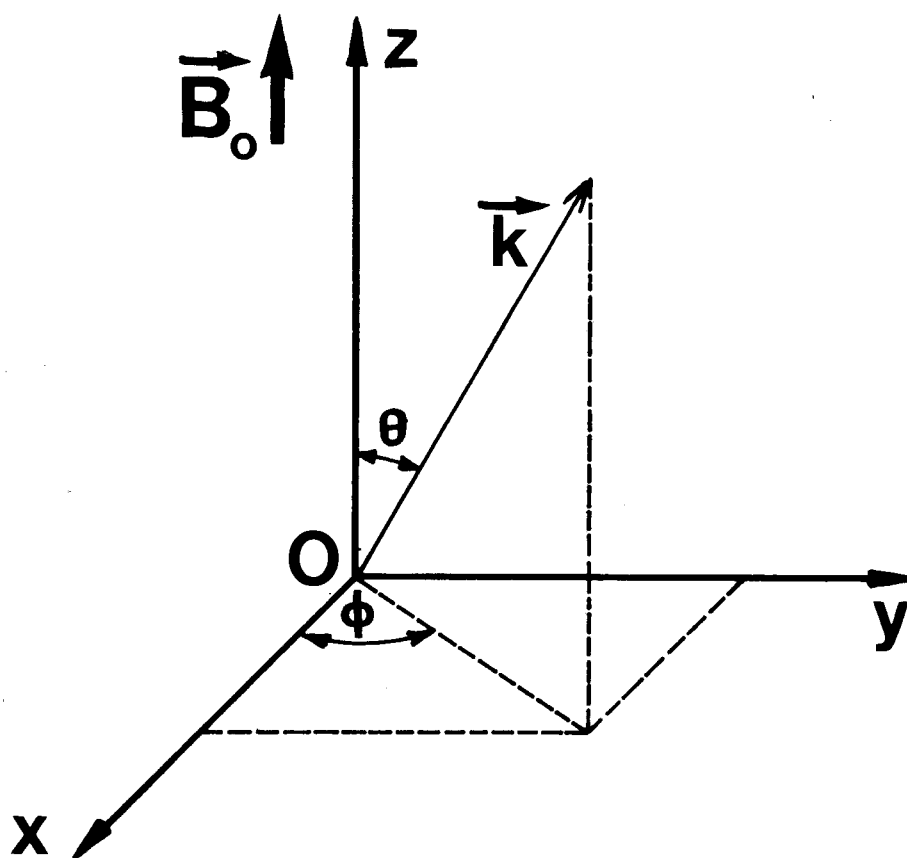
FIG. 1: COORDINATE SYSTEM USED IN THE DEFINITION OF THE WAVE DISTRIBUTION FUNCTION.
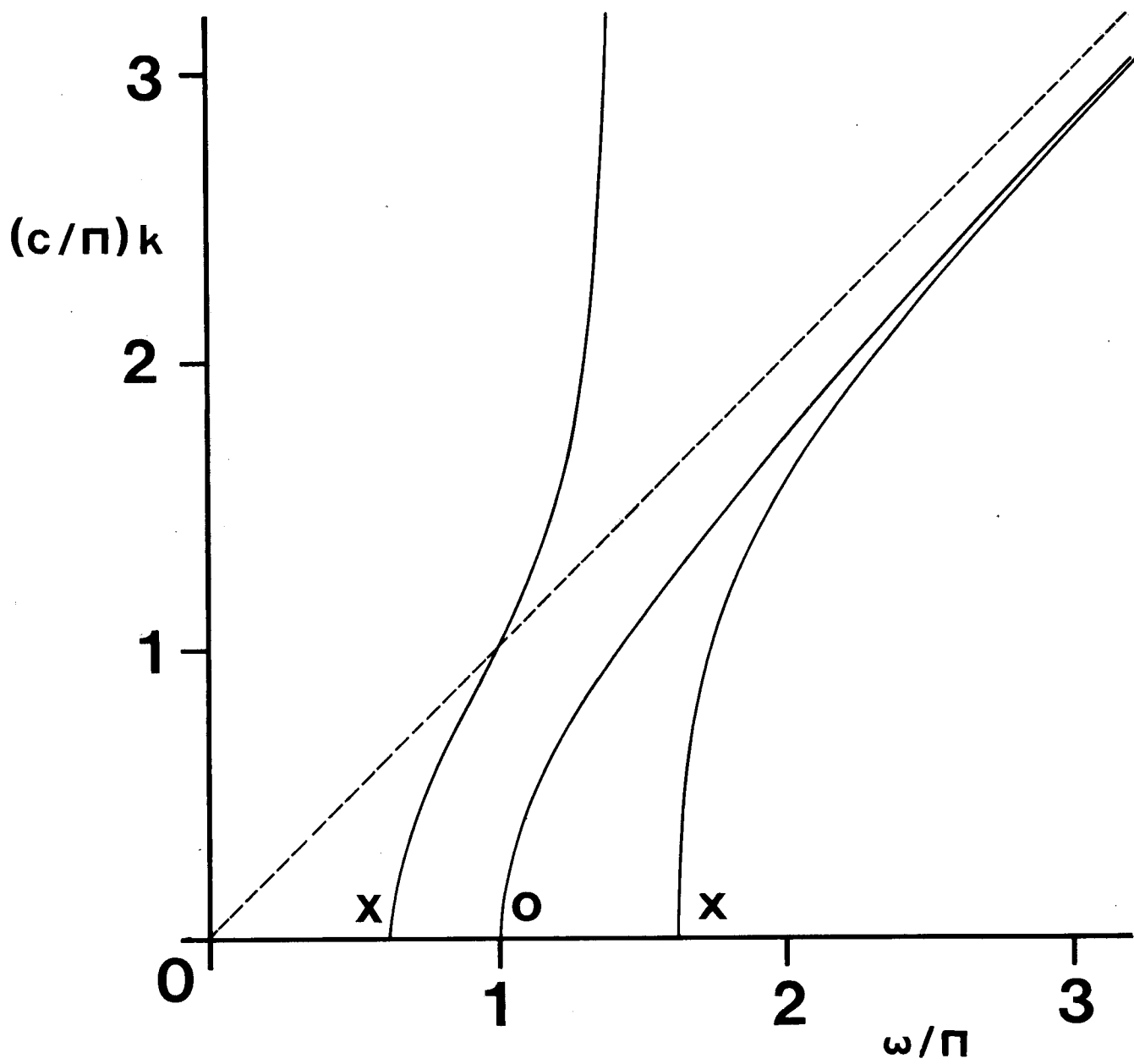
FIG. 2: DISPERSION CURVES FOR PERPENDICULAR PROPAGATION.