Effective lower and upper bounds for the Fourier coefficients of powers of the modular invariant j

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Abstract

Using an elementary approach based on careful handlings of Cauchy integrals, we give precise effective lower and upper bounds for the Fourier coefficients of powers of the modular invariant j. Moreover, we adapt an old result of Rademacher to get a convergent series expansion of these Fourier coefficients and we show that this expansion allows to find again these estimates. Our results improve previous ones by K. Mahler and O. Herrmann. In particular, we show that the Fourier coefficients of j are smaller than their asymptotically equivalent given by Petersson and Rademacher.

Keywords: modular function, modular invariant, Fourier coefficients, circle method, Hankel function

Résumé

En utilisant une approche élémentaire consistant en une manipulation précise d'intégrales de Cauchy, nous établissons des minorants et des majorants fins des coefficients de Fourier des puissances de l'invariant modulaire j. De plus, nous adaptons un résultat ancien de Rademacher pour obtenir un développement en série convergente de ces coefficients de Fourier et nous montrons que ce développement permet de retrouver ces estimations. Nos résultats améliorent les résultats de K. Mahler et O. Hermann sur ce sujet. En particulier, nous démontrons que les coefficients de Fourier de j sont inférieurs à leur équivalent en l'infini établi par Petersson et Rademacher.

Mots-clés: fonction modulaire, invariant modulaire, coefficients de Fourier, méthode du cercle, fonctions de Hankel

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Nicolas Brisebarre * **and Georges Philibert** †

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1. Introduction

Let $\mathfrak{H} = \{z \in \mathbb{C}; \Im(z) > 0\}$ denote Poincaré's half-plane. The modular invariant j can be defined by

$$j(\tau) = \frac{(1 + 240\sum_{n\geq 1}\sigma_3(n)e^{2i\pi n\tau})^3}{e^{2i\pi\tau}\prod_{n\geq 1}(1 - e^{2i\pi n\tau})^{24}}, \text{ for } \tau \in \mathfrak{H}, \tag{1}$$

where $\sigma_3(n) = \sum_{d|n} d^3$. It is a modular function[1] of weight 0. More precisely, j is holomorphic on $\mathfrak{H}$, meromorphic at infinity (with a simple pole of residue 1 at infinity) and it satisfies the following property of modular invariance

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2}$$

From (1), we can define a function J holomorphic in the punctured unit disk $D' = \{z \in \mathbb{C} : 0 < |z| < 1\}$ such that $j(\tau) = J(e^{2i\pi\tau})$. We have

$$J(q) = \frac{(1 + 240\sum_{n\geq 1}\sigma_3(n)q^n)^3}{q\prod_{n\geq 1}(1 - q^n)^{24}} \text{ for } q \in D'.$$

It follows that the coefficients of the Laurent expansion of J in 0 (which are the Fourier coefficients of j) are all nonnegative rational integers. For $m \in \mathbb{N} \setminus \{0\}$, we will denote j^m the function $\tau \in \mathfrak{H} \mapsto (j(\tau))^m$ and J^m the function $q \in D' \mapsto (J(q))^m$. We put $J^m(q) = \dfrac{1}{q^m} + \sum_{n\geq -m+1} c_m(n)q^n$ for all $m \geq 1$. When $m = 1$, we write $c(n)$ instead of $c_1(n)$. The computation of the coefficients $c(n)$ posed problems. The first seven were obtained in 1916 [Ber], the first twenty-four in 1939 [Zuc] and the first hundred in 1953[2] [VW].

In 1932, H. Petersson [Pet] proved $c(n) \sim_\infty \dfrac{e^{4\pi\sqrt{n}}}{\sqrt{2}\,n^{3/4}}$. In 1938, H. Rademacher [Rad3] obtained by a different method the same result. Later, O. Herrmann [Her] and K. Mahler [Mah], in order to give estimates about modular polynomials, established upper bounds for the $c_m(n)$. O. Herrmann obtained $c(n) \leq 6e^{4\pi\sqrt{n}}$, for $n \geq 1$, and K. Mahler showed that $c_m(n) \leq 1200e^{4\pi\sqrt{m(m+n)}}$, for n, $m \geq 1$.

These upper bounds did not seem optimal. Looking at tables of $c(n)$ that we computed, it appeared that the first $c(n)$ are smaller than the asymptotically equivalent given by Petersson and Rademacher. In this paper, we establish effective lower and upper bounds for the Fourier coefficients of j^m that give, in particular, the expected upper bound for $m = 1$: $c(n) \leq \dfrac{e^{4\pi\sqrt{n}}}{\sqrt{2}\,n^{3/4}}$ for all $n \geq 1$. We shall prove in section 4 the following result.

[1] See [Ser] for basic definitions and results about modular functions.

[2] Today, using a simple GP (the calculator of PARI [Pari]) program, we can get the first thousand in less than 26 seconds on a Celeron 466Mhz.

Theorem 1.1. *For all n, $m \in \mathbb{N}$ such that $nm \geq 1000$ and $n \geq 4m \ln^2 m$, we have*

$$c_m(n) = \frac{1}{\sqrt{2}} \frac{m^{1/4}}{n^{3/4}} e^{4\pi\sqrt{nm}} \left(1 - \frac{3}{32\pi} \frac{1}{\sqrt{nm}} + \varepsilon_{n,m}\right) \quad \text{with } |\varepsilon_{n,m}| \leq \frac{0.055}{nm}.$$

When $m = 1$, the computation of the first thousand $c(n)$ shows that the theorem is true for all $n \geq 1$.

Remark 1.2. As the reader will see in the proof of Theorem 1.1, the conditions $nm \geq 1000$ and $n \geq 4m \ln^2 m$ are a compromise. By this, we mean that we can weaken one if we strengthen the other. Nevertheless, the reader should keep in mind that these conditions have to imply the necessary (but not sufficient: cf. section 8) conditions $n \geq m$ and (6) that arise during the proofs.

Let us mention that this elementary approach applies also to the Dedekind eta function and allows to find again some results, analogous to Theorem 1.1, about the partition function that are contained in [Rad5, chap. 14] (see also [Nic] [DN]).

Reading [Pet] and [Rad3], we saw that Petersson and Rademacher had obtained, not only an asymptotically equivalent but, in fact, a convergent series expansion for $c(n)$. Petersson used Poincaré series and Rademacher, inspired by his work on the partition function [Rad1] [Rad2], used Hardy and Ramanujan's circle method combined with a method of Kloostermann [Klo] extended by Estermann [Est]. In this paper, we adapt Rademacher's work [Rad3] to give a convergent series expansion for $c_m(n)$. This expansion allows to find a slightly weaker version of Theorem 1.1 but leads to the following improvement in the case $m = 1$:

Theorem 1.3. *For $k \geq 0$, let*

$$(1, k) = \frac{\prod_{j=0}^{k-1}(4 - (2j+1)^2)}{4^k k!}.$$

For all $n \geq 1$, $p \geq 1$, we have

$$c(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2} n^{3/4}} \left(\sum_{k=0}^{p-1} \left(-\frac{1}{8\pi}\right)^k \frac{(1,k)}{n^{k/2}} + \frac{r_p(n)}{n^{p/2}}\right)$$

where

$$|r_p(n)| \leq \frac{1}{\sqrt{2}} \frac{|(1,p)|}{(4\pi)^p} + 62\sqrt{2} e^{-2\pi\sqrt{n}} n^{p/2}.$$

The outline of the paper is the following. In section 2, using a theorem by Ingham, we give an asymptotically equivalent of $c_m(n)$ as n tends to infinity. Then, we prove in section 3 some technical results necessary to the proof of Theorem 1.1 that we do in section 4, which includes also estimates of the $c_m(n)$ for nonpositive n. Following [Rad3], we give in section 5 a convergent series expansion for $c_m(n)$ from which we almost recover our main result. We notice in passing that this convergent series expansion remains valid for any modular function of weight 0 holomorphic on $\mathfrak{H}$. In section 6, we prove Theorem 1.3 and show that it implies, in particular, that $c(n) \leq \dfrac{e^{4\pi\sqrt{n}}}{\sqrt{2}\,n^{3/4}}$ for all $n \geq 1$, which improves Herrmann's bounds [Her]. Then, we give in section 7 general upper bounds for the Fourier coefficients $c_m(n)$, when $m \geq 2$, that improve those given by Mahler in [Mah] and we gather in a corollary the upper bounds we obtained in the case $m \geq 2$. Finally, we examine numerically the conditions that appear in the proof of Theorem 1.1.

2. An asymptotically equivalent of $c_m(n)$ as n tends to infinity

In 1941, A. E. Ingham proved in [Ing] a tauberian theorem that implies the following result.

Theorem 2.1. [A. E. Ingham] *Let* $f(z) = \sum_{k\geq 0} a_k z^k$ *a power series with real nonnegative coefficients and radius of convergence equal to* 1. *If there exist* $A > 0$, λ, $\alpha \in \mathbb{R}$ *such that*

$$f(x) \sim \lambda(\ln(1/x))^{\alpha} \exp\left(\frac{A}{\ln(1/x)}\right) \quad \text{as } x \to 1^-,$$

then

$$\sum_{k=0}^{n} a_k \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\alpha/2-1/4}}{n^{\alpha/2+1/4}} \exp\left(2\sqrt{An}\right) \quad \text{as } n \to +\infty.$$

We apply this theorem to the function $(1-x)x^m J^m(x)$, $m \geq 1$, to obtain an asymptotically equivalent of $c_m(n)$ as $n \to +\infty$. For $m \in \mathbb{N} \setminus \{0\}$, we have $x^m J^m(x) = 1 + \sum_{n\geq 1} \widetilde{c}_m(n) x^n$ with $\widetilde{c}_m(n) = c_m(n-m)$, $n \geq 1$. Hence, $(1-x)x^m J^m(x) = 1 + \sum_{n\geq 1} (\widetilde{c}_m(n) - \widetilde{c}_m(n-1))x^n$. But,

$$(1-x)x^m J^m(x) = \frac{Q^{3m}(x)}{(1-x)^{24m-1}\prod_{n\geq 2}(1-x^n)^{24m}}$$

where $Q(x) = 1 + 240\sum_{n\geq 1} \sigma_3(n) x^n$. Therefore, $(1-x)x^m J^m(x)$ has real nonnegative coefficients. We aim to find an asymptotically equivalent of $xJ(x)$ as x tends to 1^-. To do so, we use the modular properties of j. For $x \in]0,1[$, let $t > 0$ such that $x = e^{-2\pi t}$: we have $J(x) = j(it)$. As x tends to 1^-, t tends to 0^+ and conversely. Thus, $J(x)$ which, from (2), is equal to $j\left(\dfrac{i}{t}\right) = e^{2\pi/t} + \sum_{n\geq 0} c(n) e^{-2\pi n/t}$ is asymptotic, as x tends to 1^-, to $e^{2\pi/t} = \exp\left(\dfrac{4\pi^2}{\ln(1/x)}\right)$. Hence, as x tends to 1^-, $(1-x)x^m J^m(x)$ is asymptotic to $(1-x)\exp\left(\dfrac{4\pi^2 m}{\ln(1/x)}\right)$ or yet to $\ln(1/x)\exp\left(\dfrac{4\pi^2 m}{\ln(1/x)}\right)$. Ingham's theorem applied with $\lambda = 1$, $\alpha = 1$, $A =$

$4\pi^2 m$, gives

$$1+\sum_{p=1}^{n}(\widetilde{c}_m(p)-\widetilde{c}_m(p-1)) \underset{n\to+\infty}{\sim} \frac{1}{2\sqrt{\pi}}\frac{(4\pi^2 m)^{1/4}}{n^{3/4}}\exp(4\pi\sqrt{nm}).$$

Therefore, $\widetilde{c}_m(n) \underset{n\to+\infty}{\sim} \dfrac{1}{\sqrt{2}}\dfrac{m^{1/4}}{n^{3/4}}\exp(4\pi\sqrt{nm})$ and

$$c_m(n)=\widetilde{c}_m(n+m) \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2}}\frac{m^{1/4}}{(n+m)^{3/4}}\exp(4\pi\sqrt{(n+m)m}).$$

Finally, we obtain

$$c_m(n) \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}}\exp(4\pi\sqrt{nm}).$$

This result is also a corollary of Theorem 1.1 that we establish in the next two sections.

3. Preliminary results

3.1. An inequality.

The Fourier expansion of j gives, since its coefficients are nonnegative, $|j(\tau)|\le j(i\Im\tau)$ for all $\tau\in\mathfrak{H}$. Hence, let x, $y\in\mathbb{R}$, if $\tau=x+iy\in\mathfrak{H}$, $|j(\tau)|\le e^{2\pi y}+744+\sum_{n\ge1}c(n)e^{-2\pi ny}$. The function $y\mapsto g(y)=\sum_{n\ge1}c(n)e^{-2\pi ny}$ is a decreasing function of $y\in]0,+\infty[$. Consequently, for $y\ge1$, $g(y)\le g(1)=j(i)-e^{2\pi}-744$. Therefore, if $y\ge1$, $j(iy)\le e^{2\pi y}+1728-e^{2\pi}$ and, using the modular relation (2), if $0<y\le1$, $j(iy)=j\left(\dfrac{i}{y}\right)\le e^{2\pi/y}+1728-e^{2\pi}$. So, we showed:

Lemma 3.1. *Let x, $y\in\mathbb{R}$, for $\tau=x+iy\in\mathfrak{H}$, we have*

$$|j(\tau)|\le j(iy)\le e^{2\pi\max(y,1/y)}+1728-e^{2\pi}.$$

Now, we prove two intermediate results.

3.2. Some technical lemmata.

Lemma 3.2. *For $x>0$, we have*

$$\int_0^x e^{-2\pi u^2}\mathrm{d}u=\frac{1}{2\sqrt{2}}-r_x \quad \textit{with} \quad 0<r_x<\frac{1}{4\pi x}e^{-2\pi x^2}.$$

Proof. Since $\int_0^{+\infty}e^{-y^2}\mathrm{d}y=\sqrt{\pi}/2$, we have $\int_0^{+\infty}e^{-2\pi u^2}\mathrm{d}u=\dfrac{1}{2\sqrt{2}}$ and, for $x>0$,

$$0<r_x=\int_x^{+\infty}e^{-2\pi u^2}\mathrm{d}u<\int_x^{+\infty}\left(1+\frac{1}{4\pi u^2}\right)e^{-2\pi u^2}\mathrm{d}u=\frac{1}{4\pi x}e^{-2\pi x^2}.$$

□

Lemma 3.3. *For $\alpha > 5$, let*

$$A_\alpha = \int_0^{\alpha/2} e^{-2\pi u^2}\left(\left(\cos\frac{2\pi u^3}{\alpha\sqrt{1-\frac{u^2}{\alpha^2}}}\right)\left(1-\frac{u^2}{\alpha^2}\right)^{-3/2}-1\right)\mathrm{d}u.$$

Then, we have

$$A_\alpha = -\frac{3}{64\pi\alpha^2\sqrt{2}} - \frac{15}{4096\pi^2\alpha^4\sqrt{2}} + B_\alpha \ \textit{ with } |B_\alpha| \le \frac{0.355}{\alpha^6}.$$

Proof. For $u \in [0, \alpha/2]$, we write

$$\cos\frac{2\pi u^3}{\alpha\sqrt{1-u^2/\alpha^2}} = 1 - \frac{2\pi^2 u^6}{\alpha^2(1-u^2/\alpha^2)} + R_\alpha(u)$$

and

$$\left(1-\frac{u^2}{\alpha^2}\right)^{-3/2} = 1 + \frac{3}{2}\frac{u^2}{\alpha^2} + S_\alpha(u)$$

with

$$\frac{1}{4!}\frac{(2\pi)^4u^{12}}{\alpha^4(1-u^2/\alpha^2)^2} - \frac{1}{6!}\frac{(2\pi)^6u^{18}}{\alpha^6(1-u^2/\alpha^2)^3} \le R_\alpha(u) \le \frac{1}{4!}\frac{(2\pi)^4u^{12}}{\alpha^4(1-u^2/\alpha^2)^2}$$

and

$$\frac{15}{8}\frac{u^4}{\alpha^4} + \frac{35}{16}\frac{u^6}{\alpha^6} \le S_\alpha(u) \le \frac{15}{8}\frac{u^4}{\alpha^4} + 3.04\frac{u^6}{\alpha^6}.$$

Thus,

$$\left(\cos\frac{2\pi u^3}{\alpha\sqrt{1-u^2/\alpha^2}}\right)\left(1-\frac{u^2}{\alpha^2}\right)^{-3/2} - 1$$
$$= \frac{3}{2}\frac{u^2}{\alpha^2} + S_\alpha(u) - \frac{2\pi^2u^6}{\alpha^2(1-u^2/\alpha^2)^{5/2}} + \frac{R_\alpha(u)}{(1-u^2/\alpha^2)^{3/2}}.$$

But,

$$\left(1-\frac{u^2}{\alpha^2}\right)^{-5/2} = 1 + \frac{5}{2}\frac{u^2}{\alpha^2} + T_\alpha(u) \text{ with } \frac{35}{8}\frac{u^4}{\alpha^4} \le T_\alpha(u) \le 6.85\frac{u^4}{\alpha^4}$$

since $u \in [0, \alpha/2]$. Hence,

$$\left(\cos\frac{2\pi u^3}{\alpha\sqrt{1-u^2/\alpha^2}}\right)\left(1-\frac{u^2}{\alpha^2}\right)^{-3/2} - 1 = \frac{3}{2}\frac{u^2}{\alpha^2} - \frac{2\pi^2u^6}{\alpha^2} - \frac{5\pi^2u^8}{\alpha^4} + U_\alpha(u)$$

where

$$U_\alpha(u) = S_\alpha(u) - \frac{2\pi^2u^6}{\alpha^2}T_\alpha(u) + \frac{R_\alpha(u)}{(1-u^2/\alpha^2)^{3/2}}.$$

Then, from the inequalities above and

$$\left(1-\frac{u^2}{\alpha^2}\right)^{-7/2} = 1 + V_\alpha(u) \text{ with } \frac{7}{2}\frac{u^2}{\alpha^2} \le V_\alpha(u) \le 7\frac{u^2}{\alpha^2},$$

$$\left(1-\frac{u^2}{\alpha^2}\right)^{-9/2} = 1 + W_\alpha(u) \text{ with } \frac{9}{2}\frac{u^2}{\alpha^2} \le W_\alpha(u) \le 11\frac{u^2}{\alpha^2},$$

if $u \in [0, \alpha/2]$, we have

$$\begin{aligned}\frac{15}{8}\frac{u^4}{\alpha^4} + \frac{35}{16}\frac{u^6}{\alpha^6} - 13.7\pi^2\frac{u^{10}}{\alpha^6} + \frac{(2\pi)^4 u^{12}}{4!\alpha^4}\left(1 + \frac{7}{2}\frac{u^2}{\alpha^2}\right) \\ - \frac{(2\pi)^6 u^{18}}{6!\alpha^6}\left(1 + 11\frac{u^2}{\alpha^2}\right) \le U_\alpha(u) \le \frac{15}{8}\frac{u^4}{\alpha^4} + 3.04\frac{u^6}{\alpha^6} \\ - \frac{35\pi^2}{4}\frac{u^{10}}{\alpha^6} + \frac{(2\pi)^4 u^{12}}{4!\alpha^4}\left(1 + 7\frac{u^2}{\alpha^2}\right).\end{aligned}$$

Several integrations by parts (done with Maple) give

$$K_1(\alpha)e^{-\frac{\pi\alpha^2}{2}} + K_2(\alpha)\int_0^{\alpha/2} e^{-2\pi u^2}\mathrm{d}u \le A_\alpha \le L_1(\alpha)e^{-\frac{\pi\alpha^2}{2}} + L_2(\alpha)\int_0^{\alpha/2} e^{-2\pi u^2}\mathrm{d}u$$

with

$$\begin{aligned}K_1(\alpha) = {} & \frac{1}{1572864}\pi^5\alpha^{11} + \frac{277}{23592960}\pi^4\alpha^9 + \frac{973}{23592960}\pi^3\alpha^7 + \frac{1573}{1572864}\pi^2\alpha^5 \\ & + \frac{129471}{2621440}\pi\alpha^3 + \frac{3439207}{7864320}\alpha + \frac{8146573}{2621440}\frac{1}{\pi\alpha} + \frac{11070887}{524288}\frac{1}{\pi^2\alpha^3} \\ & + \frac{54546219}{524288}\frac{1}{\pi^3\alpha^5} + \frac{160044885}{524288}\frac{1}{\pi^4\alpha^7}, \\ K_2(\alpha) = {} & -\frac{3}{32}\frac{1}{\pi\alpha^2} - \frac{15}{2048}\frac{1}{\pi^2\alpha^4} - \frac{299481}{65536}\frac{1}{\pi^3\alpha^6} - \frac{160044885}{262144}\frac{1}{\pi^4\alpha^8}, \\ L_1(\alpha) = {} & -\frac{11}{49152}\pi^3\alpha^7 - \frac{45}{16384}\pi^2\alpha^5 + \frac{61}{49152}\pi\alpha^3 - \frac{897}{16384}\alpha \\ & - \frac{405303}{409600\pi\alpha} - \frac{424703}{81920\pi^2\alpha^3} - \frac{1275009}{81920\pi^3\alpha^5}\end{aligned}$$

and

$$L_2(\alpha) = -\frac{3}{32}\frac{1}{\pi\alpha^2} - \frac{15}{2048}\frac{1}{\pi^2\alpha^4} + \frac{1275009}{40960}\frac{1}{\pi^3\alpha^6}.$$

The function $L_2(\alpha)$ is negative when $\alpha > 5$. Hence, from Lemma 3.2, we have

$$\begin{aligned}A_\alpha & \le e^{-\frac{\pi\alpha^2}{2}}\left(L_1(\alpha) - \frac{L_2(\alpha)}{2\pi\alpha}\right) + \frac{L_2(\alpha)}{2\sqrt{2}} \\ & \le -\frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4} - \frac{1275009}{40960}\frac{1}{\pi^3\alpha^6}\right) \\ & \quad + e^{-\frac{\pi\alpha^2}{2}}\left(\frac{61}{49152}\pi\alpha^3 + \frac{3}{64}\frac{1}{\pi^2\alpha^3} + \frac{15}{4096}\frac{1}{\pi^3\alpha^5}\right).\end{aligned}$$

Moreover, as $K_1(\alpha)$ is positive and $K_2(\alpha)$ is negative for all $\alpha > 0$, we get

$$\begin{aligned}A_\alpha & \ge -\frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4} + \frac{299481}{65536}\frac{1}{\pi^3\alpha^6} + \frac{160044885}{262144}\frac{1}{\pi^4\alpha^8}\right) \\ & \ge -\frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4} + \frac{1275009}{40960}\frac{1}{\pi^3\alpha^6}\right)\end{aligned}$$

if $\alpha > 5$. Thus, for $\alpha > 5$,

$$\begin{aligned}\left|A_\alpha + \frac{1}{2\sqrt{2}}\left(\frac{3}{32}\frac{1}{\pi\alpha^2} + \frac{15}{2048}\frac{1}{\pi^2\alpha^4}\right)\right| &\leq \frac{1275009}{163840}\frac{\sqrt{2}}{\pi^3\alpha^6} \\ &\quad + e^{-\frac{\pi\alpha^2}{2}}\left(\frac{61}{49152}\pi\alpha^3 + \frac{3}{64}\frac{1}{\pi^2\alpha^3} + \frac{15}{4096}\frac{1}{\pi^3\alpha^5}\right) \\ &\leq \frac{1275009}{163840}\frac{\sqrt{2}}{\pi^3\alpha^6} + e^{-\frac{\pi\alpha^2}{2}}\left(\frac{61}{49152}\pi\alpha^3 + 4\cdot 10^{-5}\right) \leq \frac{0.355}{\alpha^6},\end{aligned}$$

which is the estimate desired. □

4. Proof of the main result

Let $m \in \mathbb{N}^*$. We put $c_m(-m) = 1$. Cauchy's formula applied to $J^m(q) = \sum_{n\geq -m} c_m(n)q^n$, $|q| < 1$, gives: for $n \geq -m$, $c_m(n) = \frac{1}{2i\pi}\int_{\gamma_r}\frac{J^m(q)}{q^{n+1}}\mathrm{d}q$ where $\gamma_r : x \in [-1/2, 1/2] \mapsto re^{2i\pi x}$, $r \in]0,1[$. Putting $r = e^{-2\pi y}$, $y > 0$, we get

$$\begin{aligned} c_m(n) &= e^{2\pi n y}\int_{-1/2}^{1/2} J^m\left(e^{2i\pi(x+iy)}\right)e^{-2i\pi n x}\mathrm{d}x \\ &= e^{2\pi n y}\int_{-1/2}^{1/2} j^m(x+iy)e^{-2i\pi n x}\mathrm{d}x. \qquad (3)\end{aligned}$$

Moreover, we know that $c_m(n) \in \mathbb{N}$ and $\overline{j(x+iy)} = j(-x+iy)$. Hence, $c_m(n) = 2e^{2\pi n y}\,\Re\left(\int_0^{1/2} j^m(x+iy)e^{-2i\pi n x}\mathrm{d}x\right)$ for all $y > 0$.

As we shall see, the coefficients of the principal part of $J^m(q)$ at 0 are involved in the estimate of the $c_m(n)$ for $n \geq 1$. Thus, we need estimates of the $c_m(n)$ for n negative that we establish in the two propositions hereafter.

We start by a general (but not necessarily good) estimate of the coefficients $c_m(n)$.

Proposition 4.1. *For all $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$, such that $n \geq -m$, we have*

$$c_m(n) \leq e^{2\pi n}1728^m.$$

Proof. If we choose y equal to 1, we get

$$c_m(n) = 2e^{2\pi n}\,\Re\left(\int_0^{1/2} j^m(x+i)e^{-2i\pi n x}\mathrm{d}x\right),$$

hence

$$c_m(n) \leq 2e^{2\pi n}\int_0^{1/2} j^m(i)\mathrm{d}x = e^{2\pi n}1728^m.$$

□

Now, we sharpen this result for some negative values of n.

Proposition 4.2. *For all $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$, such that $-m+1 \le n \le -\dfrac{me^{2\pi}}{1728}$, we have*
$$c_m(n) \le (1728 - e^{2\pi})^{m+n} \left(\frac{-n}{m+n}\right)^n \left(\frac{m}{m+n}\right)^m .$$

Proof. First, we notice that $m \ge 2$ necessarily. From Lemma 3.1 and (3), we have

$$c_m(n) \le g_{m,n}(y) = \begin{cases} e^{2\pi n y}(e^{2\pi/y} + 1728 - e^{2\pi})^m & \text{if } y \in]0,1], \\ e^{2\pi n y}(e^{2\pi y} + 1728 - e^{2\pi})^m & \text{if } y \ge 1. \end{cases} \tag{4}$$

When $n \le 0$, the function $g_{m,n}$ is decreasing on $]0,1]$. The derivative of $g_{m,n}$ on the interval $[1,+\infty[$ cancels at $y_{m,n} = \dfrac{1}{2\pi} \ln\left(\dfrac{-n}{m+n}(1728 - e^{2\pi})\right)$. The condition $y_{m,n} \ge 1$ is equivalent to $n \le -\dfrac{me^{2\pi}}{1728}$. Hence, for $-m+1 \le n \le -\dfrac{me^{2\pi}}{1728}$,
$$c_m(n) \le g_{m,n}(y_{m,n}) = (1728 - e^{2\pi})^{m+n} \left(\frac{-n}{m+n}\right)^n \left(\frac{m}{m+n}\right)^m .$$

For $-\dfrac{me^{2\pi}}{1728} < n < 0$, we notice that the best choice is $y = 1$ which gives $c_m(n) \le e^{2\pi n} 1728^m$. □

We start the proof of Theorem 1.1.

All along this section, we assume that $n \ge m$.

Inequalities (4) lead us to choose $y \in]0,1]$ so that $e^{2\pi n y} e^{2\pi m/y}$ is minimal. This is done for $y = \sqrt{\dfrac{m}{n}}$. Hence, we study $\Re(K)$ with
$$K = \int_0^{1/2} j^m \left(x + i\sqrt{\frac{m}{n}}\right) e^{-2i\pi n x} \mathrm{d}x.$$

Thanks to (2), we have
$$j\left(x + i\sqrt{\frac{m}{n}}\right) = j\left(\frac{-x + i\sqrt{m/n}}{x^2 + m/n}\right). \tag{5}$$

To take this fundamental property into account, we have to compare 1, $\sqrt{\dfrac{m}{n}}$ and $\dfrac{\sqrt{m/n}}{x^2 + m/n}$. Let $a_{m,n}$ denote $\left(\dfrac{m}{n}\right)^{1/4}\left(1 - \sqrt{\dfrac{m}{n}}\right)^{1/2}$, the inequality $\dfrac{\sqrt{m/n}}{x^2 + m/n} \le 1$ is equivalent to $x \ge a_{m,n}$. We notice that $a_{m,n} \in [0, 1/2]$, for $a_{m,n} \le 1/2$ is equivalent to $\sqrt{\dfrac{m}{n}}\left(1 - \sqrt{\dfrac{m}{n}}\right) \le \dfrac{1}{4}$ i.e. $\left(\sqrt{\dfrac{m}{n}} - \dfrac{1}{2}\right)^2 \ge 0$. Hence, we write $K = L + M$ with
$$L = \int_0^{a_{m,n}} j^m \left(x + i\sqrt{\frac{m}{n}}\right) e^{-2i\pi n x} \mathrm{d}x, \quad M = \int_{a_{m,n}}^{1/2} j^m \left(x + i\sqrt{\frac{m}{n}}\right) e^{-2i\pi n x} \mathrm{d}x.$$

4.1. **An upper bound for $|M|$.**

From (5), we have $M = \displaystyle\int_{a_{m,n}}^{1/2} j^m\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right)e^{-2i\pi nx}\mathrm{d}x$. We have just seen that, for $x \geq a_{m,n}$, $\dfrac{\sqrt{m/n}}{x^2+m/n} \leq 1$. Then,

$$|M| \leq \int_{a_{m,n}}^{1/2} j^m\left(i\frac{\sqrt{m/n}}{x^2+m/n}\right)\mathrm{d}x \leq \int_{a_{m,n}}^{1/2}\left(e^{2\pi(x^2+m/n)\sqrt{n/m}}+1728-e^{2\pi}\right)^m \mathrm{d}x$$

from Lemma 3.1. This is equivalent to

$$|M| \leq e^{2\pi m\sqrt{m/n}}\int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}}\left(1+(1728-e^{2\pi})e^{-2\pi(x^2+m/n)\sqrt{n/m}}\right)^m \mathrm{d}x,$$

which gives

$$\begin{aligned}|M| &\leq e^{2\pi m\sqrt{m/n}}\int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}}\left(1+(1728-e^{2\pi})e^{-2\pi(a_{m,n}^2+m/n)\sqrt{n/m}}\right)^m \mathrm{d}x\\ &= e^{2\pi m\sqrt{m/n}}\left(\frac{1728}{e^{2\pi}}\right)^m\int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}}\mathrm{d}x.\end{aligned}$$

As $\displaystyle\int_{a_{m,n}}^{1/2} e^{2\pi x^2\sqrt{nm}}\mathrm{d}x \leq \int_{a_{m,n}}^{1/2} e^{\pi x\sqrt{nm}}\mathrm{d}x$, we finally get

$$|M| \leq e^{2\pi m\sqrt{m/n}}\left(\frac{1728}{e^{2\pi}}\right)^m \frac{1}{\pi\sqrt{nm}}e^{\pi\sqrt{nm}/2}.$$

4.2. **An estimate of L.**

From (5), we have $L = \displaystyle\int_0^{a_{m,n}} j^m\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right)e^{-2i\pi nx}\mathrm{d}x$ i.e.

$$L = \underbrace{\int_0^{a_{m,n}} e^{-2i\pi m\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right)}e^{-2i\pi nx}\mathrm{d}x}_{L_1}$$

$$+\underbrace{\sum_{p>-m} c_m(p)\int_0^{a_{m,n}} e^{2i\pi p\left(\frac{-x+i\sqrt{m/n}}{x^2+m/n}\right)}e^{-2i\pi nx}\mathrm{d}x}_{L_2}.$$

First, we compute an upper bound for $|L_2|$. Let $p_0 = -\left\lfloor\frac{me^{2\pi}}{1728}\right\rfloor - 1$ where $\lfloor x\rfloor$ denotes the floor of x. We have $L_2 = \displaystyle\int_0^{a_{m,n}} e^{-2i\pi nx}\Big(\sum_{p=-m+1}^{p_0} + \sum_{p=p_0+1}^{-1} + \sum_{p\geq 0}\Big) = L_{21}+L_{22}+L_{23}$ where $L_{21}=0$ if $m=1$ and $L_{22}=0$ if $m=1,\,2,\,3$.

We have, for all $m \geq 2$,

$$|L_{21}| \leq \sum_{p=-m+1}^{p_0} c_m(p) \int_0^{a_{m,n}} e^{-\frac{2\pi p\sqrt{m/n}}{x^2+m/n}} \mathrm{d}x \leq \sum_{p=-m+1}^{p_0} c_m(p) a_{m,n} e^{-2\pi p\sqrt{n/m}}.$$

Hence, if λ_p denotes $-\frac{p}{m} \in \left[\frac{e^{2\pi}}{1728}, 1-\frac{1}{m}\right]$, Proposition 4.2 gives

$$|L_{21}| \leq a_{m,n} \sum_{p=-m+1}^{p_0} h^m(\lambda_p) \text{ with } \ln(h(\lambda)) = (1-\lambda)\ln(1728-e^{2\pi})$$
$$-\lambda\ln\lambda - (1-\lambda)\ln(1-\lambda) + 2\pi\lambda\sqrt{n/m}.$$

The derivative $h'(\lambda)$ cancels if and only if $\lambda = \left((1728-e^{2\pi})e^{-2\pi\sqrt{n/m}}+1\right)^{-1}$. Let us recall that $\lambda_p \in \left[\frac{e^{2\pi}}{1728}, 1-\frac{1}{m}\right]$. We have $\left((1728-e^{2\pi})e^{-2\pi\sqrt{n/m}}+1\right)^{-1} \geq 1-\frac{1}{m}$ if and only if

$$n \geq m\left(\frac{1}{2\pi}\ln((1728-e^{2\pi})(m-1))\right)^2. \tag{6}$$

In this case, for all $\lambda \in \left[\frac{e^{2\pi}}{1728}, 1-\frac{1}{m}\right]$, $m \geq 2$, we have $h(\lambda) \leq h\left(1-\frac{1}{m}\right)$ which implies $h^m(\lambda_p) \leq (1728-e^{2\pi})\, m\, e^{1+2\pi(\sqrt{mn}-\sqrt{n/m})}$.

Finally, we obtain

$$|L_{21}| \leq \frac{(1728-e^{2\pi})^2}{1728}\frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn}-\sqrt{n/m})}$$

for all m, $n \in \mathbb{N}$ satisfying $m \geq 2$ and condition (6) that we assume until the upper bound for $|L_2|$ is achieved.

Then, we turn to L_{22}. For all $m \geq 4$,

$$|L_{22}| \leq \sum_{p=p_0+1}^{-1} \int_0^{a_{m,n}} c_m(p) e^{-\frac{2\pi p\sqrt{m/n}}{x^2+m/n}} \mathrm{d}x \leq \sum_{p=p_0+1}^{-1} a_{m,n} e^{2\pi p} 1728^m e^{-2\pi p\sqrt{n/m}}$$

from Proposition 4.1. This implies

$$|L_{22}| \leq \left(\frac{m}{n}\right)^{1/4} 1728^m e^{2\pi(\sqrt{n/m}-1)} \frac{e^{2\pi(\sqrt{n/m}-1)me^{2\pi}/1728}-1}{e^{2\pi(\sqrt{n/m}-1)}-1}.$$

When $m \geq 4$, from (6), we have $e^{2\pi(\sqrt{n/m}-1)} \geq 6$ which yields

$$|L_{22}| \leq \left(\frac{m}{n}\right)^{1/4} 1728^m \frac{6}{5} e^{2\pi\frac{e^{2\pi}}{1728}\sqrt{nm}} e^{-2\pi m\frac{e^{2\pi}}{1728}} \leq \frac{6}{5}\left(\frac{m}{n}\right)^{1/4} e^{5.51m} e^{2\pi\frac{e^{2\pi}}{1728}\sqrt{nm}}.$$

Lastly, we give an upper bound for $|L_{23}|$. We have

$$|L_{23}| \leq \int_0^{a_{m,n}} \sum_{p\geq 0} c_m(p) e^{-\frac{2\pi p\sqrt{m/n}}{x^2+m/n}} \mathrm{d}x.$$

As $\frac{\sqrt{m/n}}{x^2+m/n} \geq 1$ for all $x \in [0, a_{m,n}]$, we get

$$|L_{23}| \leq \int_0^{a_{m,n}} \left(\sum_{p\geq 0} c_m(p) e^{-2\pi p} \right) \mathrm{d}x = a_{m,n} j^m(i) \leq \left(\frac{m}{n}\right)^{1/4} 1728^m.$$

We gather our results to get an upper bound for $|L_2|$.

For $m = 1$, we have $|L_2| = |L_{23}| \leq \dfrac{1}{n^{1/4}} 1728.$

For $m = 2$, 3, we obtain

$$|L_2| \leq |L_{21}| + |L_{23}| \leq \frac{m^{1/4}}{n^{1/4}} 1728^m + \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn} - \sqrt{n/m})}$$

For $m \geq 4$, we have

$$\begin{aligned} |L_2| \leq |L_{21}| + |L_{22}| + |L_{23}| \leq \frac{m^{1/4}}{n^{1/4}} 1728^m \\ + \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn} - \sqrt{n/m})} + \frac{6}{5} \frac{m^{1/4}}{n^{1/4}} e^{5.51m + 2\pi \frac{e^{2\pi}}{1728} \sqrt{nm}}. \end{aligned}$$

The conditions $m \geq 2$ and (6) imply $n \geq 1.27m$, from which follows

$$\left(\frac{m}{n}\right)^{1/4} 1728^m \leq \left(\frac{m}{n}\right)^{1/4} e^{5.51m + 2\pi \frac{e^{2\pi}}{1728} \sqrt{nm}}.$$

Thus, for $m \geq 4$, we obtain,

$$\begin{aligned} |L_2| &\leq \frac{(1728 - e^{2\pi})^2}{1728} \frac{m^{9/4}}{n^{1/4}} e^{1+2\pi(\sqrt{mn} - \sqrt{n/m})} + \frac{11}{5} \frac{m^{1/4}}{n^{1/4}} e^{5.51m + 2\pi \frac{e^{2\pi}}{1728} \sqrt{nm}} \\ &\leq 2238 \frac{m^{9/4}}{n^{1/4}} e^{2\pi(\sqrt{mn} - \sqrt{n/m})} + \frac{11}{5} \frac{m^{1/4}}{n^{1/4}} e^{5.51m + 0.62\pi\sqrt{nm}}. \end{aligned}$$

One can check immediately that this inequality remains true when $m \in \{1, 2, 3\}$.

Remark 4.3. The condition (6) is a consequence (when $m \geq 2$) of the condition $n \geq 4m \ln^2 m$.

Now we estimate L_1. We make the change of variable $u = \dfrac{n^{3/4} m^{1/4} x}{(nx^2 + m)^{1/2}}$. This gives

$$\begin{aligned} L_1 &= \frac{m^{1/4}}{n^{3/4}} e^{2\pi\sqrt{mn}} \int_0^{\sqrt{n} a_{m,n}} e^{-2i\pi \frac{u^3}{(mn)^{1/4}(1-u^2/\sqrt{mn})^{1/2}}} e^{-2\pi u^2} \frac{\mathrm{d}u}{(1 - u^2/\sqrt{mn})^{3/2}} \\ &= \frac{m^{1/4}}{n^{3/4}} e^{2\pi\sqrt{mn}} \Big(\underbrace{\int_0^{\frac{(mn)^{1/4}}{2}}}_{L_{11}} + \underbrace{\int_{\frac{(mn)^{1/4}}{2}}^{\sqrt{n} a_{m,n}}}_{L_{12}} \Big). \end{aligned}$$

First,

$$|L_{12}| \leq \int_{\frac{(mn)}{2}^{1/4}}^{\sqrt{n}a_{m,n}} e^{-2\pi u^2} \frac{\mathrm{d}u}{(1-u^2/\sqrt{mn})^{3/2}} \leq \left(\frac{n}{m}\right)^{3/4} \int_{\frac{(mn)}{2}^{1/4}}^{\sqrt{n}a_{m,n}} e^{-2\pi u^2}\mathrm{d}u$$
$$\leq \left(\frac{n}{m}\right)^{3/4} \int_{\frac{(mn)}{2}^{1/4}}^{+\infty} e^{-2\pi u^2}\mathrm{d}u \leq \frac{1}{2\pi}\frac{\sqrt{n}}{m} e^{-\pi\sqrt{mn}/2}$$

from Lemma 3.2. Then,

$$\mathrm{Re}\, L_{11} = \int_0^{\frac{(mn)}{2}^{1/4}} e^{-2\pi u^2} \left(\left(\cos \frac{2\pi u^3}{(mn)^{1/4}\sqrt{1-\frac{u^2}{\sqrt{mn}}}}\right)\left(1-\frac{u^2}{\sqrt{mn}}\right)^{-3/2} - 1\right)\mathrm{d}u$$
$$+ \int_0^{\frac{(mn)}{2}^{1/4}} e^{-2\pi u^2}\mathrm{d}u.$$

Lemmata 3.3 and 3.2 give $\mathrm{Re}\, L_{11} = \left(1 - \frac{3}{32\pi\sqrt{mn}} - \frac{15}{2048\pi^2 mn}\right)\frac{1}{2\sqrt{2}} + C_{(mn)^{1/4}}$ with $C_{(mn)^{1/4}} = B_{(mn)^{1/4}} - \int_{\frac{(mn)}{2}^{1/4}}^{+\infty} e^{-2\pi u^2}\mathrm{d}u$ and

$$|C_{(mn)^{1/4}}| \leq \frac{e^{-\pi\sqrt{mn}/2}}{2\pi(mn)^{1/4}} + \frac{0.355}{(mn)^{3/2}}. \tag{7}$$

4.3. **End of the proof of Theorem 1.1.**

For all m, $n \in \mathbb{N} \setminus \{0\}$, we have

$$c_m(n) = 2e^{2\pi\sqrt{nm}}(\mathrm{Re}\, L_1 + \mathrm{Re}\, L_2 + \mathrm{Re}\, M)$$
$$= \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}} e^{4\pi\sqrt{nm}}\left(1 - \frac{3}{32\pi}\frac{1}{\sqrt{nm}} + \frac{1}{nm}E_{n,m}\right)$$

with $E_{n,m} = 2\sqrt{2}n^{7/4}m^{3/4}e^{-2\pi\sqrt{nm}}(\mathrm{Re}\, M + \mathrm{Re}\, L_2)$

$$- \frac{15}{2048\pi^2} + 2\sqrt{2}nmC_{(nm)^{1/4}} + 2\sqrt{2}nm\mathrm{Re}\, L_{12}.$$

We shall prove in the sequel that $|E_{n,m}| \leq 0.055$ if $nm \geq 1000$ and $n \geq 4m\ln^2 m$.

First, $\left|2\sqrt{2}nm\mathrm{Re}\, L_{12}\right| \leq \frac{\sqrt{2}}{\pi}n^{3/2}e^{-\pi\sqrt{nm}/2} \leq \frac{\sqrt{2}}{\pi}(nm)^{3/2}e^{-\pi\sqrt{nm}/2} \leq 10^{-17}$ if $nm \geq 1000$.

If $nm \geq 1000$, we have from (7)

$$\left|2\sqrt{2}nmC_{(nm)^{1/4}}\right| \leq 2\sqrt{2}\left(\frac{(mn)^{3/4}e^{-\pi\sqrt{mn}/2}}{2\pi} + \frac{0.355}{(mn)^{1/2}}\right) \leq 0.03176.$$

Then,

$$\left|2\sqrt{2}n^{7/4}m^{3/4}e^{-2\pi\sqrt{nm}}\mathrm{Re}\, L_2\right| \leq 4476\sqrt{2}m^3n^{3/2}e^{-2\pi\sqrt{n/m}}$$
$$+ \frac{22\sqrt{2}}{5}n^{3/2}me^{5.51m-1.38\pi\sqrt{nm}}.$$

The function $f_m : x \mapsto x^{3/2} e^{-2\pi\sqrt{x/m}}$ is decreasing for $x \geq \max(1, 4m\ln^2 m)$.

For $m \leq 7$, as $n \geq 1000/7 \geq \max(1, 28\ln^2 7)$,

$$4476\sqrt{2} n^{3/2} m^3 e^{-2\pi\sqrt{n/m}} \leq 4476\sqrt{2} m^3 f_m\left(\frac{1000}{7}\right) \leq 4476\sqrt{2}\cdot 7^3 f_7\left(\frac{1000}{7}\right) \leq 0.0018.$$

For $m = 8$, we have $n \geq 32\ln^2 8$ i.e. $n \geq 139$ which implies

$$4476\sqrt{2} n^{3/2} m^3 e^{-2\pi\sqrt{n/m}} = 4476\sqrt{2} n^{3/2} 8^3 e^{-2\pi\sqrt{n/8}} \leq 2291712\sqrt{2} f_8(139) \leq 0.02244.$$

For $m \geq 9$, as $n \geq 4m\ln^2 m$, we have

$$4476\sqrt{2} n^{3/2} m^3 e^{-2\pi\sqrt{n/m}} \leq 4476\sqrt{2} m^3 f_m(4m\ln^2 m) = 35808\sqrt{2} m^{9/2-4\pi}\ln^3 m.$$

The function $x \mapsto x^{9/2-4\pi}\ln^3 x$ is decreasing for $x \geq \exp\left(\frac{3}{4\pi-9/2}\right)$. Hence, if $m \geq 9 \geq \left(\exp\left(\frac{3}{4\pi-9/2}\right)\right)$, we get

$$4476\sqrt{2} n^{3/2} m^3 e^{-2\pi\sqrt{n/m}} \leq 0.011.$$

Now, we give an upper bound for $\frac{22\sqrt{2}}{5} n^{3/2} m e^{5.51m - 1.38\pi\sqrt{nm}}$. We notice that

- if $m \leq 10$ then $1000 \leq nm \leq 10n$ hence $n \geq 10m$;
- if $m \geq 10$ then $n \geq 4m\ln^2 m \geq 4m\ln^2 10 \geq 20m$.

It follows

$$\frac{22\sqrt{2}}{5} n^{3/2} m e^{5.51m - 1.38\pi\sqrt{nm}} \leq \frac{22\sqrt{2}}{5}(nm)^{3/2} e^{5.51\sqrt{10}\sqrt{nm} - 1.38\pi\sqrt{nm}} \leq \frac{22\sqrt{2}}{5}(nm)^{3/2} e^{2.5\sqrt{nm}} \leq \frac{22\sqrt{2}}{5}(1000)^{3/2} e^{-2.5\sqrt{1000}} \leq 10^{-29}.$$

Finally,

$$\left|2\sqrt{2} n^{7/4} m^{3/4} e^{-2\pi\sqrt{nm}} \operatorname{Re} M\right| \leq \frac{2\sqrt{2}}{\pi} n^{5/4} m^{1/4} e^{2\pi m\sqrt{m/n}} \left(\frac{1728}{e^{2\pi}}\right)^m e^{-3\pi\sqrt{nm}/2} \leq \frac{2\sqrt{2}}{\pi}(nm)^{5/4}(1728)^m e^{-3\pi\sqrt{nm}/2}$$

since $n \geq m$. As $n \geq 10m$, we have

$$\left|2\sqrt{2} n^{7/4} m^{3/4} e^{-2\pi\sqrt{nm}} \operatorname{Re} M\right| \leq \frac{2\sqrt{2}}{\pi}(nm)^{5/4} e^{\sqrt{0.1}\ln 1728\sqrt{nm} - 3\pi\sqrt{nm}/2} \leq \frac{2\sqrt{2}}{\pi}(nm)^{5/4} e^{-2.3\sqrt{nm}} \leq \frac{2\sqrt{2}}{\pi}(1000)^{5/4} e^{-2.3\sqrt{1000}} \leq 10^{-27}.$$

These upper bounds collected give

$$|E_{n,m}| \leq \frac{15}{2048\pi^2} + 10^{-17} + 0.03176 + 0.02244 + 10^{-29} + 10^{-27} \leq 0.055$$

if $nm \geq 1000$ and $n \geq 4m\ln^2 m$, which concludes the proof.

5. The circle method, after Rademacher

5.1. A series expansion of the Fourier coefficients of j^m.

In this subsection, we establish, following step by step the proof given in [Rad3] in the case $m=1$, a series expansion of the Fourier coefficients of j^m. We give details of the key steps to make an easier reading. Let N be a positive rational integer. From Cauchy's formula, we have

$$c_m(n) = \frac{1}{2\pi i}\int_{\mathcal{C}_N} \frac{J^m(q)}{q^{n+1}}\mathrm{d}q = \sum_{\substack{\gcd(h,k)=1\\ 0\le h<k\le N}} \frac{1}{2\pi i}\int_{\xi_{h,k}} \frac{J^m(q)}{q^{n+1}}\mathrm{d}q$$

where the $\xi_{h,k}$ are the Farey[3] arcs of order N of the circle $\mathcal{C}_N = \{z\in\mathbb{C}; |z| = e^{-2\pi N^{-2}}\}$. Let h/k be a Farey fraction of order N, we consider its two neighbours h_1/k_1 and h_2/k_2: we have $\dfrac{h_1}{k_1} < \dfrac{h}{k} < \dfrac{h_2}{k_2}$ with $1\le k,\, k_1,\, k_2 \le N$ and $hk_1 - h_1k = 1$, $h_2k - hk_2 = 1$ which imply

$$hk_1 = 1 \mod k \quad\text{ and }\quad hk_2 = -1 \mod k. \tag{8}$$

Let h' be the rational integer in $[0,k[$ such that $hh' = -1 \mod k$. If we put $\theta'_{h,k} = 1/k(k_1+k)$ and $\theta''_{h,k} = 1/k(k_2+k)$, we get from (2) the equality:

$$c_m(n) = \sum_{\substack{hh'=-1 \mod k\\ 0\le h<k\le N}} e^{-\frac{2\pi i n h}{k}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} J^m\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k^2\omega}}\right) e^{2\pi n\omega}\mathrm{d}\phi$$

with $\omega = N^{-2} - i\phi$. If we write $J^m(q) = \displaystyle\sum_{\ell=1}^{m} \frac{c_m(-\ell)}{q^\ell} + D_m(q)$ with $D_m(q) = \displaystyle\sum_{\ell\ge 0} c_m(\ell)q^\ell$, we have $c_m(n) = \displaystyle\sum_{\ell=1}^{m} c_m(-\ell)Q_\ell(n) + R_m(n)$ where

$$Q_\ell(n) = \sum_{\substack{hh'=-1 \mod k\\ 0\le h<k\le N}} e^{-\frac{2\pi i}{k}(nh+\ell h')} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} e^{\frac{2\pi\ell}{k^2\omega} + 2\pi n\omega}\mathrm{d}\phi$$

and

$$R_m(n) = \sum_{\substack{hh'=-1 \mod k\\ 0\le h<k\le N}} e^{-\frac{2\pi i n h}{k}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} D_m\left(e^{\frac{2\pi i h'}{k} - \frac{2\pi}{k^2\omega}}\right) e^{2\pi n\omega}\mathrm{d}\phi.$$

[3] See [Apo, sec. 5.4] for basic definitions and properties of Farey fractions and [Apo, sec. 5.2] for the explanation of such a splitting.

For any $\ell \geq 0$, we have

$$Q_\ell(n) = \underbrace{\sum_{k=1}^{N} \sum_{\substack{hh'=-1 \mod k \\ 0\leq h<k}} e^{-\frac{2\pi i}{k}(nh+\ell h')} \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\phi}_{Q_{\ell,0}(n)}$$
$$+ \underbrace{\sum_{k=1}^{N} \sum_{\substack{hh'=-1 \mod k \\ 0\leq h<k}} e^{-\frac{2\pi i}{k}(nh+\ell h')} \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\phi}_{Q_{\ell,1}(n)}$$
$$+ \underbrace{\sum_{k=1}^{N} \sum_{\substack{hh'=-1 \mod k \\ 0\leq h<k}} e^{-\frac{2\pi i}{k}(nh+\ell h')} \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\phi}_{Q_{\ell,2}(n)}.$$

We set $S(a,b;c) = \sum_{\substack{1\leq u,v\leq c \\ uv=1 \mod c}} e^{2i\pi\left(\frac{au+bv}{c}\right)}$ where a, b, c are rational integers with $c>0$: it is a Kloosterman sum and, from [Wei], we know that

$$|S(a,b;c)| \leq (a,b,c)^{1/2} c^{1/2} \tau(c) \tag{9}$$

where (a,b,c) is the g.c.d. of a, b and c and $\tau(c)$ denotes the number of positive divisors of c. Thus, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|S(a,b;c)| \leq C_\varepsilon (a,b,c)^{1/2} c^{1/2+\varepsilon}. \tag{10}$$

It is easy to see that $S(a,b;c)$ is a real number. In other words, $S(-a,-b;c) = S(a,b;c)$. So we have

$$Q_{\ell,1}(n) = \sum_{k=1}^{N} S(n,-\ell;k) \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\phi$$
$$= \sum_{k=1}^{N} S(n,-\ell;k) \frac{1}{i} \int_{\gamma_0} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\omega$$

where γ_0 denotes the path $u \in [-1,1] \mapsto N^{-2} + \frac{iu}{k(N+k)}$. We define the paths $\gamma_1 : u \in [-1,1] \mapsto -uN^{-2} + \frac{i}{k(N+k)}$, $\gamma_2 : u \in [-1,1] \mapsto -N^{-2} - \frac{iu}{k(N+k)}$, $\gamma_3 : u \in [-1,1] \mapsto uN^{-2} - \frac{i}{k(N+k)}$ and R is the path $\gamma_0\gamma_1\gamma_2\gamma_3$ i.e. the closed rectangular path, surrounding 0 in the positive sense, with the vertices $\pm N^{-2} \pm \frac{i}{k(N+k)}$.

Hence, we write

$$Q_{\ell,1}(n) = 2\pi \sum_{k=1}^{N} S(n,-\ell;k) \frac{1}{2i\pi} \int_R e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\omega$$
$$-\frac{1}{i}\sum_{k=1}^{N} S(n,-\ell;k)\Bigg(\underbrace{\int_{\gamma_1} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega}\mathrm{d}\omega}_{J_1} + \underbrace{\int_{\gamma_2} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega}\mathrm{d}\omega}_{J_2} + \underbrace{\int_{\gamma_3} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega}\mathrm{d}\omega}_{J_3}\Bigg).$$

It is easy to prove that, for all $\omega \in \gamma_1([-1,1]) \cup \gamma_3([-1,1])$, $\left|e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega}\right| \le e^{8\pi\ell+2\pi n N^{-2}}$ and, for all $\omega \in \gamma_2([-1,1])$, $\left|e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega}\right| < 1$. Therefore,

$$\max(|J_1|,|J_3|) \le 2N^{-2}e^{8\pi\ell+2\pi n N^{-2}} \text{ and } |J_2| \le 2/(kN). \tag{11}$$

Now, we assume $n \ge 1$: thus $(n,-\ell,k) \le n$. The inequalities (11) combined with (10) yield, for any $\varepsilon > 0$,

$$\sum_{k=1}^{N} S(n,-\ell;k)(J_1+J_2+J_3) = O_\varepsilon(e^{8\pi\ell+2\pi n N^{-2}} n^{1/2} N^{-1/2+\varepsilon})$$

Finally, the residue at 0 of the function $z \mapsto e^{a\left(z+\frac{1}{z}\right)}$, a being any complex number, is $I_1(2a)$ where $I_1(z) = \sum_{\nu=0}^{+\infty} \frac{(z/2)^{2\nu+1}}{\nu!(\nu+1)!}$ is the Bessel function of first order with purely imaginary argument. Hence,

$$Q_{\ell,1}(n) = \frac{2\pi\sqrt{\ell}}{\sqrt{n}} \sum_{k=1}^{N} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right) + O_\varepsilon(e^{8\pi\ell+2\pi n N^{-2}} n^{1/2} N^{-1/2+\varepsilon}). \tag{12}$$

Then, we give an upper bound for $Q_{\ell,0}(n)$. We write

$$Q_{\ell,0}(n) = \sum_{k=1}^{N} \sum_{\substack{hh'=-1 \mod k \\ 0\le h<k}} e^{-\frac{2\pi i}{k}(nh+\ell h')} \sum_{j=k_1+k}^{N+k-1} \int_{-\frac{1}{kj}}^{-\frac{1}{k(j+1)}} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\phi.$$

As the fraction $(h_1+h)/(k_1+k)$ which is the mediant of the fractions h_1/k_1 and h/k does not belong to the Farey series of order N, we have $k_1 + k > N$. Thus, we can write

$$Q_{\ell,0}(n) = \sum_{k=1}^{N} \sum_{j=N+1}^{N+k-1} \int_{-\frac{1}{kj}}^{-\frac{1}{k(j+1)}} e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega} \mathrm{d}\phi \underbrace{\sum_{\substack{hh'=-1 \mod k \\ 0\le h<k \text{ and } N<k_1+k\le j}} e^{-\frac{2\pi i}{k}(nh+\ell h')}}_{L_1}$$

From (8), the restriction on k_1 means a restriction on h' : the sum L_1 is an incomplete Kloosterman sum. We get from [Hoo, chap. 2, §5] the estimate $L_1 = O_\varepsilon(k^{1/2+\varepsilon}(n,k)^{1/2})$.

For $\phi \in \left[-\frac{1}{k(N+1)}, -\frac{1}{k(N+k)}\right]$, we have $\left|e^{\frac{2\pi\ell}{k^2\omega}+2\pi n\omega}\right| \leq e^{8\pi\ell+2\pi nN^{-2}}$. Therefore, we get

$$Q_{\ell,0}(n) = O_\varepsilon\left(e^{8\pi\ell+2\pi nN^{-2}} n^{1/2} \sum_{k=1}^{N} \sum_{j=N+1}^{N+k-1} \left(\frac{1}{kj} - \frac{1}{k(j+1)}\right) k^{1/2+\varepsilon}\right)$$
$$= O_\varepsilon\left(e^{8\pi\ell+2\pi nN^{-2}} n^{1/2} \sum_{k=1}^{N} \frac{1}{k^{1/2-\varepsilon}N}\right) = O_\varepsilon\left(e^{8\pi\ell+2\pi nN^{-2}} n^{1/2} N^{-1/2+\varepsilon}\right).$$

We can prove in the same way that $Q_{\ell,2}(n) = O_\varepsilon(e^{8\pi\ell+2\pi nN^{-2}} n^{1/2} N^{-1/2+\varepsilon})$.

Combining (12) with the last two equalities, we obtain
(13)
$$Q_\ell(n) = \frac{2\pi\sqrt{\ell}}{\sqrt{n}} \sum_{k=1}^{N} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right) + O_\varepsilon(e^{8\pi\ell+2\pi nN^{-2}} n^{1/2} N^{-1/2+\varepsilon}).$$

We refer the reader to [Rad3, pp. 508-510] for the proof of

$$R_m(n) = O_\varepsilon(e^{8\pi m+2\pi nN^{-2}} n^{1/2} N^{-1/2+\varepsilon}). \tag{14}$$

The only change is the replacement of $\sum_{\ell=0}^{+\infty} c(\ell)e^{-\pi\ell}$ by $\sum_{\ell=0}^{+\infty} c_m(\ell)e^{-\pi\ell}$. We notice that $\left|\sum_{\ell=0}^{+\infty} c_m(\ell)e^{-\pi\ell}\right| \leq j^m\left(\frac{i}{2}\right) \leq e^{8\pi m}$.

Then, we deduce, from (13) and (14), that, for all $n, m \geq 1$,

$$c_m(n) = \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{m} c_m(-\ell)\sqrt{\ell} \left(\sum_{k=1}^{N} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right)\right)$$
$$+ O_\varepsilon(m e^{8\pi m+2\pi nN^{-2}} n^{1/2} N^{-1/2+\varepsilon}).$$

If we fix m, n and let N tend to infinity, the remainder term tends to zero. So, we achieved the following result:

Theorem 5.1. [H. Rademacher] *For $m, n \geq 1$, we have*

$$c_m(n) = \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{m} c_m(-\ell)\sqrt{\ell} \sum_{k=1}^{+\infty} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right)$$

with $S(n,-\ell;k) = \sum_{\substack{1\leq u,v\leq k \\ uv=1 \mod k}} e^{2i\pi\left(\frac{nu-\ell v}{k}\right)}$ *and* $I_1(z) = \sum_{\nu=0}^{+\infty} \frac{(z/2)^{2\nu+1}}{\nu!(\nu+1)!}$.

In fact, Rademacher proved only the case $m = 1$ but, as we follow step by step his proof, we chose to credit him with the general case of J^m.

Reading carefully the proof we have just done, we can see that, in fact, we obtained the following more general statement.

Theorem 5.2. *Let f a modular function of weight 0, holomorphic on $\mathfrak{H}$, having the Fourier expansion*

$$f(\tau) = \sum_{n \geq -m_f} a(n) q^n, \text{ with } q = e^{2i\pi\tau},$$

for all $\tau \in \mathfrak{H}$. Then, we have, for $n \geq 1$,

$$a(n) = \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{m_f} a(-\ell)\sqrt{\ell} \sum_{k=1}^{+\infty} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right).$$

5.2. **An estimate of the Fourier coefficients of j^m with an arbitrarily small error term.**

For $n \geq 1$, $\ell \geq 1$, $N \geq 1$, we set $R_{\ell,N}(n) = \sum_{k=N+1}^{+\infty} \frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right)$.

We first prove the following result.

Lemma 5.3. *For all positive integers ℓ, n, k, we have*

$$|S(n,-\ell;k)| \leq 9(n,\ell,k)^{1/2}\, k^{3/4}.$$

Proof. Inequality (9) gives $|S(n,-\ell;k)| \leq k^{1/2}(n,\ell,k)^{1/2}\, \tau(k)$. Thus, it remains to prove that $\tau(k) \leq 9k^{1/4}$, which we do following [HW, §18.1]. Let $\mathcal{P}$ denote the set of prime numbers and, for all $p \in \mathcal{P}$, let v_p denote the p-adic valuation. We have $k = \prod_{p\in\mathcal{P}} p^{v_p(k)}$ As $\tau(k) = \prod_{p\in\mathcal{P}}(v_p(k)+1)$, we have to prove $\frac{\tau(k)}{k^{1/4}} = \prod_{p\in\mathcal{P}}\left(\frac{v_p(k)+1}{p^{v_p(k)/4}}\right) \leq 9$. If $p \geq 16$, we have $p^{1/4} \geq 2$ and then $\frac{v_p(k)+1}{p^{v_p(k)/4}} \leq \frac{v_p(k)+1}{2^{v_p(k)}} \leq 1$. Hence,

$$\frac{\tau(k)}{k^{1/4}} \leq \prod_{\substack{p\in\mathcal{P},\\ p\leq 16}} \frac{v_p(k)+1}{p^{v_p(k)/4}}.$$

Let u_p be the function $x \mapsto \frac{x+1}{p^{x/4}}$ for $p \in \mathcal{P}$. When $p \leq 16$, the respective maxima of the u_p over $\mathbb{N}$ are $u_2(5)$, $u_3(3)$, $u_5(2)$, $u_7(1)$, $u_{11}(1)$ and $u_{13}(1)$, which implies $\frac{\tau(k)}{k^{1/4}} \leq 9$. □

Then, following [Rad5, §121], we get, for $\ell \geq 1$, $N \geq 1$,

$$\begin{aligned}
|R_{\ell,N}(n)| &\leq 9(n,\ell)^{1/2} \sum_{k=N+1}^{+\infty} k^{-1/4} \left| I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right)\right| \\
&= 9(n,\ell)^{1/2} \sum_{k=N+1}^{+\infty} k^{-1/4} \sum_{\nu=0}^{+\infty} \frac{(2\pi\sqrt{n\ell}/k)^{2\nu+1}}{\nu!(\nu+1)!} \\
&< 9(n,\ell)^{1/2} \sum_{\nu=0}^{+\infty} \frac{(2\pi\sqrt{n\ell})^{2\nu+1}}{\nu!(\nu+1)!} \int_N^{+\infty} \frac{\mathrm{d}x}{x^{2\nu+5/4}} \\
&= 9(n,\ell)^{1/2} \sum_{\nu=0}^{+\infty} \frac{(2\pi\sqrt{n\ell})^{2\nu+1}}{\nu!(\nu+1)!(2\nu+1/4)} \frac{1}{N^{2\nu+1/4}} \\
&\leq 36(n,\ell)^{1/2} N^{3/4} I_1\left(\frac{4\pi\sqrt{n\ell}}{N}\right).
\end{aligned}$$

Thus, for m, $n \geq 1$, $(N_1,\ldots,N_m) \in (\mathbb{N}\setminus\{0\})^m$, we have

$$c_m(n) = \frac{2\pi}{\sqrt{n}}\sum_{\ell=1}^{m} c_m(-\ell)\sqrt{\ell}\left(\sum_{k=1}^{N_\ell}\frac{S(n,-\ell;k)}{k} I_1\left(\frac{4\pi\sqrt{n\ell}}{k}\right) + R_{\ell,N_\ell}(n)\right) \tag{15}$$

with $|R_{\ell,N_\ell}(n)| \leq 36(n,\ell)^{1/2} {N_\ell}^{3/4} I_1\left(\dfrac{4\pi\sqrt{n\ell}}{N_\ell}\right)$.

Now, we prove that a slightly weaker version of Theorem 1.1 is a consequence of this estimate.

5.3. **Theorem 1.1 as a corollary of estimate** (15).

For m, $n \geq 1$, we fix $N_1 = \cdots = N_{m-1} = 1$ and $N_m = 2$. This gives

$$\begin{aligned}
c_m(n) = &2\pi\sqrt{\frac{m}{n}}\left(I_1\left(4\pi\sqrt{nm}\right) + \frac{(-1)^{n-m}}{2} I_1\left(2\pi\sqrt{nm}\right) + R_{m,2}(n)\right) \\
&+ \frac{2\pi}{\sqrt{n}}\sum_{\ell=1}^{m-1} c_m(-\ell)\sqrt{\ell}\left(I_1\left(4\pi\sqrt{n\ell}\right) + R_{\ell,1}(n)\right) \\
= &\frac{2\pi}{\sqrt{n}}\left(\sqrt{m} I_1\left(4\pi\sqrt{nm}\right) + R(n)\right)
\end{aligned}$$

with

$$|R(n)| \leq 62\sqrt{m}\,(n,m)^{1/2} I_1\left(2\pi\sqrt{nm}\right) + 37\sum_{\ell=1}^{m-1} c_m(-\ell)\sqrt{\ell}(n,\ell)^{1/2} I_1\left(4\pi\sqrt{n\ell}\right).$$

For all $z \in \mathbb{C}$, we have $I_1(z) = -\dfrac{i}{2}\left(H_1^{(1)}(iz) + H_1^{(2)}(iz)\right)$ where $H_1^{(1)}$ and $H_1^{(2)}$ are respectively the first and second Hankel functions [4] [Wat, §3.6]. As $H_1^{(1)}(ix)$ is real for all $x > 0$ (cf. [Erd, §7.2.2] or [Wat, §7.2, p. 196]), we know that $I_1(x) = \Im\left(H_1^{(2)}(ix)\right)/2$ for $x > 0$. Following [Wat, Chap. 7], we establish

[4] also called Bessel functions of the third kind.

an asymptotic expansion of $I_1(x)$ (where $x > 0$) with an effective remainder term.

For all $x > 0$, we have (see [Wat, p. 168])

$$H_1^{(2)}(ix) = i\sqrt{\frac{2}{\pi x}}\frac{e^x}{\Gamma\left(\frac{3}{2}\right)}\int_0^{\infty \exp i\beta} e^{-u}u^{1/2}\left(1-\frac{u}{2x}\right)^{1/2}\mathrm{d}u \tag{16}$$

where $0 < \beta < \pi/2$. For all $p \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}\left(1-\frac{u}{2x}\right)^{1/2} &= \sum_{k=0}^{p-1}\frac{\left(-\frac{1}{2}\right)_k}{k!}\left(\frac{u}{2x}\right)^k \\ &\quad + \frac{\left(-\frac{1}{2}\right)_p}{(p-1)!}\left(\frac{u}{2x}\right)^p\int_0^1(1-t)^{p-1}\left(1-\frac{ut}{2x}\right)^{1/2-p}\mathrm{d}t\end{aligned} \tag{17}$$

where $(\alpha)_n$ is the Pochhammer symbol defined by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. We choose (for simplicity) $\beta = \pi/4$. For all $t \in [0,1]$ and u on the half-line $[0, \infty e^{i\pi/4})$, we have $\left|1-\frac{ut}{2x}\right| \geq \sin(\pi/4) = \frac{\sqrt{2}}{2}$ hence $\left|\left(1-\frac{ut}{2x}\right)^{1/2-p}\right| \leq 2^{\frac{2p-1}{4}}$. Then, in (16), we replace $\left(1-\frac{u}{2x}\right)^{1/2}$ by its expansion (17) and integrate term-by-term. This gives

$$H_1^{(2)}(ix) = i\sqrt{\frac{2}{\pi x}}\,e^x\left(\sum_{k=0}^{p-1}\frac{\left(-\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!(2x)^k} + \frac{r_{H_1^{(2)},p}}{x^p}\right)$$

with

$$\begin{aligned}\left|r_{H_1^{(2)},p}\right| &\leq \frac{2^{\frac{2p-1}{4}}}{(p-1)!}\frac{\left|\left(-\frac{1}{2}\right)_p\right|}{\Gamma\left(\frac{3}{2}\right)2^p}\left|\int_0^1(1-t)^{p-1}\mathrm{d}t\int_0^{\infty\exp\left(i\frac{\pi}{4}\right)}e^{-u}u^{p+1/2}\mathrm{d}u\right| \\ &\leq \frac{2^{\frac{2p-1}{4}}}{p!}\frac{\left|\left(-\frac{1}{2}\right)_p\right|}{\Gamma\left(\frac{3}{2}\right)2^p}2^{\frac{2p+3}{4}}\Gamma\left(p+\frac{3}{2}\right) = \frac{\sqrt{2}}{p!}\left|\left(-\frac{1}{2}\right)_p\left(\frac{3}{2}\right)_p\right|.\end{aligned}$$

If we write, after Hankel [Han, p. 494],

$$(1,k) = (-1)^k\frac{\left(-\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!} = \frac{\Gamma\left(k+\frac{3}{2}\right)}{k!\Gamma\left(-k+\frac{3}{2}\right)} = \frac{\prod_{j=0}^{k-1}(4-(2j+1)^2)}{4^k k!}, \tag{18}$$

we obtained, for all $x > 0$ and $p \in \mathbb{N}\setminus\{0\}$,

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left(\sum_{k=0}^{p-1}\frac{(-1)^k(1,k)}{(2x)^k} + \frac{r_{I_1,p}}{x^p}\right) \tag{19}$$

with $|r_{I_1,p}| \leq \frac{1}{\sqrt{2}}|(1,p)|$. Notice that this formula does not give a convergent series expansion. This was done by Hadamard [Had] whose work implies, for all $x > 0$,

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_k\gamma\left(k+\frac{3}{2}, 2x\right)}{\Gamma\left(\frac{3}{2}\right)k!(2x)^k}$$

where γ denotes Legendre's incomplete Gamma-function defined by $\gamma(a,u) = \int_0^u t^{a-1}e^{-t}\mathrm{d}t$. As $\left(-\frac{1}{2}\right)_k \leq 0$ for all $k \geq 1$, we get, in particular, for all $x > 0$,

$$I_1(x) \leq \frac{e^x}{\sqrt{2\pi x}}. \tag{20}$$

Until the end of the section, we assume that $n, m \in \mathbb{N}$, $nm \geq 1000$ and $n \geq 4m\ln^2 m$.

Let $p_1 = \left\lfloor \frac{me^{2\pi}}{1728} \right\rfloor$. We have $p_1 \geq 1$ if and only if $m \geq 4$.

For all $m \geq 2$, we have, from inequality (20),

$$\begin{aligned}
S_1 &:= \frac{2\pi}{\sqrt{n}} \sum_{\ell=p_1+1}^{m-1} 37c_m(-\ell)\sqrt{\ell}(n,\ell)^{1/2} I_1(4\pi\sqrt{n\ell}) \\
&\leq \sum_{\ell=p_1+1}^{m-1} \frac{37\sqrt{2}}{2} c_m(-\ell)\left(\frac{\ell}{n}\right)^{3/4} e^{4\pi\sqrt{n\ell}} \\
&= \frac{\sqrt{2}}{2}\frac{m^{1/4}}{n^{3/4}}\frac{e^{4\pi\sqrt{nm}}}{nm} \sum_{\ell=p_1+1}^{m-1} 37nc_m(-\ell)(m\ell)^{3/4} e^{4\pi\sqrt{n}(\sqrt{\ell}-\sqrt{m})} \\
&\leq \frac{\sqrt{2}}{2}\frac{m^{1/4}}{n^{3/4}}\frac{e^{4\pi\sqrt{nm}}}{nm} \underbrace{37nm^{3/2}e^{-2\pi\sqrt{nm}} \sum_{\ell=p_1+1}^{m-1} c_m(-\ell)e^{2\pi\ell\sqrt{n/m}}}_{S_1'}.
\end{aligned}$$

If we proceed as in subsection 4.2, we obtain, under the assumptions $m \geq 2$ and (6),

$$S_1' \leq 37nm^{7/2}\frac{(1728-e^{2\pi})^2}{1728}e^{1-2\pi\sqrt{n/m}} \leq 82771nm^{7/2}e^{-2\pi\sqrt{n/m}}.$$

The function $g_m : x \mapsto xe^{-2\pi\sqrt{x/m}}$ is decreasing for $x \geq \max(1, 4m\ln^2 m)$.

For $m \leq 7$, as $n \geq 1000/7 \geq \max(1, 28\ln^2 7)$,

$$\begin{aligned}
82771nm^{7/2}e^{-2\pi\sqrt{n/m}} &\leq 82771m^{7/2}g_m\left(\frac{1000}{7}\right) \\
&\leq 82771 \cdot 7^{7/2} g_7\left(\frac{1000}{7}\right) \leq 0.0051.
\end{aligned}$$

For $m = 8$, we have $n \geq 32\ln^2 8$ i.e. $n \geq 139$ which implies

$$82771nm^{7/2}e^{-2\pi\sqrt{n/m}} = 82771 \cdot 8^{7/2}ne^{-2\pi\sqrt{n/8}} \leq 84757504\sqrt{2}g_8(139) \leq 0.0704.$$

For $m \geq 9$, as $n \geq 4m\ln^2 m$, we have

$$82771nm^{7/2}e^{-2\pi\sqrt{n/m}} \leq 82771m^{7/2}g_m(4m\ln^2 m) = 331084m^{9/2-4\pi}\ln^2 m.$$

The function $x \mapsto x^{9/2-4\pi}\ln^2 x$ is decreasing for $x \geq \exp\left(\frac{2}{4\pi-9/2}\right)$. Hence, if $m \geq 9 \geq \left(\exp\left(\frac{2}{4\pi-9/2}\right)\right)$, we get

$$82771nm^{7/2}e^{-2\pi\sqrt{n/m}} \leq 331084 \cdot 9^{9/2-4\pi}\ln^2 9 \leq 0.0321$$

Finally, we obtained $S_1' \leq 0.0704$ under the assumptions $nm \geq 1000$ and $n \geq 4m\ln^2 m$.

For all $m \geq 4$,

$$S_2 := \frac{2\pi}{\sqrt{n}} \sum_{\ell=1}^{p_1} 37 c_m(-\ell)\sqrt{\ell}(n,\ell)^{1/2} I_1(4\pi\sqrt{n\ell})$$

$$\leq \sum_{\ell=1}^{p_1} \frac{37\sqrt{2}}{2} \frac{1728^m}{e^{2\pi\ell}} \left(\frac{\ell}{n}\right)^{3/4} e^{4\pi\sqrt{n\ell}}$$

$$= \frac{\sqrt{2}}{2} \frac{m^{1/4}}{n^{3/4}} \frac{e^{4\pi\sqrt{nm}}}{nm} \underbrace{\sum_{\ell=1}^{p_1} 37n \frac{1728^m}{e^{2\pi\ell}} (m\ell)^{3/4} e^{4\pi\sqrt{n}\left(\sqrt{\ell}-\sqrt{m}\right)}}_{S_2'}$$

from Proposition 4.1 and (20). As $e^{4\pi\sqrt{n}\left(\sqrt{\ell}-\sqrt{m}\right)} = e^{\frac{4\pi\sqrt{n}(\ell-m)}{\sqrt{\ell}+\sqrt{m}}} \leq e^{\frac{4\pi\sqrt{n}(\ell-m)}{2\sqrt{m}}}$, it follows that

$$S_2' \leq 37nm^{3/2}1728^m e^{-2\pi\sqrt{nm}} \sum_{\ell=1}^{p_1} e^{2\pi\ell(\sqrt{n/m}-1)}.$$

Here again, under the assumptions $m \geq 4$ and (6), subsection 4.2 gives (cf. the estimate of $|L_{22}|$)

$$S_2' \leq \frac{222}{5} nm^{3/2} e^{5.51m} e^{2\pi\left(\frac{e^{2\pi}}{1728}-1\right)\sqrt{nm}} \leq \frac{222}{5} \frac{(nm)^{3/2}}{n^{1/2}} e^{5.51m-1.38\pi\sqrt{nm}}.$$

As $m \geq 4$, the condition $n \geq 4m\ln^2 m$ implies $\sqrt{n} \geq 4\sqrt{m}\ln 2 \geq 8\ln 2$. Then, we get

$$S_2' \leq \frac{111}{20\ln 2}(nm)^{3/2} e^{(5.51/4\ln 2 - 1.38\pi)\sqrt{nm}} \leq 2\cdot 10^{-27}$$

if $nm \geq 1000$.

For $m \geq 1$, we have

$$S_3 := 124\pi\sqrt{\frac{m}{n}}\,(n,m)^{1/2} I_1(2\pi\sqrt{nm})$$

$$\leq 62\left(\frac{m}{n}\right)^{3/4} e^{2\pi\sqrt{nm}} = \frac{\sqrt{2}}{2}\frac{m^{1/4}}{n^{3/4}}\frac{e^{4\pi\sqrt{nm}}}{nm} \underbrace{62\sqrt{2}nm^{3/2}e^{-2\pi\sqrt{nm}}}_{S_3'}.$$

But $S_3' \leq 62\sqrt{2}(nm)^{3/2}e^{-2\pi\sqrt{nm}} \leq 2\cdot 10^{-80}$ if $nm \geq 1000$.

As $(1,1) = \frac{3}{4}$ and $(1,2) = -\frac{15}{32}$, the expression (19) gives

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{3}{8x} + \frac{r_{I_1,2}}{x^2}\right) \tag{21}$$

with $|r_{I_1,2}| \leq \frac{15\sqrt{2}}{64}$. It follows that

$$c_m(n) = \frac{1}{\sqrt{2}}\frac{m^{1/4}}{n^{3/4}} e^{4\pi\sqrt{nm}}\left(1 - \frac{3}{32\pi}\frac{1}{\sqrt{nm}} + \frac{1}{nm}\rho_{n,m}\right)$$

with $|\rho_{n,m}| \leq \frac{15\sqrt{2}}{1024\pi^2} + S_1' + S_2' + S_3' \leq 0.0725$ for all $m, n \in \mathbb{N}$ such that $nm \geq 1000$ and $n \geq 4m \ln^2 m$.

The result obtained here is slightly weaker than the one proved in the previous section. But in the case $m = 1$, Rademacher's approach leads to a better result than the one provided by our approach, as we are going to see in the next section.

6. The case of j

According to subsection 5.3, we have

$$c(n) = \frac{2\pi}{\sqrt{n}} I_1\left(4\pi\sqrt{n}\right) + R'(n)$$

with

$$|R'(n)| \leq \frac{124\pi}{\sqrt{n}} I_1\left(2\pi\sqrt{n}\right).$$

From (19) and (20), we get, for $n \geq 1$, $p \geq 1$,

$$c(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(\sum_{k=0}^{p-1} \frac{(-1)^k (1,k)}{(8\pi\sqrt{n})^k} + \frac{r_p(n)}{n^{p/2}} \right) \tag{22}$$

where $(1,k)$ is given by (18) and

$$|r_p(n)| = \left| \frac{r_{I_1,p}}{(4\pi\sqrt{n})^p} + 62\sqrt{2}e^{-2\pi\sqrt{n}} \right| \leq \frac{\sqrt{2}}{2} \frac{|(1,p)|}{(4\pi)^p} + 62\sqrt{2}e^{-2\pi\sqrt{n}} n^{p/2}.$$

Note that the maximum over $\mathbb{R}^+$ of the function $x \mapsto 62\sqrt{2}e^{-x}(2x)^p$ is reached at $x = p$.

For example, if $p = 3$, we have

$$|r_3(n)| = \frac{\sqrt{2}}{2} \frac{105}{128(4\pi)^3} + 62\sqrt{2}e^{-2\pi\sqrt{n}} (\sqrt{n})^3 \leq 0.0003$$

if $n \geq 10$. Hence, for all $n \geq 10$, we have

$$c(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} + \varepsilon_n \right) \text{ with } |\varepsilon_n| \leq \frac{3 \cdot 10^{-4}}{n^{3/2}}. \tag{23}$$

The computation of the first ten $c(n)$ (cf. section 8) shows that this remains true for $n \geq 3$.

A careful analysis of section 4 would give a similar result. But, when dealing with parameter $p \geq 4$, we need to get a more precise estimate in Lemma 3.3 to provide with our approach a result similar to the one obtained from (22).

To end this section, we want to check that $c(n) \leq \dfrac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$ for all $n \geq 1$ as claimed in the introduction. The relation (23) implies $c(n) \leq \dfrac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}$ for all $n \geq 10$. The computation of the first ten $c(n)$ shows that this is indeed satisfied for all $n \geq 1$. Note that this result can also be obtained from Theorem 1.1. This improves the result $c(n) \leq 6e^{4\pi\sqrt{n}}$, for $n \geq 1$, given in [Her].

7. Upper bounds for the coefficients $c_m(n)$ when $m \geq 2$

We obtained in Proposition 4.1 some general upper bounds for the $c_m(n)$ and an improvement of these bounds when $-m+1 \leq n \leq -\dfrac{me^{2\pi}}{1728}$ in Proposition 4.2 and when $nm \geq 1000$ and $n \geq 4m\ln^2 m$ in Theorem 1.1. Nevertheless, some cases remain where the bounds $c_m(n) \leq 1200e^{4\pi\sqrt{m(m+n)}}$ given in [Mah] are better than the ones we gave until now. The aim of this section is, first, to give upper bounds for the $c_m(n)$ that improve Mahler's ones in these remaining cases and then, to give a general statement providing the best upper bounds known for the coefficients $c_m(n)$ when $m \geq 2$.

We start by another general estimate.

Proposition 7.1. *For all $m \in \mathbb{N}^*$, $n \in \mathbb{Z}$, we have*

$$c_m(n) \leq e^{2\pi\sqrt{m(m+n)}} \left(\frac{1728}{e^{2\pi}}\right)^{m+n} \quad \text{if } -m+1 \leq n \leq 0$$

and

$$c_m(n) \leq e^{4\pi\sqrt{m(m+n)}} \text{ if } n \geq 0.$$

Proof. Let us recall that we have for all $y > 0$

$$c_m(n) = e^{2\pi ny}\int_{-1/2}^{1/2} j^m(x+iy)e^{-2i\pi nx}\mathrm{d}x.$$

We fix $y = \sqrt{\dfrac{m}{m+n}}$.

First, we assume that $n \geq 0$, Hence, we have $y \leq 1$ and Lemma 3.1 gives

$$\begin{aligned} c_m(n) &\leq e^{2\pi n\sqrt{\frac{m}{m+n}}} \left(e^{2\pi\sqrt{\frac{m+n}{m}}} + 1728 - e^{2\pi}\right)^m \\ &\leq e^{2\pi n\sqrt{\frac{m}{m+n}}+2\pi\sqrt{m(m+n)}} \left(1 + (1728 - e^{2\pi})e^{-2\pi\sqrt{\frac{m+n}{m}}}\right)^m \\ &\leq e^{4\pi\sqrt{m(m+n)}} \left(e^{-2\pi\sqrt{\frac{m}{m+n}}} \left(1 + (1728 - e^{2\pi})e^{-2\pi\sqrt{\frac{m+n}{m}}}\right)\right)^m. \end{aligned}$$

Let $f : x \in]0,1] \longmapsto e^{-2\pi x}(1 + (1728 - e^{2\pi})e^{-2\pi/x})$. The derivative of f is negative on $]0,1]$. As $\lim_{x\to 0^+} f(x) = 1$, we thus have $f(x) < 1$ for all $x \in]0,1]$. Consequently, $c_m(n) \leq e^{4\pi\sqrt{m(m+n)}} f(y)^m \leq e^{4\pi\sqrt{m(m+n)}}$ since $y \in]0,1]$.

Then, we assume $-m+1 \leq n \leq 0$. We have $y \geq 1$. We get from Lemma 3.1

$$\begin{aligned} c_m(n) &\leq e^{2\pi n\sqrt{\frac{m}{m+n}}} \left(e^{2\pi\sqrt{\frac{m}{m+n}}} + 1728 - e^{2\pi}\right)^m \\ &\leq e^{2\pi\sqrt{m(m+n)}} \left(1 + (1728 - e^{2\pi})e^{-2\pi\sqrt{\frac{m}{m+n}}}\right)^m. \end{aligned}$$

We study $\left(1 + (1728 - e^{2\pi})e^{-2\pi\sqrt{\frac{m}{m+n}}}\right)^m$ when $n+m$ is fixed. Therefore, we study $g : x \in [m+n, +\infty[\longmapsto \left(1 + (1728 - e^{2\pi})e^{-2\pi\sqrt{\frac{x}{m+n}}}\right)^x$. The derivative of g is nonpositive on $[m+n,+\infty]$. Hence, $g(x) \leq g(m+n) = \left(\frac{1728}{e^{2\pi}}\right)^{m+n}$. It follows that

$$c_m(n) \leq e^{2\pi\sqrt{m(m+n)}} \left(\frac{1728}{e^{2\pi}}\right)^{m+n}.$$

Note that $e^{2\pi\sqrt{m(m+n)}} \left(\frac{1728}{e^{2\pi}}\right)^{m+n} \leq e^{4\pi\sqrt{m(m+n)}}$. □

Let $\alpha = 2 - \frac{\ln(1728)}{2\pi} + 2\sqrt{2 - \frac{\ln(1728)}{2\pi}} = 2.617\ldots$. We have, for $0 \leq n \leq \alpha m$,

$$e^{2\pi n} 1728^m \leq e^{4\pi\sqrt{m(m+n)}}.$$

Moreover, for $-m+1 \leq n \leq 0$, the bounds given in Propositions 4.1 and 4.2 are smaller than those from Proposition 7.1.

We collect the upper bounds that we obtained for the $c_m(n)$ with $m \geq 2$ in the following assertion.

Corollary 7.2. *For all $m \in \mathbb{N}$, $m \geq 2$, we have*

- *for* $-m+1 \leq n \leq -m\frac{e^{2\pi}}{1728}$,
$$c_m(n) \leq (1728 - e^{2\pi})^{m+n} \left(\frac{-n}{m+n}\right)^n \left(\frac{m}{m+n}\right)^m;$$
- *for* $-m\frac{e^{2\pi}}{1728} \leq n \leq \alpha m$,
$$c_m(n) \leq e^{2\pi n} 1728^m;$$
- *for* $\alpha m \leq n \leq \max\left(\frac{1000}{m}, 4m\ln^2 m\right)$,
$$c_m(n) \leq e^{4\pi\sqrt{m(m+n)}};$$
- *for* $n \geq \max\left(\frac{1000}{m}, 4m\ln^2 m\right)$,
$$c_m(n) \leq \frac{1}{\sqrt{2}} \frac{m^{1/4}}{n^{3/4}} e^{4\pi\sqrt{nm}} \left(1 - \frac{3}{32\pi}\frac{1}{\sqrt{nm}} + \frac{0.055}{nm}\right).$$

These upper bounds improve Mahler's ones [Mah]. The reader should also keep in mind the exact expression given in Theorem 5.1 and the estimate of subsection 5.2.

8. Some numerical experiments

First, we deal with the case $m = 1$. In Table 1, we compare the first ten values of $c(n)$ with the bounds obtained from (23) : for $n \geq 1$, let

$$\mathrm{Min}(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} - \frac{3\cdot 10^{-4}}{n^{3/2}}\right)$$

and

$$\mathrm{Maj}(n) = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} + \frac{3\cdot 10^{-4}}{n^{3/2}}\right).$$

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

n	$c(n)$	$c(n) - \lceil \mathrm{Min}(n) \rceil$	$\lfloor \mathrm{Maj}(n) \rfloor - c(n)$
1	196884	382	-261
2	21493760	-217	4876
3	864299970	52342	49232
4	20245856256	582090	959653
5	333202640600	7600670	10525532
6	4252023300096	71384200	104367475
7	44656994071935	601906544	861509359
8	401490886656000	4409439269	6351251025
9	3176440229784420	29283637831	42019206145
10	22567393309593600	177496731279	254801748021

TABLE 1

Then, we assume $m \geq 2$ and we want to check if the condition $n \geq 4m \ln^2 m$ is necessary in Theorem 1.1 and, in case of a negative answer, if the condition (6) is sufficient. It is easy to see that the condition $nm \geq 1000$ is stronger than the condition $n \geq 4m \ln^2 m$ if and only if $m \leq 7$. Hence, we only examine the case $m \geq 8$. We put, from condition (6), $f(m) = m \left(\frac{1}{2\pi} \ln((1728 - e^{2\pi})(m-1)) \right)^2$ for $m \geq 2$. In Table 2, $n_{\min}(m)$ denotes the least rational integer such that $c_m(n)$ satisfies the result of Theorem 1.1 for all $n \geq n_{\min}(m)$. For $m \in \{30, 35, 40, 45, 50\}$, because of a lack of memory of our computer, we can only give a lower bound for $n_{\min}(m)$ (we believe that these bounds are indeed the correct values). We see in this table that, for none of the values of m considered, condition $n \geq 4m \ln^2 m$ is a necessary one and condition (6) is a sufficient one.

m	$\lceil f(m) \rceil$	$\lceil 4m \ln^2 m \rceil$	$n_{\min}(m)$
8	17	139	60
9	20	174	71
10	22	213	83
11	25	253	95
15	36	441	146
20	51	718	215
25	67	1037	290
30	84	1389	≥ 369
35	100	1770	≥ 451
40	118	2178	≥ 537
45	135	2609	≥ 625
50	153	3061	≥ 716

TABLE 2

The GP programs used and tables of $c(n)$ and $c_m(n)$ computed can be obtained from the web page `http://www.ens-lyon.fr/~nbriseba/fuji.html`.

References

[Apo] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, G.T.M. **41**, Springer-Verlag, 1976.

[Ber] W. E. H. Berwick, *An invariant modular equation of the fifth order*, Quart. J. of Math. **47** (1916), 94-103.

[BDGP] K. Barré-Sirieix, G. Diaz, F. Gramain et G. Philibert, *Une preuve de la conjecture de Mahler-Manin*, Invent. Math. **124** (1996), 1-9.

[DN] J. Dixmier and J.-L. Nicolas, *Partitions without small parts*, Number theory, Vol. I (Budapest, 1987), 9-33, Colloq. Math. Soc. János Bolyai, 51, North-Holland, Amsterdam, 1990.

[Erd] A. Erdélyi et al., *Higher transcendental functions*, vol. 2, McGraw-Hill, New York, 1953.

[Est] T. Estermann, *Vereinfachter Beweis eines Satzes von Kloostermann*, Abhandlungen Hamburg. Math. Seminar **7** (1929), 82-98.

[Had] J. Hadamard, *Sur l'expression asymptotique de la fonction de Bessel*, Bull. Soc. Math. France **36** (1908), 77-85.

[Han] H. Hankel, *Die Cylinderfunctionen erster und zweiter Art*, Math. Ann. **1** (1869), 467-501.

[Her] O. Herrmann, *Über die Berechnung der Fourierkoeffizienten der Funktion $j(\tau)$*, J. reine angew. Math. **274/275** (1975), 187-195.

[Hoo] C. Hooley, *Applications of sieve methods to the theory of numbers*, Cambridge Tracts in Mathematics **70**, Cambridge University Press, Cambridge-New York-Melbourne, (1976).

[HW] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fifth edition, The Clarendon Press, Oxford University Press, New York, 1979.

[Ing] A. E. Ingham, *A tauberian theorem for partitions*, Ann. of Math. **42**-5 (1941), 1075-1090.

[Klo] H. D. Kloostermann, *Asymptotische Formeln für die Fourierkoeffizienten ganzer Modulformen*, Abhandlungen Hamburg. Math. Seminar **5** (1927), 337-352.

[Mah] K. Mahler, *On the coefficients of transformation polynomials for the modular function*, Bull. Austral. Math. Soc. **10** (1974), 197-218.

[Nic] J.-L. Nicolas, *Sur les entiers n pour lesquels il y a beaucoup de groupes abéliens d'ordre n*, Ann. Inst. Fourier **28**-4, (1974), 1-16.

[Pari] C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier, *User's Guide to PARI-GP*, available by anonymous `ftp` from the site `ftp://megrez.math.u-bordeaux.fr/pub/pari`.

[Pet] H. Petersson, *Über die Entwicklungskoeffizienten der automorphen Formen*, Acta Math. **58** (1932), 169-215.

[Rad1] H. Rademacher, *A convergent series for the partition function $p(n)$*, Proc. Nat. Acad. Sci. U.S.A. **23** (1937), 78-84.

[Rad2] H. Rademacher, *On the partition function $p(n)$*, Proc. London Math. Soc. **43** (1937), 241-254.

[Rad3] H. Rademacher, *The Fourier coefficients of the modular invariant $j(\tau)$*, Amer. J. Math. **60** (1938), 501-512.

[Rad4] H. Rademacher, *The Fourier series and the functional equation of the absolute modular invariant $j(\tau)$*, Amer. J. Math. **61** (1939), 237-248.

[Rad4'] H. Rademacher, *Correction*, Amer. J. Math. **64** (1942), 456.

[Rad5] H. Rademacher, *Topics in Analytic Number Theory*, Die Grundlehren der mathematischen Wissenschaften, Bd. 169, Springer-Verlag (1973).

[RaZuc] H. Rademacher and H. S. Zuckerman, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. **39** (1938), 433-462.

[Ser] J.-P. Serre, *Cours d'arithmétique*, P.U.F., Paris, 1970 ; *A Course in Arithmetic*, G.T.M. **7**, Springer-Verlag, 1973.

[VW] A. van Wijngaarden, *On the coefficients of the modular invariant* $j(\tau)$, Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. **15** (1953), 389-400.

[Wat] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1995.

[Wei] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204-207.

[Zuc] H. S. Zuckerman, *The computation of the smaller coefficients of* $j(\tau)$, Bull. Am. Math. Soc. **45** (1939), 917-919.