Generalized Tilings with Height Functions

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Abstract

In this paper, we introduce a generalization of a class of tilings which appear in the literature: the tilings over which a height function can be defined (for example, the famous tilings of polyominoes with dominoes). We show that many properties of these tilings can be seen as the consequences of properties of the generalized tilings we introduce. In particular, we show that any tiling problem which can be modelized in our generalized framework has the following properties: the tilability of a region can be constructively decided in polynomial time, the number of connected components in the undirected flip-accessibility graph can be determined, and the directed flip-accessibility graph induces a distributive lattice structure. Finally, we give a few examples of known tiling problems which can be viewed as particular cases of the new notions we introduce.

Keywords: Tilings, Height Functions, Tilability, Distributive Lattices, Random Sampling, Potentials, Flows

Résumé

Nous introduisons dans cet articl une classe général de pavages : les pavages sur lesquel une fonction hauteur peut être définie (par exemple, le célèbre pavage des polyominos par des dominos). Nous montrons que la plus part des propriétés de ces pavages peuvent être vu des concéquences des propriétés des pavages généraux que nous introduisons. En particulier, nous montrons que tout pavage que l'on peut modéliser dans notre cadre généralisé a les propriétés suivantes : la pavabilité d'une région peut être décidé en temps polynômial, le nombre de composantes connexes dans le graphe non orienté d'accessibilité par flip peut être déterminé algorithmiquement, et le graphe orienté d'accessibilité par flip a une structure de réseau distributif. Pour conclure, mous donnons quelques exemples de problèmes de pavages classiques que l'on peut voir comme des cas particuliers de la nouvelle classe que nous avons définie.

Mots-clés: Pavage, fonction hauteur, réseau distributif, flot, potentiel, tirage aléatoire

1 Preliminaries

Given a finite set of elementary shapes, called *tiles*, a *tiling* of a given region is a set of translated tiles such that the union of the tiles covers exactly the region, and such that there is no overlapping between any tiles. See for example Figure 1 for a tiling of a polyomino (set of squares on a two-dimensional grid) with dominoes (1×2 and 2×1 rectangles). Tilings are widely used in physics to modelize natural objects and phenomena. For example, quasicrystals are modelized by Penrose tilings [?] and dimers on a lattice are modelized by domino tilings [?]. Tilings appeared in computer science with the famous undecidability of the question of whether the plane is tilable or not using a given finite set of tiles [?]. Since then, many studies appeared concerning these objects, which are also strongly related to many important combinatorial problems [?].

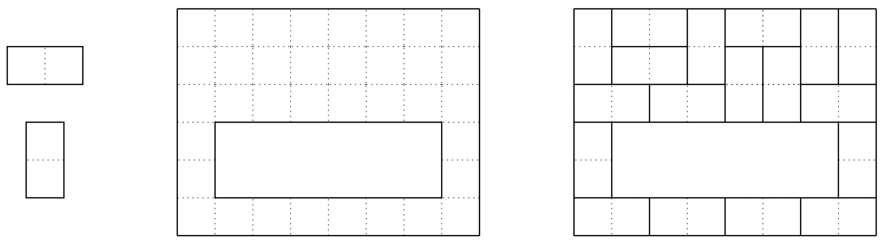

Figure 1: *From left to right: the two possible tiles (called* dominoes*), a polyomino (*i.e. *a set of squares) to tile, and a possible tiling of the polyomino with dominoes.*

A local transformation is often defined over tilings. This transformation, called *flip*, is a local rearrangement of some tiles which makes it possible to obtain a new tiling from a given one. One then defines the (undirected) *flip-accessibility graph* of the tilings of a region R, denoted by $\overline{A_R}$, as follows: the vertices of $\overline{A_R}$ are all the tilings of R, and $\{t, t'\}$ is an (undirected) edge of $\overline{A_R}$ if and only if there is a flip between t and t'. See Figure 2 for an example. The flip notion is a key element for the generation and enumeration of the tilings of a given region, and for many algorithmical questions. For example, we will see in the following that the structure of $\overline{A_R}$ may give a way to sample randomly a tiling of R with the uniform distribution, which is crucial for physicists. This notion is also a key element to study the entropy of the physical objects [?], and to examine some of their properties like frozen areas, weaknesses, and others [?].

On some classes of tilings which can be drawn on a regular grid, it is possible to define a *height function* which associates an integer to any node of the grid (it is called the *height* of the node). For example, one can define such a function over domino tilings as follows. As already noticed, a domino tiling can be drawn on a two dimensional square grid. We can draw the

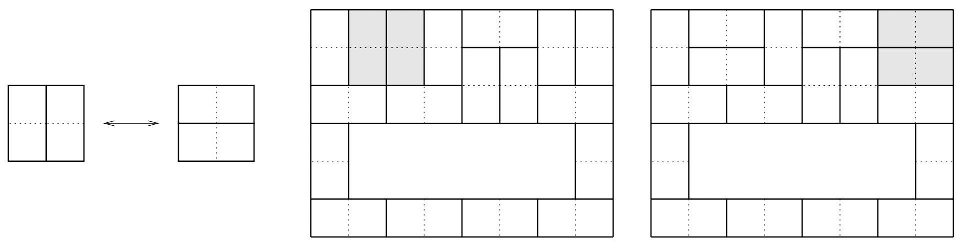

Figure 2: *From left to right: the flip operation over dominoes, and two examples of tilings which can be obtained from the one shown in Figure 1 by one flip. In these tilings, we shaded the tiles which moved during the flip.*

squares of the grid in black and white like on a chessboard. Let us consider a polyomino P and a domino tiling T of P, and let us distinguish a particular node p on the boundary of P, say the one with smaller coordinates. We say that p is of height 0, and that the height of any other node p' of P is computed as follows: initialize a counter to zero, and go from p to p' using a path composed only of edges of dominoes in T, increasing the counter when the square on the right is black and decreasing it when the square is white. The height of p' is the value of the counter when one reaches p'. One can prove that this definition is consistent and can be used as the height function for domino tilings [?]. See Figure 3 for an example.

These height functions make it possible to define A_R, the *directed* flip-accessibility graph of the tilings of a region R: the vertices of A_R are the tilings of R and there is a directed edge (t, t') if and only if t can be transformed into t' by a flip which decreases the sum of the heights of all the nodes. See Figure 3 for an example with domino tilings. The generalized tilings we introduce in this paper are based on these height functions, and most of our results are induced by them.

These notions of height functions are close to classical notions of flows theory in graphs. Let $G = (V, E)$ be a directed graph. A *flow* on G is a map from E into $\mathbb{C}$ (actually, we will only use flows with values in $\mathbb{Z}$). Given two vertices v and v' of G, a *travel* from s to s' is a set of edges of G such that, if one forgets their orientations, then one obtains a path from s to s'. Given a flow C, the *flux* of C on the travel T is

$$F_T(C) \;=\; \sum_{e \in T^+} C(e) \;-\; \sum_{e \in T^-} C(e)$$

where T^+ is the set of oriented edges of T which are traveled in the right direction when one goes from s to s', and T^- is the set of oriented edges traveled in the reverse direction. One can easily notice that the flux is additive by concatenation of travels: if T_1 and T_2 are two travels such that the ending point of T_1 is equal to the starting point of T_2, then $F_{T_1 \cdot T_2}(C) = F_{T_1}(C) + F_{T_2}(C)$. See [?] for more details about flows theory in graphs.

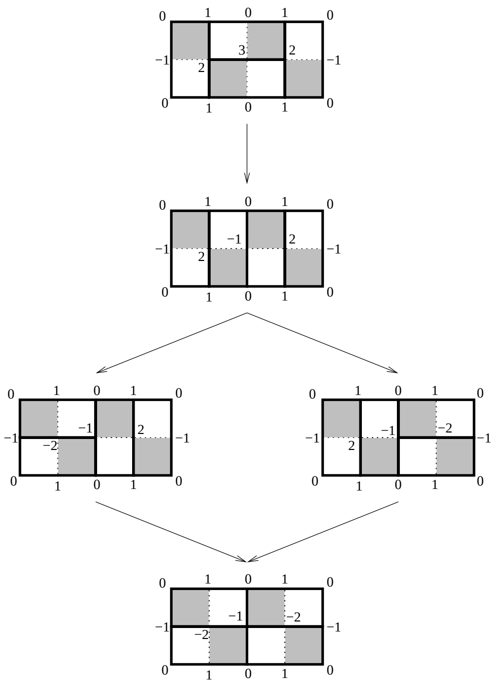

Figure 3: *The directed flip-accessibility graph of the tilings of a polyomino by dominoes. The height of each node of the polyomino is shown for each tiling. The set of all the tilings of this polyomino is ordered by the flip relation directed with respect to the height functions.*

Since there is no circuit in the graph A_R (there exists no nonempty sequence of directed flips which transforms a tiling into itself), it induces an order relation over all the tilings of R: $t \leq t'$ if and only if t' can be obtained from t by a sequence of (directed) flips. In Section 3, we will study A_R under the order theory point of view, and we will meet some special classes of orders, which we introduce now. A lattice is an order L such that any two elements x and y of L have a greatest lower bound, called the *infimum* of x and y and denoted by $x \wedge y$, and a lowest greater bound, called the *supremum* of x and y and denoted by $x \vee y$. The infimum of x and y is nothing but the greatest element among the ones which are lower than both x and y. The supremum is defined dually. A lattice L is *distributive* if for all x, y and z in L, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. For example, it is known that the flip-accessibility graph of the domino tilings of a polyomino without holes is always a distributive lattice [**?**]. Therefore, this is the case of the flip-accessibility graph shown in Figure 3 (notice that the maximal element of the order is at the bottom, and the minimal one is at the top of the diagram since we used the discrete dynamical models convention: the flips go from top to bottom). Lattices (and especially distributive lattices) are strongly structured sets. Their study is an important part of order theory, and many results about them exist. In particular, various codings and algorithms are known about lattices and distributive lattices. For example, there exists a generic algorithm to sample randomly an element of any distributive lattice with the uniform distribution [?]. For more details about orders and lattices, we refer to [?].

Finally, let us introduce a useful notation about graphs. Given a directed graph $G = (V, E)$, the undirected graph $\overline{G} = (\overline{V}, \overline{E})$ is the graph obtained from G by removing the orientations of the edges. In other words, $\overline{V} = V$, and $\overline{E}$ is the set of undirected edges $\{v, v'\}$ such that $(v, v') \in E$. We will also call $\overline{G}$ the *undirected version* of G. Notice that this is consistent with our definitions of A_R and $\overline{A_R}$.

In this paper, we introduce a generalization of tilings on which a height function can be defined, and show how some known results may be understood in this more general context. All along this paper, like we did in the present section, we will use the tilings with dominoes as a reference to illustrate our definitions and results. We used this unique example because it is very famous and simple, and permits to give clear figures. We emphasize however on the fact that our definitions and results are much more general, as explained in the last section of the paper.

2 Generalized tilings.

In this section, we give all the definitions of the generalized notions we introduce, starting from the objects we tile to the notions of tilings, height functions, and flips. The first definitions are very general, therefore we will only consider some classes of the obtained objects, in order to make the more specific notions (mainly height functions and flips) relevant in this context. However, the general objects introduced may be useful in other cases.

Let G be a simple directed graph (G has no multiple edges, no loops, and if (v, v') is an edge then (v', v) can not be an edge). We consider a set Θ of elementary circuits of G, which we will call *cells*. Then, a *polycell* is any set of cells in Θ. Given a polycell P, we call the edges of cells in P the *edges of* P, and their vertices the *vertices of* P. A polycell P is k*-regular* if and only if there exists an integer k such that each cell of P is a circuit of length k. Given a polycell P, the *boundary* of P, denoted by ∂P, is an arbitrarily not empty partial subgraph of P. A polycell P is *full* if ∂P is connected.

Given an edge e of P which is not on the boundary, we call the set of all the cells in P which have e in common a *tile*. A set of edges of $P \backslash \partial P$ such that the associated tiles constitute a partition of the cells of P is called a *tiling* Q. An edge in Q is by definition a *tiling edge*. A polycell P which admits at least a tiling Q is *tilable*. See Figure 4 and Figure 5 for some examples. Notice that if we distinguish exactly one edge of each cell of a polycell P, in such a way that none of them is on the boundary of P, then the distinguished edges can be viewed as the tiling edges of a tiling of P. Indeed, each edge induces a tile (the set of cells which have this edge in common), and each cell is in exactly one tile.

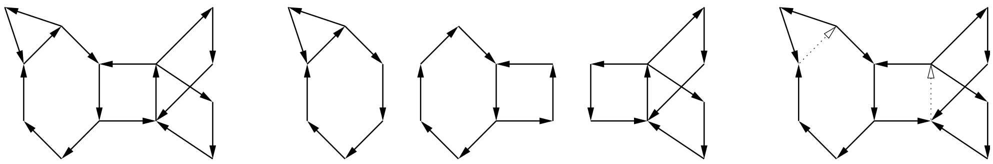

Figure 4: *From left to right: a polycell P (the boundary ∂P being the set of all the vertices of P and Θ being the set of all the elementary circuits), the three tiles of P, and a tiling of P represented by its tiling edges (the dotted edges). This polycell is full, tilable, and is not k-regular for any k. Notice that there are two tiles composed of two cells, and another one composed of three cells. Notice also that the tiling given in this figure is the only possible one.*

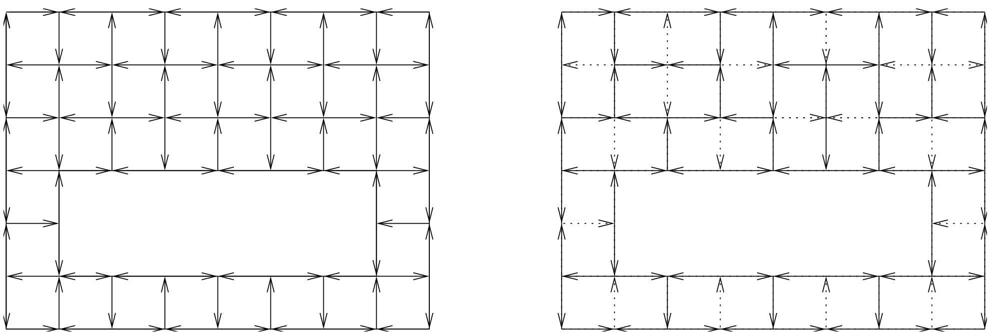

Figure 5: *Left: a 4-regular polycell P (the cells in Θ are the circuits of length 4), the boundary of which is composed of those vertices which belong to at most three edges. Right: a tiling of P represented by its tiling edges (the dotted edges). Notice that this figure is very similar to Figure 1.*

Let P be a k-regular tilable polycell and Q be a tiling of P. We associate to Q a flow C_Q on Θ (seen as a graph):

$$C_Q(e) = \begin{cases} 1-k & \text{if the edge } e \text{ is a tiling edge of } Q \\ 1 & \text{otherwise.} \end{cases}$$

For each cell c, we define T_c as the travel which contains exactly the edges of c (in other words, it consists in turning around c). Notice that the flux of C_Q on the travel T_c is always null: $F_{T_c}(C_Q) = 0$ since each cell contains exactly a tiling edge, valued $1-k$, and $k-1$ other edges, valued 1. Moreover, for each edge $e \in \partial P$, we have $C_Q(e) = 1$ since from the definition e cannot be a tiling edge.

Let us consider a polycell P and a flow C on the edges of P. If for all *closed* travel T (*i.e.* a cycle when one forgets the orientation of each edge) in P we have $F_T(C) = 0$, then the flow C is called a *tension*. A polycell P is *contractible* if the fact that $F_{T_c}(C) = 0$ for all cell c implies that C is a tension. Since the converse is always true, we then have that C is a tension if and only if for all cell c, $F_{T_c}(C) = 0$. Notice that if P is a contractible k-regular polycell and Q is a tiling of P, then the flow C_Q is a tension, since for all cell c, $F_{T_c}(C_Q) = 0$.

Now, if we (arbitrarily) distinguish a vertex ν on the boundary of P, we can associate to the tension C_Q a *potential* φ_Q, defined over the vertices of P:

- $\varphi_Q(\nu) = 0$.
- for all vertices x and y of P, $\varphi_Q(y) - \varphi_Q(x) = F_{T_{x,y}}(C_Q)$ where $T_{x,y}$ is a travel from x to y.

The distinguished vertex is needed else φ_Q would only be defined at almost a constant, but one can choose any vertex on the boundary. Notice that this potential can be viewed as a *height function* associated to Q, and we will see that it indeed plays this role in the following. Therefore, we will call the potential φ_Q the *height function* of Q. See Figure 6 for an example.

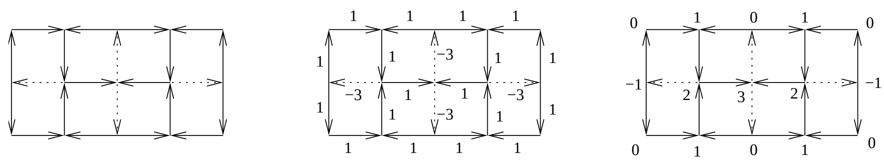

Figure 6: *From left to right: a tiling Q of a polycell (represented by its tiling edges, the dotted ones), the tension C_Q and the height function (or potential) φ_Q it induces. Again, this figure may be compared to Figure 3 (topmost tiling).*

We now have all the main notions we need about tilings of polycells, including height functions, except the notion of flips. In order to introduce it, we need to prove the following:

Theorem 2.1 *Let P be a k-regular contractible polycell. There is a bijection between the tilings of P and the tensions C on P which verify:*

- *for all edge e in ∂P, $C(e) = 1$,*
- *and for all edge e of P, $C(e) \in \{1 - k, 1\}$.*

Preuve. For all tiling Q of P, we have defined above a flow C_Q which verifies the property in the claim, and such that for all cell c, $F_{T_c}(C_Q) =$

0. Since P is contractible, this last point implies that C_Q is a tension. Conversely, let us consider a tension C which satisfies the hypotheses. Since each cell is of length k, and since $C(e) \in \{1-k, 1\}$, the fact that $F_{T_c}(C) = 0$ implies that each cell has exactly one negative edge. These negative edges can be considered as the tiling edges of a tiling of P, which ends the proof. $\square$

Given a k-regular contractible polycell P defined over a graph G, this theorem allows us to make no distinction between a tiling Q and the associated tension C_Q. This makes it possible to define the notion of flip as follows. Suppose there is a vertex x in P which is not on the boundary and such that its height, with respect to the height function of Q, is greater than the height of each of its neighbors in $\overline{G}$. We will call such a vertex a *maximal* vertex. The neighbors of x in $\overline{G}$ have a smaller height than x, therefore the outgoing edges of x in G are tiling edges of Q and the incoming edges of x in G are not. Let us consider function $C_{Q'}$ defined as follows:

$$C_{Q'}(e) = \begin{cases} 1-k & \text{if } e \text{ is an outgoing edge of } x \\ 1 & \text{if } e \text{ is an incoming edge of } x \\ C_Q(e) & \text{else.} \end{cases}$$

Each cell c which contains x contains exactly one outgoing edge of x and one incoming edge of x, therefore we still have $F_{T_c}(C_{Q'}) = 0$. Therefore, $C_{Q'}$ is a tension, and so it induces from Theorem 2.1 a tiling Q'. We say that Q' is obtained from Q by a *flip around* x, or simply by a *flip*. Notice that Q' can also be defined as the tiling associated to the height function obtained from the one of Q by decreasing the height of x by k, and without changing anything else. This corresponds to what happens with classical tilings (see for example [**?**]). See Figure 7 for an example.

We now have all the material needed to define and study A_P, the (directed) flip-accessibility graph of the tilings of P: $A_P = (V_P, E_P)$ is the directed graph where V_P is the set of all the tilings of P and (Q, Q') is an edge in E_P if Q can be transformed into Q' by a flip. We will also study the undirected flip-accessibility graph $\overline{A_P}$. The properties of these graphs are crucial for many questions about tilings, like enumeration, generation and sampling.

3 Structure of the flip-accessibility graph.

Let us consider a k-regular contractible polycell P and a tiling Q of P. Let h be the maximal value among the heights of all the vertices with respect to the height function of Q. If Q is such that all the vertices of height h are on the boundary of P, then it is said to be a *maximal tiling*. For a given P, we denote by Tmax_P the set of the maximal tilings of P. We will see that these tilings play a particular role in the graph A_P. In particular, we will give an

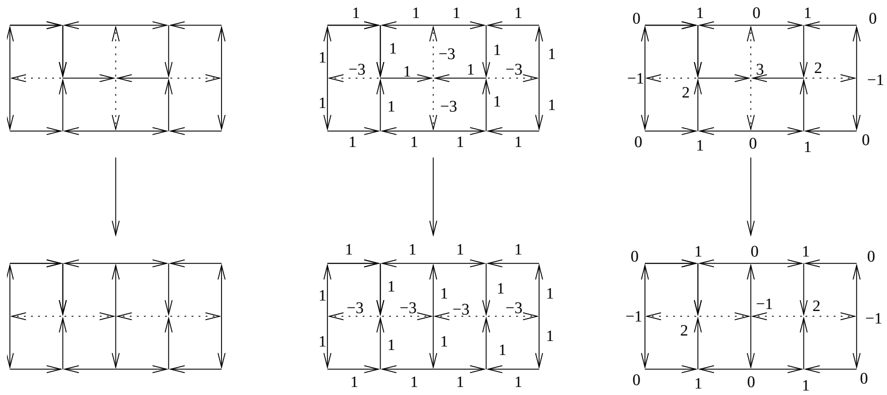

Figure 7: *A flip which transforms a tiling Q of a polycell P into another tiling Q' of P. From left to right, the flip is represented between the tilings, then between the associated tensions, and finally between the associated height functions.*

explicit relation between them and the number of connected components of $\overline{A_P}$. Recall that we defined the maximal vertices of Q as the vertices which have a height greater than the height of each of their neighbors, with respect to the height function of Q (they are *local* maximals).

Lemma 3.1 *Let P be a k-regular tilable contractible polycell (P is not necessarily full). There exists a maximal tiling Q of P.*

Preuve. Let V be the set of vertices of P, and let Q be a tiling of P such that for all tiling Q' of P, we have:

$$\sum_{x \in V} \varphi_Q(x) \leq \sum_{x \in V} \varphi_{Q'}(x).$$

We will prove that Q is a maximal tiling. Suppose there is a maximal vertex x_m which is not on the boundary. Therefore, one can transform Q into Q' by a flip around x_m. Then $\sum_{x \in V} \varphi_{Q'}(x) = \sum_{x \in V} \varphi_Q(x) - k$, which is in contradiction with the hypothesis. □

Lemma 3.2 *For all tiling Q of a k-regular contractible polycell P, there exists a unique tiling in $Tmax_P$ reachable from Q by a sequence of flips.*

Preuve. It is clear that at least one tiling in Tmax_P can be reached from Q by a sequence of flips, since the flip operation decreases the sum of the heights, and since we know from the proof of Lemma 3.1 that a tiling such that this sum is minimal is always in Tmax_P. We now have to prove that the tiling in Tmax_P we obtain does not depend on the order in which we flip around the successive maximal vertices. Since making a flip around a maximal vertex x is nothing but decreasing its height by k and keeping the other values, if we have two maximal vertices x and x' then it is equivalent to make first the flip around x and after the flip around x' or the converse. $\square$

Lemma 3.3 *Let P be a k-regular contractible and tilable polycell. A tiling Q in $Tmax_P$ is totally determined by the values of φ_Q on ∂P.*

Preuve. The proof is by induction over the number of cells in P. Let x be a maximal vertex for φ_Q in ∂P. For all outgoing edges e of x, $C_Q(e) = 1 - k$ (otherwise $\varphi(x)$ would not be maximal). Therefore, these edges can be considered as tiling edges, and determine some tiles of a tiling Q of P. Iterating this process, one finally obtains Q. See Figure 8 for an example. $\square$

Theorem 3.4 *Let P be a k-regular contractible and tilable polycell. The number of connected components in $\overline{A_P}$ is equal to the cardinal of $Tmax_P$.*

Preuve. Immediate from Lemma 3.2. $\square$

This theorem is very general and can explain many results which appeared in previous papers. We obtain for example the following corollary, which generalizes the one saying that any domino tiling of a polyomino can be transformed into any other one by a sequence of flips [**?**].

Corollary 3.5 *Let P be a* full *k-regular contractible and tilable polycell. There is a unique element in $Tmax_P$, which implies that $\overline{A_P}$ is connected.*

Preuve. Since $\overline{\partial P}$ is connected, the heights of the vertices in ∂P are totally determined by the orientation of the edges of ∂P and do not depend on any tiling Q. Therefore, from Lemma 3.3, there is a unique tiling in Tmax_P. $\square$

As a consequence, if P is a full tilable and contractible polycell, the height of a vertex x on the boundary of P is independent of the considered tiling. In the case of full polyominoes, this restriction of φ_Q to the boundary of P is called *height on the boundary* [**?**] and has been introduced in [**?**].

Notice also that the proof of Lemma 3.3 gives an algorithm to build the unique maximal tiling of any k-regular contractible and tilable full polycell P, since the height function on the boundary of P can be computed without knowing any tiling of P. See Algorithm 1 and Figure 8. This algorithm gives in polynomial time a tiling of P if it is tilable. It can also be used to decide whether P is tilable or not. Therefore, it generalizes the result of Thurston [**?**] saying that it can be decided in polynomial time if a given polyomino is tilable with dominoes.

With these results, we obtained much information concerning a central question of tilings: the connectivity of the undirected flip-accessibility graph. We did not only give a condition under which this graph is connected, but we also gave a relation between the number of its connected components and

Algorithm 1 Computation of the maximal tiling of a full k-regular contractible polycell.

Input: A full k-regular contractible polycell P, its boundary ∂P and a distinguished vertex ν on this boundary.

Output: An array *tension* on integers indexed by the edges of P and another one *height* indexed by the vertices of P. The first gives the tension associated to the maximal tiling, and the second gives its height function.

```
begin
    P' ← P;
    height[ν] ← 0;
    for each edge e = (v, v') on the boundary of P' do
        tension[e] ← 1;
    for each vertex v in ∂P' do
        Compute height[v] using the values in tension;
    repeat
        for each vertex v in ∂P' which has the maximal height
        among the heights of all the vertices in ∂P' do
            for each incoming edge e of v do
                tension[e] ← 1 − k;
                for each edge e' in a cell containing e do
                    tension[e'] ← 1;
            for each edge e = (v, v') such that tension[e] has newly
            been computed do
                Compute height[v] and height[v'] using the values in
                tension;
        Remove in P' the cells which contain a negative edge;
        Compute the boundary of P': it is composed of all the
        vertices of P' which have a computed height;
    until P' is empty;
end
```

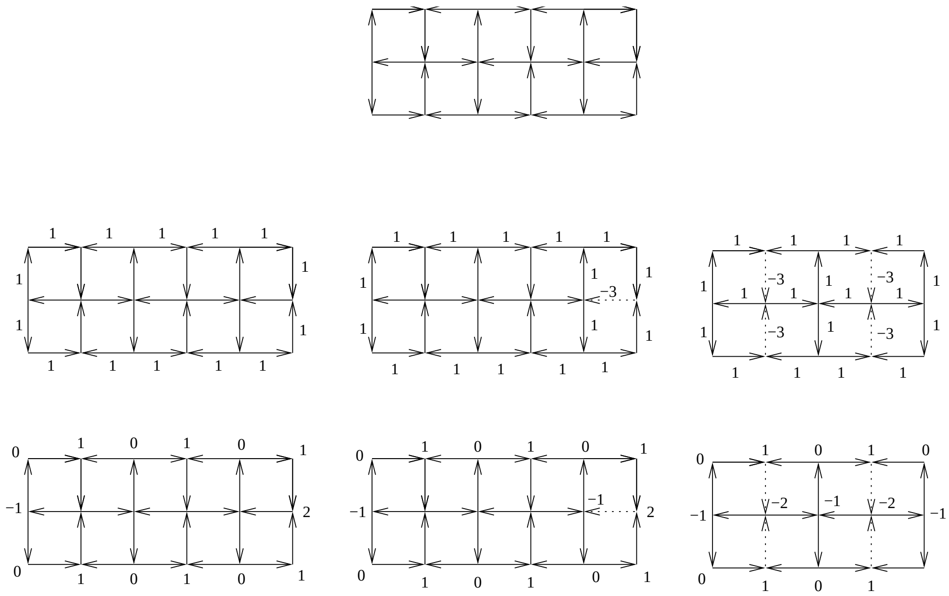

Figure 8: *An example of execution of Algorithm 1. The polycell we want to tile is drawn on the top. Its boundary is composed of the vertices which belong to at most three edges. Below, from left to right, we show the result of each iteration of the algorithm (computation of the tension on the first line, and of the height function on the second line). In this example, the first iteration of the algorithm gives one vertical tile, and the second (and last) iteration gives four horizontal tiles.*

some special tilings. We will now deepen the study of the structure induced by the flip relation by studying the directed flip-accessibility graph, and in particular the partial order it induces over the tilings: $t \leq t'$ if and only if t' can be obtained from t by a sequence of (directed) flips.

Lemma 3.6 *Let Q and Q' be two tilings in the same connected component of A_P of a polycell P. By definition, we put $X_{\neq} := \{x$ such that $\varphi_Q(x) \neq \varphi'_Q(x)\}$, we take $x_m \in X_{\neq}$ such that $\max(\varphi_Q(x_m), \varphi_{Q'}(x_m)) = \max\{\varphi_Q(x), \varphi_{Q'}(x); x \in X_{\neq}\}$, We can suppose that $\varphi_Q(X_{\neq}) < \varphi_{Q'}(X_{\neq})$ - else we reverse Q and Q' -, then the potential φ_1 defined by : $\varphi_1(x) := \varphi_{Q'}(X_{\neq}) - k$, when $x := X_{\neq}$ and $\varphi_1(x) := \varphi_{Q'}(x)$, otherwise, is associated to a tiling of P.*

Let Q and Q' be two tilings in the same connected component of A_P for a given k-regular contractible polycell P. Let us consider x_m such that $\left|\varphi_Q(x_m) - \varphi_{Q'}(x_m)\right|$ is maximal in $\{\left|\varphi_Q(x) - \varphi_{Q'}(x)\right|,\ x$ is a vertex of $P\}$. Then, one can make a flip around x_m from Q or Q'.

Preuve. We can suppose that $\varphi_{Q'}(x_m) < \varphi_Q(x_m)$ (otherwise we exchange Q and Q'). We will show that the height function φ defined by $\varphi(x_m) = \varphi_Q(x_m) - k$ and $\varphi(x) = \varphi_Q(x)$ for all vertex $x \neq x_m$ defines a tiling of P

(which is therefore obtained from Q by a flip around x_m). Let us consider any circuit which contains x_m. Therefore, it contains an incoming edge (x_p, x_m) and an outgoing edge (x_m, x_s) of x_m. We will prove that $\varphi_Q(x_p) = \varphi_Q(x_m) - 1$ and $\varphi_Q(x_s) = \varphi_Q(x_m) - k + 1$, which will prove the claim since it proves that x_m is a maximal vertex.

The couple $(\varphi_Q(x_p), \varphi_Q(x_s))$ can have three values: $(\varphi_Q(x_m)-1, \varphi_Q(x_m)+1)$, $(\varphi_Q(x_m) - 1, \varphi_Q(x_m) - k + 1)$, or $(\varphi_Q(x_m) + k - 1, \varphi_Q(x_m) + 1)$. But, if $\varphi_Q(x_s) = \varphi_Q(x_m)+1$ then $\varphi_{Q'}(x_s) = \varphi(x_m)+1$, and so $\varphi_{Q'}(x_m) = \varphi(x_m)+k$, which is a contradiction. If $\varphi_Q(x_p) = \varphi_Q(x_m) + k - 1$ then $\varphi_{Q'}(x_p) = \varphi_Q(x_m) + k - 1$, and so $\varphi_{Q'}(x_m) > \varphi_Q(x_m)$, which is a contradiction again. Therefore, $(\varphi_Q(x_p), \varphi_Q(x_s))$ must be equal to $(\varphi_Q(x_m) - 1, \varphi_Q(x_m) - k + 1)$ for all circuit which contain x_m, which is what we needed to prove. $\square$

Let us now consider two tilings Q and Q' of a k-regular contractible polycell P. Let us define $\max(\varphi_Q, \varphi_{Q'})$ as the height function such that its value at each vertex is the maximal between the values of φ_Q and $\varphi_{Q'}$ at this vertex. Let us define $\min(\varphi_Q, \varphi_{Q'})$ dually. Then, we have the following result:

Lemma 3.7 *Given two tilings Q and Q' in the same connected component of A_P for a k-regular contractible polycell P, $\max(\varphi_Q, \varphi_{Q'})$ and $\min(\varphi_Q, \varphi_{Q'})$ are the height functions of tilings of P.*

Preuve. We can see that $\max(\varphi_Q, \varphi_{Q'})$ is the height function of a tiling of P by iterating Lemma 3.6: $\sum\limits_x \left|\varphi_Q(x) - \varphi_{Q'}(x)\right|$ can be decreased until we reach $\max(\varphi_Q, \varphi_{Q'})$. The proof for $\min(\varphi_Q, \varphi_{Q'})$ is symmetric. $\square$

Theorem 3.8 *If P is a k-regular contractible polycell, then each connected component of A_P induces a distributive lattice structure over the tilings of P.*

Preuve. Given two tilings Q and Q' in the same connected component of A_P, let us define the following binary operations: $\varphi_Q \wedge \varphi_{Q'} = \min(\varphi_Q, \varphi_{Q'})$ and $\varphi_Q \vee \varphi_{Q'} = \max(\varphi_Q, \varphi_{Q'})$. It is clear from the previous results that this defines the infimum and supremum of Q and Q'. To show that the obtained lattice is *distributive*, it suffices now to verify that these relations are distributive together. $\square$

As already discussed, this last theorem gives much information on the structure of the flip-accessibility graphs of tilings of polycells. It also gives the possibility to use in the context of tilings the numerous results known about distributive lattices, in particular the generic random sampling algorithm described in [?].

To finish this section, we give another proof of Theorem 3.8 using only discrete dynamical models notions. This proof is very simple and has the advantage of putting two combinatorial object in a relation which may help understanding them. However, the reader not interested in discrete dynamical models may skip the end of this section.

An Edge Firing Game (EFG) is defined by a connected undirected graph G with a distinguished vertex ν, and an orientation O of G. In other words, $\overline{O} = G$. We then consider the set of obtainable orientations when we iterate the following rule: if a vertex $v \neq \nu$ only has incoming edges (it is a *sink*) then one can reverse all these edges. This set of orientations is ordered by the reflexive and transitive closure of the evolution rule, and it is proved in [?] that it is a distributive lattice. We will show that the set of tilings of any k-regular contractible polycell P is isomorphic to configuration space of an EFG, which implies Theorem 3.8.

Let us consider a k-regular contractible polycell P defined over a graph G. Let G' be the sub-graph of G which contains exactly the vertices and edges in P plus a new vertex ν and a edge (v, ν) for all v in ∂P. This vertex will be the distinguished vertex of our EFG. Let us now consider the height function φ_Q of a tiling Q of P, and let us define the orientation $\pi(Q)$ of $\overline{G'}$ as follows: the edges involving ν are directed towards ν, and each other undirected edge $\{v, v'\}$ in $\overline{G'}$ is directed from v to v' in $\pi(Q)$ if $\varphi_Q(v') > \varphi_Q(v)$. Then, the maximal vertices of Q are exactly the ones which have only incoming edges in $\pi(Q)$, and applying the EFG rule to a vertex of $\pi(Q)$ is clearly equivalent to making a flip around this vertex in Q. Moreover, one can never apply the EFG rule to a vertex in ∂P, since it always has an outgoing edge to ν, which can never be reversed. Finally, the configuration space of the EFG is isomorphic to the connected component of A_P which contains Q, which proves Theorem 3.8 again. An example is given in Figure 9.

4 Some applications.

In this section, we present some examples which appear in the literature, and we show how these tiling problems can be seen as special cases of k-regular contractible polycells tilings. We therefore obtain as corollaries some known results about these problems, as well as some new results.

4.1 Polycell drawn on the plane or the sphere.

Let us consider a set of vertices V and a set Θ of elementary (undirected) cycles of length k, with vertices in V, such that any couple of cycles in Θ have at most one edge in common. Now let us consider the undirected graph $G = (V, E)$ such that e is an edge of G if and only if it is an edge of a cycle in Θ. Moreover, let us restrict ourselves to the case where G is a planar graph which can be drawn in such a way that no cycle of Θ is drawn inside another one. G is 2-dual-colorable if one can color in black and white each bounded face in such a way that two faces which have an edge in common have different colors. See for example Figure 10.

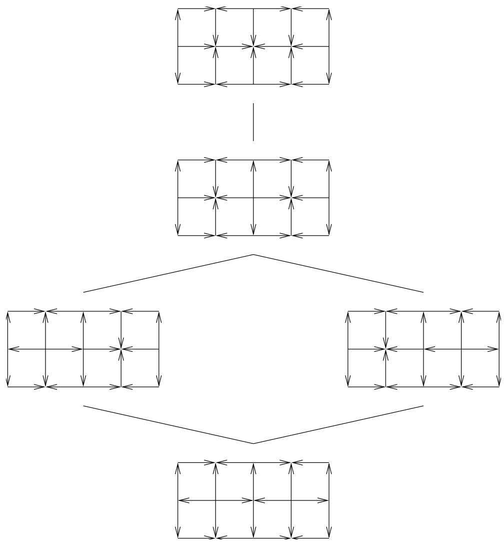

Figure 9: *The configuration space of the EFG obtained from Figure 3 (the distinguished vertex ν is not represented: there is an additional outgoing edge from each vertex on the boundary to ν). The two orders are isomorphic.*

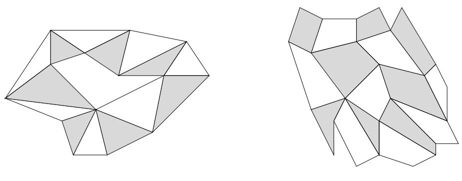

Figure 10: *Two examples of graphs which satisfy all the properties given in the text. The leftmost is composed of cycles of length* 3 *and has a hole. The rightmost one is composed of cycles of length* 4.

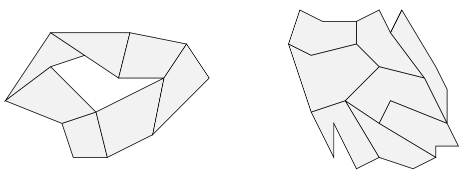

Figure 11: *A tiling of each of the objects shown in Figure 10, obtained using the polycells formalism.*

The fact that G has the properties above, including being 2-dual-colorable, makes it possible to encode tilings with bifaces (the tiles are two adjacent faces) as tilings of polycells. This includes tilings with dominoes, and tilings with calissons. Following Thurston [Thu90], let us define an oriented version of G as follows: the edges which constitute the white cycles boundaries are directed to travel the cycle in the clockwise orientation, and the edges which constitute the black cycles boundaries are directed counterclockwise. If for all *closed* travel (its origin and its extremity coincide) on the boundary of polycell P we have $F_T(C) = 0$ where C is a flow such that $C(e) = 1$ for all $e \in \partial P$, then we say that P has a *balanced boundary*. One can verify that a polycell with a balanced boundary defined in this way is always contractible. Therefore, our results can be applied, which generalizes some results of Chaboud [**?**] and Thurston [**?**].

4.2 Rhombus tiling in higher dimension.

Let us consider the canonical basis $\{e_1, \ldots, e_d\}$ of the d-dimensional affine space $\mathbb{R}^d$, and let us define $e_{d+1} = \sum_{i=1}^{d} e_i$. For all α between 1 and $d+1$, let us define the zonotope $Z^{\alpha}_{d,d}$ as the following set of points:

$$Z^{\alpha}_{d,d} \;=\; \{x \in \mathbb{R}^d \text{ such that } x = \sum_{i=1, i\neq\alpha}^{d+1} \lambda_i e_i, \text{ with } -1 \leq \lambda_i \leq 1\}.$$

In other words, the $Z^{\alpha}_{d,d}$ is the zonotope defined by all the vectors e_i except the α-th. We are interested in the tilability of a given solid S when the set of allowed tiles is $\{Z^{\alpha}_{d,d},\ 1 \leq \alpha \leq d+1\}$. These tilings are called *codimension one rhombus tilings*, and they are very important as a physical model of quasicristals [**?**]. If $d = 2$, they are nothing but the tilings of regions of the plane with three parallelograms which tile an hexagon, which have been widely studied. See Figure 12 for an example in dimension 2, and Figure 13 for an example in dimension 3.

In order to encode this problem by a problem over polycells, let us consider the directed graph G with vertices in $\mathbb{Z}^d$ and such that $e = (x, y)$ is an edge if and only if $y = x + e_j$ for an integer j between 1 and d or $y = x - e_{d+1}$. We will call *diagonal edges* the edges which correspond to the second case. This graph can be viewed as a d-dimensional directed grid to which we add a diagonal edge in the reverse direction, at each point of the grid. An example in dimension 3 is given in Figure 14.

Each edge is in a one-to-one correspondence with a copy of a $Z^{\alpha}_{d,d}$ translated by an integer vector, namely the one of which it is the diagonal edge. The set Θ of the cells we will consider is the set of all the circuits of length $d+1$ which contain exactly one diagonal edge. Therefore, each edge belongs to a $d!$ cells, and so the tiles will be themselves composed of $d!$ cells. See Figure 14 for an example in dimension 3. Given a polycell P defined over Θ, we define ∂P as the set of the edges of P which do not belong to $d!$ circuits of P.

First notice that a full polycell defined over G is always contractible. Therefore, our previous results can be applied, which generalizes some results presented in [**?**] and [**?**, **?**]. We also generalize some results about the 2-dimensional case, which has been widely studied.

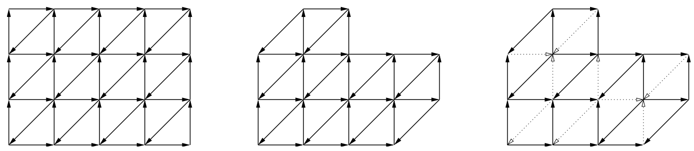

Figure 12: *If one forgets the orientations and removes the dotted edges, then the rightmost object is a classical codimension one rhombus tiling of a part of the plane ($d = 2$). From the polycells point of view, the leftmost object represents the underlying graph G, the middle object represents a polucell P (the boundary of which is the set of the edges which belong to only one cell), and the rightmost object represents a tiling of P (the dottes edges are the tiling edges).*

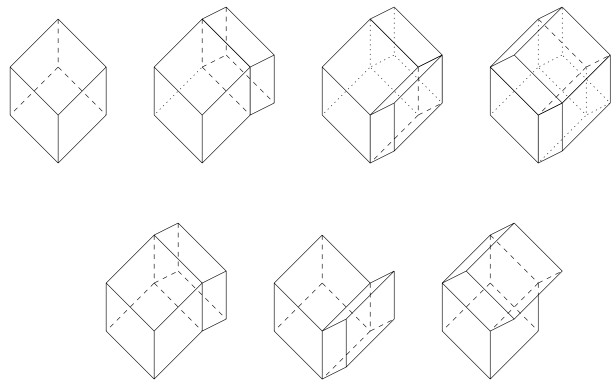

Figure 13: *A codimension one rhombus tiling with $d = 3$ (first line, rightmost object). It is composed of four different three dimensional tiles, and the first line shows how it can be constructing by adding successive tiles. The second line shows the position of each tile with respect to the cube.*

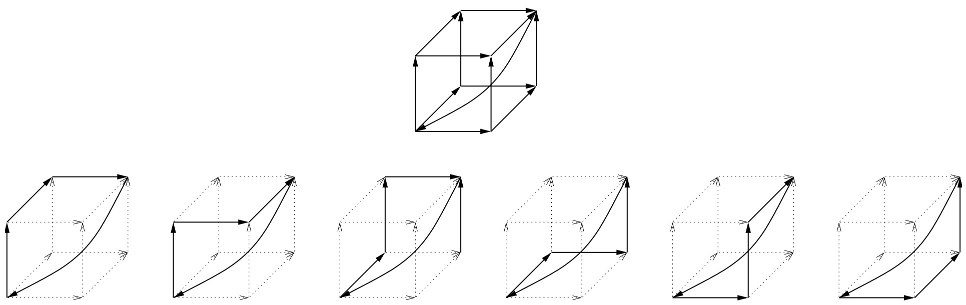

Figure 14: *Top: the 3-dimensional grid is obtained by a concatenation of cubes with reverse diagonal edges, like this one. Bottom: the cells in Θ. Each tile is composed of six such cells, since each edge belongs to exactly six cells.*

5 Conclusion and Perspectives.

In conclusion, we gave in this paper a generalized framework to study some tiling problems over which a height function can be defined. This includes the famous tilings of polyominoes with dominoes, as well as various other classes, like codimension one rhombus tilings, tilings with holes, tilings on torus, on spheres, three-dimensional tilings, and others we did not detail here. We gave some results on our generalized tilings which made it possible to obtain a large set of known results as corollaries, as well as to obtain new results on tiling problems which appear in the scientific literature. Many other problems may exist which can be modelized in the general framework we have introduced, and we hope that this paper will help understanding them.

Many tiling problems, however, do not lead to the definition of any height function. The key element to make such a function exist is the presence of a strong underlying structure (the k-regularity of the polycell, for example). Some important tiling problems (for example tilings of zonotopes) do not have this property, and so we can not apply our results in this context. Some of these problems do not have the strong properties we obtained on the tilings of k-regular contractible polycells, but may be included in our framework, since our basic definitions of polycells and tilings being very general. This would lead to general results on more complex polycells, for example polycells which are not k-regular, or with cells which have more than one edge in common.

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