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► **To cite this version:**

Bruno Gaujal, Eric Thierry. Markov chain analysis of an agent-based growth model. [Research Report] LIP RR-2005-24, Laboratoire de l'informatique du parallélisme. 2006, 13p. hal-02102132

**HAL Id: hal-02102132**

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Submitted on 17 Apr 2019

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growth model***

Bruno Gaujal (ID-IMAG, INRIA) ,  
Eric Thierry

Mai 2005

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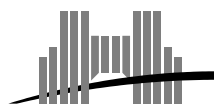
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**INRIA**



# Markov chain analysis of an agent-based growth model

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Mai 2005

## Abstract

In this paper we investigate the asymptotic behavior of a discrete and probabilistic dynamical system which can be described as a growth model where autonomous agents aggregates.

The aim of this paper is to give a mathematical analysis of the dynamics. The analysis uses face homogeneous Markov chains and thanks to this study we validate a conjecture set by Laszlo Gulyas concerning a growth model for cities where simulations had shown that the sizes of the cities asymptotically distribute as a Zipf's law. In light of our analysis, we discuss how the emergence of such a Zipf's law could be expected in Gulyas' model and in its variants.

**Keywords:** discrete probabilistic dynamics, face homogeneous Markov chains, Zipf's law.

## Résumé

Cet article étudie le comportement asymptotique d'un système dynamique discret et probabiliste qui peut être vu comme un processus de croissance où des agents se regroupent sur différents sites.

Le but de ce travail est de fournir une analyse mathématique de la dynamique. Cette analyse utilise des chaînes de Markov homogènes par zones, grâce à elle nous démontrons une conjecture posée par Laszlo Gulyas concernant un modèle de croissance des villes où les simulations ont montré que les tailles de villes finissent par être distribuée selon une loi de Zipf. Nous discutons de l'émergence d'une telle loi dans le contexte du modèle de Gulyas et dans ses variantes.

**Mots-clés:** dynamique de modèles probabilistes discrets, face homogeneous Markov chains, loi de Zipf.

# 1 Introduction

Many growth phenomena in human or natural multi-site systems exhibit a quite remarkable property, known as Zipf's law, which says that if one calculates the logarithm of the rank and of the site sizes and plot the resulting data in a diagram you will get a remarkable log-linear pattern. In other words, Zipf's law states that the size  $v$  of the  $r$ 'th largest site is a power of its rank:  $v \sim Cr^{-\alpha}$ , where  $\alpha$  is a constant greater than or equal to 1 and  $C$  a positive constant (in Zipf's original work [17],  $\alpha$  was 1).

For instance, it appears that Zipf's law governs many features of the Internet. It has been measured that distributions of the sizes of websites in term of the number of pages they include or the number of links given to other sites follow a Zipf's law [1]. This behavior of the World Wide Web has also been observed on the node degree distribution of the graph underlying the Internet backbone (the physical network) [13]. It has lead to the design of growth models in order to explain these features such as growth models by Huberman and Adamic [10] with intuitive and simple assumptions or new random graph growth models by Albert and Barabasi [2] which can be seen as agents aggregating with preferential attachments. In economics, some studies suggest that several types of ranking of companies follow Zipf's laws and that some other economic data is also concerned [3]. Another famous example is the distribution of city populations. After a seminal work of Zipf [18], city sizes growth has been investigated by many authors. One may mention the survey papers by Gabaix [5, 6, 7] which also provide their own explanations of the emergence of Zipf's laws. In another recent paper [15], Soo uses a new data set on 75 countries and shows that the Zipf's law applies (with different parameters according to the countries).

The reader interested by Zipf's laws is referred to the very comprehensive bibliography gathered by Li on <http://linkage.rockefeller.edu/wli/zipf/>.

However, in many cases, either this remarkable feature remains unexplained or rather complicated models are used to derive it.

During the 2003 Exystence Thematic Institute on Complex Systems [11], a simple probabilistic model solely based on individual decisions (autonomous agents) was presented by Laszlo Gulyas. It was introduced as a agent based model to study the growth of city sizes [9, 14]. This model consists in a finite number of sites, which can be seen as the cities, which are ranked by their size, then some agents arrive sequentially and join the cities with the following rule: when arriving, each agent has a random set of eligible cities to join, he then decides to join the largest city proposed to him (and thus increase by one its size). While this process is only based on individual decisions of the agents, Gulyas observed on simulations that the distribution of the city sizes asymptotically behaves like a Zipf's law. He conjectured that its parameter  $\alpha$  was equal to 2.

Motivated by this conjecture, our work focusses on the mathematical analysis of a discrete probabilistic dynamics for which Gulyas' model is a special case. As a result, we prove that Gulyas' conjecture is true and that some variants of its model also follow Zipf's law.

As mentioned earlier, a large amount of work has been done to try explaining the recurrent apparition of Zipf's laws when studying city sizes growth. Note that our aim is not to compare all these previous models with Gulyas' new model or to confront its behavior with real data. Our work presents some mathematical tools that prove to be useful in the analysis of such discrete probabilistic models. In particular our proofs are based on a face homogeneous Markov chain description of the dynamics of the system.

In Section 2, we present a general growth model for which we provide an asymptotic

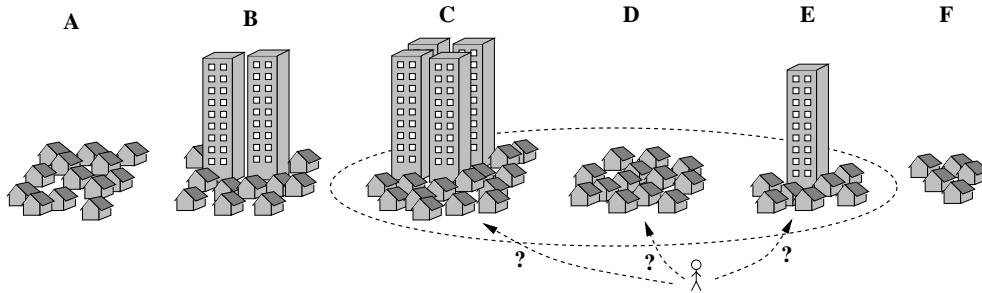


Figure 1: Once the set of possible choices is fixed ( $\{C, D, E\}$ ), the new inhabitant joins the largest city ( $\{C\}$ ).

analysis. In Section 3, we precisely describe Gulyas' model and we show how it falls into the framework presented before. Applying the preceding asymptotic analysis, we prove that his conjecture about the distribution of sizes is true. We then show that the general growth model apply to variants of Gulyas' model yielding some other Zipf's laws. Section 4 concludes the paper by summing up features that made the analysis work and by presenting some open problems.

## 2 The growth model and its asymptotic analysis

### 2.1 Presentation of the model

We consider the following growth model in discrete time, where agents sequentially aggregate with several sites. We will consider that these are new inhabitants joining cities. Let  $C = \{c_1, \dots, c_n\}$  be a finite set on  $n$  cities. The respective populations in the cities at time  $t$  are denoted  $s_1(t), \dots, s_n(t)$ . The initial population is set to 0 (i.e.  $s_i(0) = 0$  for all  $1 \leq i \leq n$ ).

A set of probabilities  $\{p_1, \dots, p_n\}$  is also given ( $p_1 + \dots + p_n = 1$ ).

At time  $t$ , the cities can be ranked by their respective populations, let  $\sigma_t$  be one permutation of  $\{1, \dots, n\}$  such that  $s_{\sigma_t(1)}(t) \geq \dots \geq s_{\sigma_t(n)}(t)$ .

At time  $t + 1$ , a new person arrives. This person joins one of the cities with the following probabilities: he joins the  $i$ 'th largest city  $c_{\sigma_t(i)}$  with probability  $p_i$ . In other words,  $s_{\sigma_t(i)}(t + 1) = s_{\sigma_t(i)}(t) + 1$  with probability  $p_i$ .

Note that this may induce a new permutation  $\sigma_{t+1}$  if this arrival has altered the order of the sizes of the cities. Note also that the behavior of this system is not completely well defined when several cities have the same size. Indeed, in such cases, several permutations  $\sigma_t$  verify  $s_{\sigma_t(1)}(t) \geq \dots \geq s_{\sigma_t(n)}(t)$  and the growth of each city at time  $t$  depends on  $\sigma_t$ . However it should be clear that if we focuss on the sequence of city sizes  $s_{\sigma_t(1)}(t) \geq \dots \geq s_{\sigma_t(n)}(t)$  at time  $t$ , it does not depend on the choices of  $\sigma_1, \dots, \sigma_t$  made at all steps up to  $t$ .

Our aim is to study the evolution of the city sizes when  $t$  goes to infinity and under the assumption that the sequence  $p_1, \dots, p_n$  is strictly positive and **decreasing** ("people prefer larger cities").

For notation purposes, we also set  $p_{n+1} = 0$  and we introduce  $I(x) = \max\{i \mid 1/p_i \leq x\}$  that will be used later.

## 2.2 An equivalent model

We consider another representation of the model where each city is replaced by a token and its size is replaced by the position (or slot) of the token over the integer line. If several tokens have the same position, they are piled up on their slot. An example of the old and new representations is given in Figure 2.

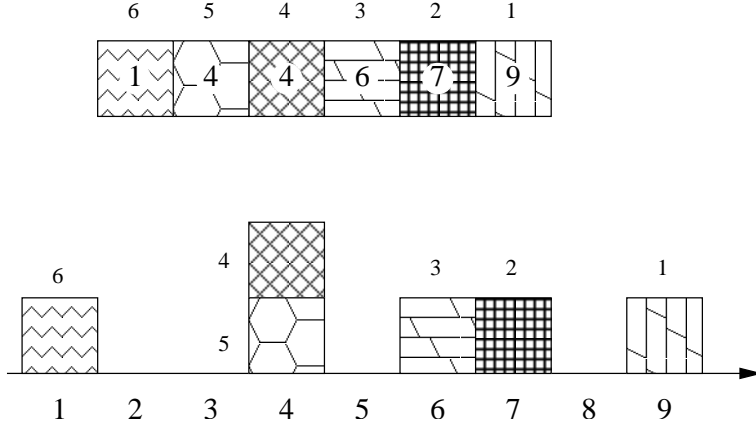


Figure 2: The old representation (size on each token) and the new one representation (each token at a position equal to its size).

Note that the selection process for the new model makes one token (the right most among all the picked tokens) move to the right by one slot. The models differ slightly in the presence of ties. In the new model, if two tokens are in the same position, we decide to select the one on "top", rather than selecting arbitrarily. Similarly when one token moves to the right and joins other tokens at its new position, we choose to insert it at the bottom of the stack. This convention has one main feature which will be used in the following: the relative order between the tokens never changes: the tokens cannot overtake each other. Without loss of generality, we number them in decreasing order: the rightmost token is labeled by 1 and the last token (leftmost) is labeled by  $n$  (in a stack of tokens, the labels are decreasing from top to bottom). The position (or value) of token  $i$  is denoted by  $v_i$ .

It should be obvious that the probabilistic laws of both models are the same. In the remaining, we will rather use the token model where no overtaking can occur because it makes the model easier to describe, however, all the analysis below could have been made using the original model.

## 2.3 Markov chain analysis

The state of the system at step  $t$   $V(t) \stackrel{\text{def}}{=} (v_1(t), \dots, v_n(t))$  obviously forms a Markov chain with a state space included in  $\mathbb{N}^n$ , since the future evolution only depends on the present state. However, its structure is rather complicated and cannot to be analyzed directly. It should be clear that all the tokens "tend" to move to the right during the evolution. However, to

compute the speed of this shift to the right, one needs to look first at second order quantities such as the relative positions of the tokens.

Let  $g_i, 1 \leq i \leq n$ , be the size of the gap between token  $i$  and token  $i + 1$ . For all  $1 \leq i \leq n - 1$ ,

$$g_i = v_i - v_{i+1} \text{ and } g_n = v_n$$

We also denote by  $h_i$  the smallest label of all tokens piled above token  $i$ .

$$h_i = \min\{j \leq i, v_j = v_i\} = \min\{j \leq i, \prod_{k=i}^j g_k = 0\}.$$

It should also be obvious that  $G = (g_1, \dots, g_n)$  is a Markov chain living in  $\mathbb{N}^n$ , where the space can be divided in  $2^n$  zones  $Z_S = \{(g_1, \dots, g_n) \in \mathbb{N}^n \mid \forall i \in S, g_i = 0\}$  for all  $S \subseteq \{1, \dots, n\}$  on which the chain is homogeneous. It is a face-homogeneous Markov chain [4].

The chains  $V = (v_1, \dots, v_n)$  and  $G = (g_1, \dots, g_n)$  have the following transition kernels, respectively denoted by  $O$  and  $P$ :

At step  $t + 1$ , with probability  $p_i, 1 \leq i \leq n$ ,

$$\begin{aligned} (v_1, \dots, v_n) &\xrightarrow{p_i} (v_1, \dots, v_{h_i} + 1, \dots, v_n) \\ (g_1, \dots, g_n) &\xrightarrow{p_i} (g_1, \dots, g_{h_i-1} - 1, g_{h_i} + 1, \dots, g_n) \end{aligned}$$

The kernel  $P$  of the chain  $G$  is displayed in Figure 3 for  $n = 2$ .

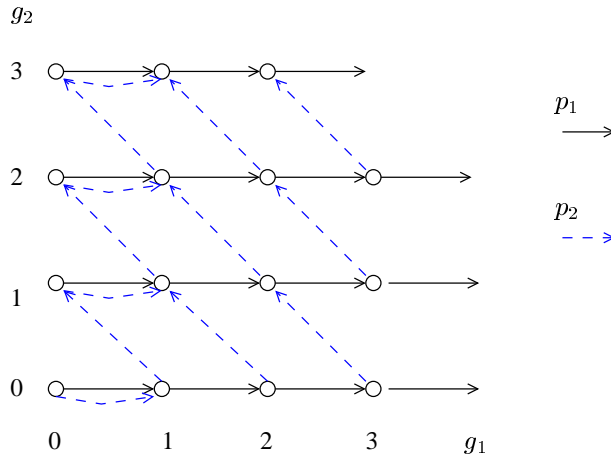


Figure 3: The kernel  $P$  for  $n = 2$ .

**Lemma 1.** *The chain  $G = (g_1, \dots, g_n)$  reaches states where  $g_i > 0$  for all  $1 \leq i \leq n$ , in finite time, almost surely.*

*Proof.* The proof involves the construction of a new chain  $B(t) = (b_1(t), \dots, b_n(t))$  with a bounded state space. As shown below, this new chain will reach states with no null components with probability one and this will imply that  $G$  reaches states where  $g_i > 0$  for all  $i$ , with probability one.

Here is the description of  $B(t) = (b_1, \dots, b_n)$ . The state space is  $\{0, 1, 2\}^n$ . We define a function  $f_i$  similar to  $h_i$  but for the chain  $B$ .

$$f_i = \min\{j \leq i, \prod_{k=i}^j b_k = 0\}.$$

The transition kernel  $W$  of  $B$  is the following. For each  $1 \leq i \leq n$ ,

$$\begin{aligned} (b_1, \dots, b_n) &\xrightarrow{p_i} (b_1, \dots, b_{f_i-1} - 1, b_{f_i} + 1, \dots, b_n) && \text{if } f_i = i \text{ and } b_{f_i} < 2, \\ (b_1, \dots, b_n) &\xrightarrow{p_i} (b_1, \dots, b_{f_i-1} - 1, b_{f_i}, \dots, b_n) && \text{if } f_i = i \text{ and } b_{f_i} = 2, \\ (b_1, \dots, b_n) &\xrightarrow{p_i} (b_1, \dots, b_{f_i-1} - 1, b_{f_i}, \dots, b_n) && \text{if } f_i < i. \end{aligned} \quad (1)$$

The finite chain  $B$  is displayed in Figure 4. It should be clear that the chain  $B(t)$  can reach state  $(2, \dots, 2)$  from any starting point, with positive probability.

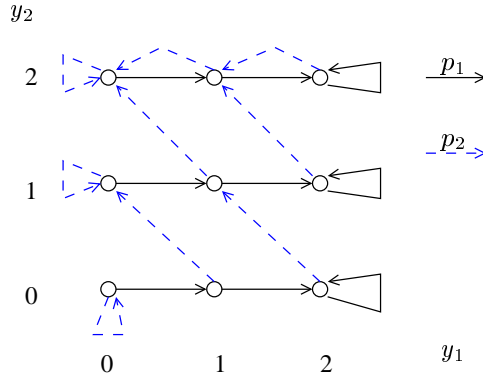


Figure 4: The kernel  $W$  when  $n = 2$ .

Now, we claim that if in the initial state,  $B(0) \leq G(0)$ , component-wise, then at any time  $t$ ,  $B(t) \leq_{st} G(t)$ . This can be shown by induction on  $t$ . Assume that  $B(t) \leq_{st} G(t)$ , then, by coupling the transitions in both chains and looking at the transition kernel, we will show that  $B(t+1) \leq_{st} G(t+1)$ .

With probability  $p_i$ ,

$$G(t+1) = (g_1(t), \dots, g_{h_i-1}(t) - 1, g_{h_i}(t) + 1, \dots, g_n(t))$$

$$B(t+1) = (b_1(t), \dots, b_{f_i-1}(t) - 1, b_{f_i}(t) + 1, \dots, b_n(t)) \quad (2)$$

$$\text{or } (b_1(t), \dots, b_{f_i-1}(t) - 1, b_{f_i}(t), \dots, b_n(t)). \quad (3)$$

First, note that if  $f_i = h_i$ , then it is straightforward to check that in both cases, if  $B(t) \leq_{st} G(t)$ , then  $B(t+1) \leq_{st} G(t+1)$ .

Since  $B(t) \leq_{st} G(t)$ , the only case remaining is  $f_i < h_i \leq i$ . This implies that  $b_{h_i-1}(t) = 0$  and the transition kernel of  $B$  uses case (3).

Since  $G(t+1)$  remains non-negative by construction, this implies

$$\begin{aligned} g_{h_i}(t+1) &= g_{h_i}(t) + 1 \geq b_{h_i}(t) + 1 \geq b_{h_i}(t+1) \\ g_{h_i-1}(t+1) &= g_{h_i-1}(t) - 1 \geq 0 = b_{h_i-1}(t) = b_{h_i-1}(t+1) \\ g_{f_i}(t+1) &= g_{f_i}(t) \geq b_{f_i}(t) = b_{f_i}(t+1) \quad (\text{if } f_i < h_i - 1) \\ g_{f_i-1}(t+1) &= g_{f_i-1}(t) \geq b_{f_i-1}(t) \geq b_{f_i-1}(t+1). \end{aligned}$$



All the other coordinates are unchanged in  $G$  and in  $B$  with probability  $p_i$ .

This coupling shows by induction that  $G(0) \leq_{st} B(0)$  implies for all  $t$ ,  $G(t) \leq_{st} B(t)$ .

Now, to finish the proof, it is enough to note that since  $B(t)$  reaches states where  $b_i > 0$  for all  $i$  in finite time, then so does  $G(t)$  with probability one.  $\square$

The second step of the analysis of the chain  $G(t)$  is to show that there is a drift towards states where the gaps between the tokens grow.

**Lemma 2.** *After some finite time  $t_0$ , the chain  $G(t)$  has all its components that remain positive almost surely. Furthermore, for all  $i$ ,*

$$\lim_{t \rightarrow \infty} \frac{g_i(t)}{t} = p_i - p_{i+1}.$$

*Proof.* Let us first construct a Markov chain  $S(t) = (s_1(t), \dots, s_n(t))$  with the following kernel  $Q$ .

If  $X$  has all its components positive, then  $Q(X, Y) = Pr(G(1) = Y | G(0) = X) = P(X, Y)$ .

If  $X$  has some null coordinates then using Lemma 1, the time

$$\nu(X) = \inf\{t \geq 0, s.t. G(t) \geq (1, \dots, 1), G(0) = X\}$$

is finite almost surely and uniformly bounded. Indeed, for all  $X$  with null coordinates, let us denote by  $X_2 \stackrel{\text{def}}{=} (\max(X_1, 2), \dots, \max(X_n, 2))$  the state  $X$  with all its components truncated at 2. Then

$$\begin{aligned} \nu(X) &\leq_{st} \inf\{t \geq 0, s.t. B(t) \geq (1, \dots, 1), B(0) = X_2\} \\ &\leq_{st} \max_{Z \in \{0,1,2\}^n} \inf\{t \geq 0, s.t. B(t) \geq (1, \dots, 1), B(0) = Z\}, \end{aligned}$$

which does not depend on  $X$ . Then, we set  $Q(X, Y) = Pr(G(\nu) = Y | G(0) = X) = P^{\nu(X)}(X, Y)$ .

Now, let us consider the Lyapounov function  $\min : \mathbb{R}^n \rightarrow \mathbb{R}$ , which has bounded average increments for the kernel  $Q$ . For all  $X \in \mathbb{N}^n$ , if  $X$  has some null components

$$\mathbb{E}(\min(s_1(1), \dots, s_n(1)) - \min(s_1(0), \dots, s_n(0)) | (s_1(0), \dots, s_n(0)) = X) \geq 1,$$

and if  $X$  has no null components

$$\mathbb{E}(\min(s_1(1), \dots, s_n(1)) - \min(s_1(0), \dots, s_n(0)) | (s_1(0), \dots, s_n(0)) = X) \geq \min_{i=1}^n (p_i - p_{i+1}).$$

An adequate version of Foster's theorem, (see for example [12]), shows that the chain  $S$  is transient and  $\min(s_1(t), \dots, s_n(t))$  goes to infinity when  $t$  goes to infinity. This implies that there exists a finite time  $\tau$  such that for all  $t \geq \tau$ ,  $s_i(t)$  remains strictly positive for all  $i$ . Also, the transition kernels of  $S$  and  $G$  coincide as soon as  $X > 0$  for all components. Therefore, there exists two finite random variables  $n_0$  and  $m_0$  such that  $S(t) = G(t + m_0)$  for all  $t \geq n_0$ . This means that after a finite time  $t_0$ , all the components of  $G$  remain positive. This ends the proof the first statement.

As for the second statement, consider  $(\delta_j)_{j \in \mathbb{N}}$ , an i.i.d. sequence such that  $\delta_j = 1$  with probability  $p_i$ ,  $\delta_j = -1$  with probability  $p_{i+1}$ , and  $\delta_j = 0$  with probability  $1 - p_{i+1} - p_i$ .

Since  $g_i(t)$  is strictly positive almost surely after a finite time  $t_0$  and since for all  $t \geq t_0$ ,  $g_i(t) - g_i(t_0) = \sum_{j=t_0}^t \delta_j$ , the strong law of large numbers yields

$$\lim_{t \rightarrow \infty} \frac{g_i(t)}{t} = \lim_{t \rightarrow \infty} \frac{\sum_{j=1}^t \delta_j}{t} = p_i - p_{i+1} \quad a.s.$$

□

Using the same method as in the previous lemma, one can use the central limit theorem to get a more precise result on the behavior of  $g_i(t)$  when  $t$  goes to infinity (in distribution).

$$\lim_{t \rightarrow \infty} \frac{|g_i(t) - t(p_i - p_{i+1})|}{\sigma^2 \sqrt{t}} = \mathcal{N}(0, 1)$$

Where  $\mathcal{N}(0, 1)$  is the normal distribution with mean 0 and variance 1 and  $\sigma^2 = p_i + p_{i+1} - p_i^2 - p_{i+1}^2 + 2p_i p_{i+1}$ .

Let  $N_\ell(t)$  be the number of tokens whose positions are larger than  $\ell$  after  $t$  steps.

$$N_\ell(t) = \max\{i, \sum_{j=i}^n g_j(t) \geq \ell\}.$$

Using the central limit theorem,

$$\sum_{j=i}^n g_j(t) = p_i t + \sqrt{t} S_i(t), \quad (4)$$

where  $S_i(t)$  is a centered random variable such that  $\lim_{t \rightarrow \infty} S_i(t)$  is a finite random variable with zero mean.

If one denotes  $k = N_\ell(t)$ , then by definition of  $k$ ,  $p_k t + \sqrt{t} S_k(t) \geq \ell > p_{k+1} t + \sqrt{t} S_{k+1}(t)$ . If  $\ell = \Omega(t^{1/2+\varepsilon})$ , for some  $\varepsilon > 0$  and since  $\lim_{t \rightarrow \infty} \sqrt{t} S_i(t) / (t^{1/2+\varepsilon}) = 0$  a.s., one gets by inverting

$$1/p_k \leq t/\ell \leq 1/p_{k+1},$$

as long as  $t$  is large enough. Finally we get the following result

**Theorem 3.** *When  $\ell = \Omega(t^{1/2+\varepsilon})$ , for some  $\varepsilon > 0$ , the number  $N_\ell(t)$  of token whose position are larger than  $\ell$  after  $t$  steps verifies in distribution*

$$N_\ell(t) = I(t/\ell) + o(I(t/\ell)), \quad (5)$$

where  $I(t/\ell) = \max\{i \mid 1/p_i \leq t/\ell\}$ .

**Remark 4.** *Let  $p_i$  be a Zipf's law with parameter  $\alpha$ . In the above discussion, Equation (4) shows that when  $t$  is large, the sizes of the cities follow a Zipf's law (with parameter  $\alpha$ ). As for Equation (5), it gives the related cumulative distribution function, which is Pareto (with parameter  $1/\alpha$ ), i.e. at time  $t$ , the probability for city sizes to be larger than  $\ell$  is approximatively  $A\ell^{-1/\alpha}$  where  $A = (Ct)^{1/\alpha}/n$  since  $p_i \sim Ci^{-\alpha}$  and  $I(t/\ell) = \max\{i \mid 1/p_i \leq t/\ell\} \sim (\frac{\ell}{Ct})^{-1/\alpha}$ .*

**Remark 5.** *It should also be clear from all the reasoning that initializing the process with city sizes different from zero does not change the asymptotic behavior.*

**Remark 6.** *We have used the fact that the sequence  $(p_i)_{1 \leq i \leq n}$  is decreasing to apply Forster's theorem which ensures that from a certain time the gap between city sizes remains strictly positive and then the city of rank  $i$  grows almost independently from others at speed  $p_i$ .*

*If the sequence  $(p_i)_{1 \leq i \leq n}$  is not decreasing, then the asymptotic behavior of the face-homogeneous Markov chain is more difficult to analyze. In general the cities group into clusters, each with the same growing speed. However, given the  $p_i$ 's, it is an open question to know, for example, if all sites form a single cluster (which is the same as assessing the ergodicity of the corresponding face-homogeneous Markov Chain, see [4] for discussions about this).*

### 3 Application to Gulyas' Model

The next model was proposed by Laszlo Gulyas as an alternative simple model for city growth. Given a set of  $n$  cities with initial populations equal to 0, new inhabitants arrive sequentially. When a new inhabitant arrives, a random sample of cities is drawn by first picking at random an integer  $k$  between 1 and  $n$  (with uniform distribution) and then a random subset of  $k$  cities from the  $n$  ones (with uniform distribution). The person joins the city with the largest size in the selected subset (if there are several ones, choose one of them arbitrarily) and it increments its size by 1.

Running some simulations, Laszlo Gulyas observed that the distribution of city sizes follows a Zipf's law. He conjectured that its parameter was 2.

We will first show that Gulyas' model is a special case of our general model presented in Section 2, and then apply our analysis to validate Gulyas' conjecture.

#### 3.1 The probability to choose one city

With this model, the probability  $p_i$  that the  $i$ 'th largest city is chosen and has its size incremented by 1 is equal to:

$$\frac{1}{n} \sum_{k=0}^{n-i} \frac{\binom{n-i}{k}}{\binom{n}{k+1}},$$

since this city can be chosen only if it belongs to a subset of cardinality  $k+1$  with  $1 \leq k+1 \leq n-i+1$  and once  $k+1$  is fixed (with probability  $1/n$ ), among the  $\binom{n}{k+1}$  subsets of cardinality  $k+1$  only the ones containing the  $i$ 'th largest city and  $k$  cities taken from the  $n-i$  smaller ones will lead to its choice (thus it occurs with probability  $\binom{n-i}{k}/\binom{n}{k+1}$ ).

This sum has a closed and very simple expression that we may compute thanks to formal series. We will use the equality:

$$\sum_{k \geq 0} \binom{k+m}{k} x^k = \frac{1}{(1-x)^{m+1}}$$

We have:

$$\begin{aligned}
\frac{1}{n} \sum_{k=0}^{n-i} \frac{\binom{n-i}{k}}{\binom{n}{k+1}} &= \frac{1}{n} \sum_{k=0}^{n-i} \frac{(n-i)!}{n!} (k+1) \frac{(n-k-1)!}{(n-i-k)!} \\
&= \frac{1}{n} \frac{(n-i)!}{n!} \sum_{k=0}^{n-i} (k+1) \frac{(n-k-1)!}{(n-i-k)!} \\
&= \frac{(i-1)!}{n} \frac{(n-i)!}{n!} \sum_{k=0}^{n-i} (k+1) \frac{(n-k-1)!}{(n-i-k)!(i-1)!} \\
&= \frac{(i-1)!}{n} \frac{(n-i)!}{n!} \underbrace{\sum_{k=0}^{n-i} (k+1) \binom{n-k-1}{i-1}}_{(*)}
\end{aligned}$$

By denoting  $[x^n]S(x)$  the coefficient of  $x^n$  in the series  $S(x)$ , the under-braced sum  $(*)$  is equal to:

$$\begin{aligned}
\sum_{k=1}^{n-i+1} k \binom{n-k}{i-1} &= [x^n] \left( \sum_{k \geq 1} k x^k \right) \left( \sum_{j \geq 0} \binom{j}{i-1} x^j \right) \\
&= [x^n] \left( \sum_{k \geq 1} k x^k \right) \left( \sum_{j \geq 0} \binom{j+i-1}{i-1} x^{j+i-1} \right) \\
&= [x^n] x^i \underbrace{\left( \sum_{k \geq 1} k x^{k-1} \right)}_{\frac{1}{(1-x)^2}} \underbrace{\left( \sum_{j \geq 0} \binom{j+i-1}{j} x^j \right)}_{\frac{1}{(1-x)^i}} \\
&= [x^n] \frac{x^i}{(1-x)^{i+2}} \\
&= [x^{n-i}] \frac{1}{(1-x)^{i+2}} \\
&= \binom{n+1}{i+1}
\end{aligned}$$

By reintroducing this value in the initial sum, we obtain for the probability to increment the  $i^{\text{th}}$  greatest integer:

$$p_i = \frac{(i-1)!}{n} \frac{(n-i)!}{n!} \binom{n+1}{i+1} = \frac{n+1}{n} \frac{1}{i(i+1)}$$

The sequence of  $p_i$  is strictly positive and decreasing, thus we can apply all the results of Section 2. Since the  $p_i$ 's follow a Zipf's law with parameter 2 and  $I(x) = \max\{i \mid 1/p_i \leq x\} \sim ((n+1)x/n)^{1/2}$ , with Remark 4, we prove Gulyas' conjecture:

**Theorem 7.** *In Gulyas' growth model, the city sizes asymptotically distribute as a Zipf's law with parameter 2. More precisely when  $t$  tends to infinity, the size of the  $i^{\text{th}}$  largest city  $v_i(t)$  verifies  $v_i(t) = \frac{n+1}{n} \frac{t}{i(i+1)} + o(\sqrt{t})$  and the number  $N_\ell(t)$  of cities of size larger or equal to  $\ell$  verifies  $N_\ell(t) \sim \frac{n+1}{nt}^{1/2} t^{-1/2}$  when  $\ell = \Omega(t^{1/2+\varepsilon})$ .*

### 3.2 Simulation

The Zipf's law in the previous model was observed by Gulyas thanks to simulation. For a bench of experimental results, the reader is referred to his work on such models [9, 14].

We have also run some simulations of the system, mainly to estimate the speed of convergence towards the asymptotic regime given above. Section 2 gives some clues about it, but there are several hidden functions or constants which asymptotically disappear but play a role in the convergence speed. For now, a precise quantification of this speed remains a difficult problem.

Figure 5 presents an example of the evolution of the system when  $n = 100$  for a sequence of random draws. At time  $t = 100, 1000, 10000, 100000$  and  $1000000$ , we have plotted the distribution of the 100 city sizes, each city having as coordinates its rank and its size. We have also plotted the ideal curves where the points should be located under the asymptotic regime, namely  $p_i t \approx t / (i * (i + 1))$  for the point of rank  $i$ . We have used log-scaled axes to put the data all together and to point out the power law behavior.

It can be clearly seen that the output data match the mathematical analysis: it converges towards the ideal curves which correspond to Zipf's laws with parameter 2.

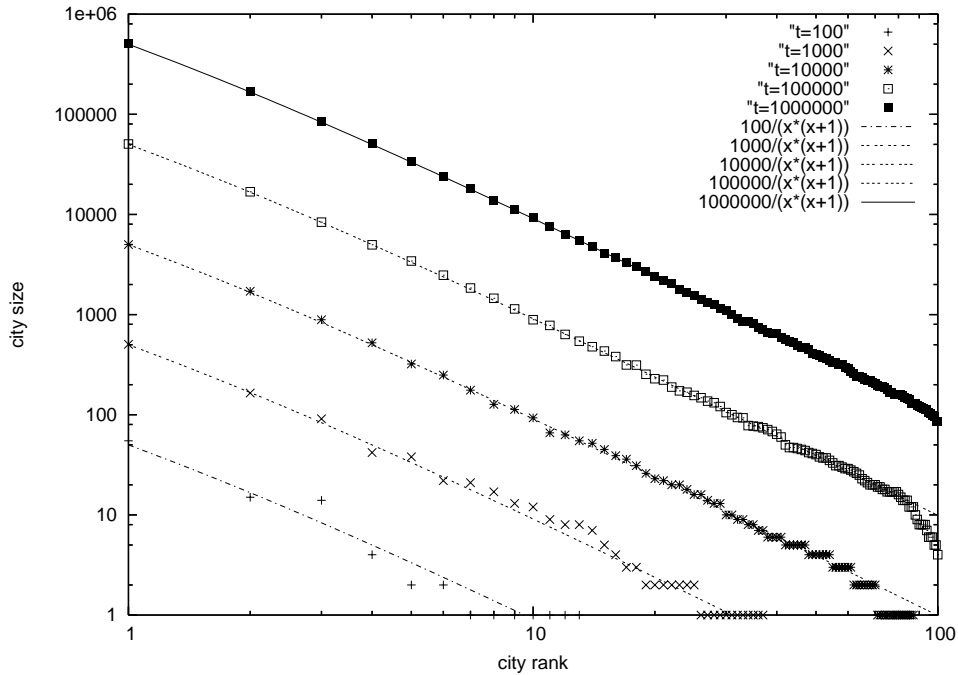


Figure 5: The sizes of cities depending on their rank at various times during the process, with log-scaled axes.

The simulation uses aliasing techniques to select which city will be joined by the new person in constant time [16].

### 3.3 Variations on this agent-based model

A natural way to generalize Gulyas' model is to keep the scheme that a newcomer chooses the largest city proposed to him but to change the probability distribution used to pick the

random sample of cities.

Suppose that we do not wish that the city names play any role in the random sampling, i.e. we do not wish to impose before starting some correlated growth of cities. Then given the  $n$  initial cities  $c_1, \dots, c_n$ , for all  $1 \leq k \leq n$ , all the subsets of  $\{c_1, \dots, c_n\}$  of cardinality  $k$  should be picked with the same probability. Then the system is fully defined by a set of probabilities  $(\alpha_k)_{1 \leq k \leq n}$ ,  $\sum_{1 \leq k \leq n} \alpha_k = 1$ , where  $\alpha_k$  is the probability to choose the cardinality  $k$  and thus the probability to pick a fixed subset of cardinality  $k$  is  $\alpha_k / \binom{n}{k}$ . Gulyas' original model is when you take  $\alpha_k = 1/n$  for all  $1 \leq k \leq n$ .

Given the sequence  $(\alpha_k)_{1 \leq k \leq n}$ , this model is clearly equivalent to the general model of Section 2 where the probability  $p_i$  to join the  $i$ 'th largest city is equal to:

$$\sum_{k=0}^{n-i} \alpha_{k+1} \frac{\binom{n-i}{k}}{\binom{n}{k+1}}.$$

Moreover the sequence  $(p_i)_{1 \leq i \leq n}$  is decreasing if and only if there exists  $2 \leq k_0 \leq n-1$  such that  $\alpha_{k_0} > 0$ . In that case, all the analysis of Section 2 applies. We have seen that asymptotically the size of the  $i$ 'th largest city is approximately  $p_i t$ , thus the emergence of a Zipf's law is directly linked to the form of the  $p_i$ 's and not to intricate combinatorics due to cities overtaking each other.

Note that formal series may be also used here to compute the values of  $p_i$ . All the reasoning of Section 3.1 applies and following the same steps, it can be checked that :

$$p_i = \frac{(i-1)!(n-i)!}{n!} [x^{n-i}] \frac{S'(x)}{(1-x)^i},$$

where  $[x^{n-i}]$  is the coefficient of  $x^{n-i}$  in the series and  $S(x) = \sum_{k=1}^n \alpha_k x^k$ ,  $S'(x)$  being its derivative.

## 4 Conclusion

The main feature of this paper is to provide an example of face homogeneous Markov chain that may appear in the modelization of real discrete systems. The mathematical analysis matches simulation results and offers a powerful prediction tool. As shown in the preceding section, the general model we have presented includes some agent-based and multi-site growth models where each site has a value (which may be seen as its power of attraction) and where new agents preferentially choose the more attractive sites among the ones proposed to them (and then increase the attraction of the chosen sites).

Several variations of this type of dynamics may appear in models of real systems.

For instance, if the sequence  $(p_i)_{1 \leq i \leq n}$  is not decreasing, the behavior of the asymptotic distribution of sizes is more difficult to describe and the mathematical analysis is much more difficult. Even worse, it might be undecidable to predict some features of the dynamics [8].

Of course, many other features could be added to the general model described in this paper. One may cite: some synchronicity of the arrivals, some departures from the sites also depending on the site ranks, non-uniform distribution when drawing subsets of the same

cardinal e.g. to take into account the geography of the sites and the fact that newcomers will choose their destination among close sites which leads to a correlated evolution of their sizes ... Some of these variations may drastically change the long run behavior and thus the study of each of them constitutes a new interesting challenge.

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