On connectedness and dimension of a Besicovitch space over SZ.

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Abstract

We prove that the topological space $S^{\mathbb{Z}}/_{\dot{=}}$, proposed in [5] is path-connected and has infinite dimension. The latter property makes of this space a more natural setting for cellular automata when they are considered as a solutions of difference equations. In fact, difference equations are defined on an infinite dimensional space. On the contrary the classical product topology on $S^{\mathbb{Z}}$ is zero-dimensional. Moreover we present a transitive dynamical system on $S^{\mathbb{Z}}/_{\dot{=}}$, whose existence was given as an open problem in [5]. Another interesting property that we prove is that $S^{\mathbb{Z}}/_{\dot{=}}$ is not separable. This property partially explain the "difficulty" of finding transitive systems on such a space. We also prove that some properties of Toeplitz sequences on $S^{\mathbb{Z}}/_{\dot{=}}$ and as a byproduct we obtain a "weak fixed point" theorem for continuous mappings on $S^{\mathbb{Z}}/_{\dot{=}}$. Finally we sketch an interesting connection between infinite Sturmian words and $S^{\mathbb{Z}}/_{\dot{=}}$.

Keywords: Shift-invariant metrics, dimension, discrete dynamical systems

Résumé

On prouve que l'espace topologique $S^{\mathbb{Z}}/_{\dot{=}}$, proposé dans [5], est connexe et que sa dimension topologique est infinie. Cette dernière proprieté rend cette espace plus naturel pour l'étude des automates cellulaires, par exemple quand ils sont considerés comme solutions des équations aux différences. En effet, l'espace des équations aux différences a une dimension infinie. Alorque la topologie produit classique sur $S^{\mathbb{Z}}/_{\dot{=}}$ donne un espace de dimension zéro. De plus, nous exhibons un système dynamique topologiquement transitif sur $S^{\mathbb{Z}}/_{\dot{=}}$; l'existence d'un tel système a été donnée comme problème ouvert dans [5]. Une autre propriété intéressante de $S^{\mathbb{Z}}/_{\dot{=}}$ est la non-separabilité, qui explique en part "la difficulté" de trouver des systemes transitifs sur cet espace. On prouve aussi quelques propriétés des suites de Toeplitz sur $S^{\mathbb{Z}}/_{\dot{=}}$. Comme corollaire, on obtient un theoreme faible de point fixe. Nous montrons aussi quelques relations entre $S^{\mathbb{Z}}/_{\dot{=}}$ et l'ensemble des mots Sturmiens infinis.

Mots-clés: métriques invariantes par translation, dimension, systèmes dynamiques discrets

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Abstract

We prove that the topological space $S^{\mathbb{Z}}/_{\doteq}$, proposed in [5] is path-connected and has infinite dimension. The latter property makes of this space a more natural setting for cellular automata when they are considered as solutions of difference equations. In fact the space of difference equations is infinite dimensional. On the contrary the classical product topology on $S^{\mathbb{Z}}$ is zero-dimensional. Moreover we present a transitive dynamical system on $S^{\mathbb{Z}}/_{\doteq}$, whose existence was given as an open problem in [5]. Another interesting property that we prove is that $S^{\mathbb{Z}}/_{\doteq}$ is not separable. The proofs of many of the results are made using properties of Toeplitz configurations. In particular, we prove that every two Toeplitz configurations are in different classes of $\doteq$ and that this set does not coincide with $S^{\mathbb{Z}}/_{\doteq}$. As a byproduct we obtain a "weak fixed point" theorem for continuous mappings on $S^{\mathbb{Z}}/_{\doteq}$. Finally we sketch an interesting connection between infinite Sturmian words and $S^{\mathbb{Z}}/_{\doteq}$.

1 Introduction

Shift-invariant metrics have been proposed in [5] as a possible solution to the problem of distinguishing strong chaotic behavior of some cellular automata (CA) from the chaotic behavior of systems topologically conjugated with the shift map. In [5] the authors proposed the following approach.

For all configurations $x, y \in S^{\mathbb{Z}}$ and $\forall k \in \mathbb{N}$ let

$$\#^{\neq}_{-k,k}(x,y) = |\{i \in \{-k, -k+1, \ldots, 0, 1, \ldots, k\} \text{ s.t. } x_i \neq y_i\}| \quad .$$

Roughly speaking, the quantity $\#^{\neq}_{-k,k}(x,y)$ is the Hamming distance between the two segments of length $2k+1$ (and centered in zero) of the configurations x and y.

Let us consider the mapping $d: S^{\mathbb{Z}} \times S^{\mathbb{Z}} \to \mathbb{R}^+$ defined as follows

$$\forall x, y \in S^{\mathbb{Z}},\ d(x,y) = \limsup_{k \to +\infty} \left\{ \frac{\#^{\neq}_{-k,k}(x,y)}{2k+1} \right\} \quad .$$

In other words, $d(x,y)$ quantifies the percentage of different cells in the configurations x and y.

Example 1 *Let us consider the configuration b defined as follows*

$$b = \ldots 0^{q^4} 1^{q^3} 0^{q^2} 1^{q^1} 0^{q^0} \mid 0^{q^0} 1^{q^1} 0^{q^2} 1^{q^3} 0^{q^4} \ldots$$

where $q \in \mathbb{N}$ and $q > 1$. The symbol '$|$' indicates the position of the cell of index zero. It is not difficult to prove that

$$\limsup_{k \to \infty} \frac{\#_{-k,k}(b, \underline{0})}{2k+1} \geq \frac{2q}{q+1} \quad \text{and} \quad \liminf_{k \to \infty} \frac{\#_{-k,k}(b, \underline{0})}{2k+1} \leq \frac{2}{q+1}.$$

We remark that b does not contain any arithmetic progression of cells with identical values.

Unfortunately, d is only a pseudo-metric. In literature d is known as Besicovitch pseudo-metric and it is usually defined on $\mathbb{R}$ or $\mathbb{C}$ (see for example [3, 2]). If we consider the relation $x \doteq y$ if and only if $d(x,y) = 0$, then d restricted to $S^{\mathbb{Z}}/_{\doteq}$ is a metric. When not confusion is likely, let us denote by d both the pseudo-metric on $S^{\mathbb{Z}}$ and the metric on $S^{\mathbb{Z}}/_{\doteq}$. We will also denote the equivalence class of $x \in S^{\mathbb{Z}}$ w.r.t. $\doteq$ by $x_{\doteq}$ or simply x.

If we consider $S^{\mathbb{Z}}/_{\doteq}$ endowed with the topology induced by d then one can prove the following.

Theorem 1 **([5, 6])** *The space $S^{\mathbb{Z}}/_{\dot{=}}$ is not discrete, not compact, but perfect.*

In literature one may find an analogous pseudo-metric, called *Weyl pseudo-metric* [10, 12], which is defined as follows:

$$\forall x, y \in S^{\mathbb{Z}}, \quad d_W(x, y) = \limsup_{L \to \infty} \left\{ \sup_{k \in \mathbb{Z}} \frac{\#^{\neq}_{k,k+L-1}(x, y)}{L} \right\}.$$

We prefer to use the Besicovitcz pseudo-metric other than the Weyl's one because of many reasons. First, the space $\langle S^{\mathbb{Z}}, d \rangle$ is complete (see [14]), while $\langle S^{\mathbb{Z}}, d_W \rangle$ is not (see [10] for a proof). Some results of the present paper, for example Theorem 2, have already been proved for the Weyl pseudometric in [10]. Second, it has been proved that the space $\langle S^{\mathbb{Z}}/_{\dot{=}}, d \rangle$ is suitable for the study of cellular automata and that at least the concept of sensitivity to initial conditions fits some intuitive requirements.

There are many open questions on this subject. For example, we were not aware of the existence of any transitive system on such a space. This fact is also due to a more deep topological problem. We had not yet proved or disproved the existence of a dense set in $S^{\mathbb{Z}}/_{\dot{=}}$. Remark 3 answers to this last question.

We think that a deep understanding of the topological properties of $S^{\mathbb{Z}}/_{\dot{=}}$ may shed new light on the chaotic behavior of cellular automata. Along this line of thought we prove the following.

Theorem 2 *The space $S^{\mathbb{Z}}/_{\dot{=}}$ is simply connected.*

As a trivial consequence of Theorem 2 we have the following.

Corollary 1 *The space $S^{\mathbb{Z}}/_{\dot{=}}$ is perfect.*

Theorem 3 gives a quite interesting justification for using the new topology on $S^{\mathbb{Z}}$. Cellular automata can be thought as a solution of a difference equation. The space of difference equations is infinite dimensional. It is for this reason that we think that the new topology is more "natural" then the classical product topology, which is well known to give a zero-dimensional space.

Theorem 3 *The space $S^{\mathbb{Z}}_{\dot{=}}$ has infinite dimension.*

Theorem 4 *The space $S^{\mathbb{Z}}/_{\dot{=}}$ is not separable.*

Let us recall that a dynamical system $\langle X, f\rangle$ is *strongly transitive* (resp. *transitive*) if for all non-void open sets A, it holds $\cup_{n\in\mathbb{N}} f^n(A) = X$ (resp. $\mathrm{cl}(\cup_{n\in\mathbb{N}} f^n(A)) = X$, where $\mathrm{cl}(.)$ is the topological closure operator).
For dynamical systems in compact spaces the property of having a dense orbit is equivalent to topological transitivity, and it is often easier to prove than transitivity [9]. From Theorem 4 one deduces that no dynamical system on $S^{\mathbb{Z}}/_{\doteq}$ can have a dense orbit. This fact explain in part the "difficulty" in finding transitive systems on $S^{\mathbb{Z}}/_{\doteq}$. In Example 2 we show an example of such a system. This settles, in part, a question in [5].

Example 2 *Let us consider the following mapping* $f\colon S^{\mathbb{Z}}/_{\doteq} \to S^{\mathbb{Z}}/_{\doteq}$ *defined as follows* $\forall x \in S^{\mathbb{Z}}/_{\doteq}$

$$f(\ldots x_{-4}, x_{-3}, x_{-2}, x_{-1}|x_0, x_1, x_2, x_3, x_4 \ldots) = \ldots x_{-4}, x_{-2}|x_0, x_2, x_4 \ldots$$

Let us prove that f is continuous. Remark that

$$\forall k \in \mathbb{N}, \quad \frac{\#^{\neq}_{-k,k}(f(x), f(y))}{2k+1} \leq 2 \cdot \frac{\#^{\neq}_{-2k,2k}(x, y)}{4k+1} \tag{1}$$

and therefore $d(f(x), f(y)) \leq 2 \cdot d(x, y)$. This implies that f is continuous. From (1) we have that if $x \doteq y$ then $f(x) \doteq f(y)$.

Recall that a dynamical system $\langle X, g\rangle$ is strongly transitive *if and only if $\forall x, y \in X$, $\forall \epsilon > 0$ there exists $z \in B_\epsilon(x)$ and $n \in \mathbb{N}$ such that $g^n(z) = y$. A strongly transitive system is also transitive (but in general the converse is not true).*
For any $x, y \in S^{\mathbb{Z}}/_{\doteq}$, let $\epsilon > 0$ and $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \epsilon$. We build a configuration $z \in B_\epsilon(x)$ as follows.

$$\forall i \in \mathbb{Z}, z_i = \begin{cases} y_j & \text{if } i = j \cdot 2^k \text{ for some } j \in \mathbb{Z} \\ x_i & \text{otherwise.} \end{cases}$$

It is not difficult to see that $f^k(z) = y$. Hence, $\langle S^{\mathbb{Z}}/_{\doteq}, f\rangle$ is strongly transitive.
Consider the same system when $S^{\mathbb{Z}}$ is equipped with the classical product topology. For simplicity, let $S = \{0, 1\}$. Let $C_0 = \left\{c \in \{0,1\}^{\mathbb{Z}} \mid c_0 = 0\right\}$, C_1 is defined similarly. C_0 and C_1 are clopen set such that $\{0,1\}^{\mathbb{Z}} = C_0 \cup C_1$. Moreover it is easy to see that $f(C_i) \subseteq C_i$, $i = 0, 1$. From these facts we deduce that the system is not transitive; in fact, for all non-void open sets $U \subseteq C_0$, $cl(f(U)) \subseteq C_0 \neq \{0,1\}^{\mathbb{Z}}$.

A configuration c is *spatial periodic* if and only if

$$\exists p \in \mathbb{N}\, \text{such that}\ \forall m, n \in \mathbb{N},\ (m \equiv n \bmod p) \Rightarrow c(m) = c(n).$$

It is well known that the set of spatial periodic configurations is dense in $S^{\mathbb{Z}}$ when $S^{\mathbb{Z}}$ is given the product topology. In [6], it has been proved that the same set is not dense when $S^{\mathbb{Z}}$ is given the topology that we have presented above. Moreover every two spatial periodic configurations are in different equivalence classes. Here we try to extend these results to Toeplitz configurations.
We say that a configuration c is *Toeplitz* if and only if

$$\forall n \in \mathbb{N},\ \exists p \in \mathbb{N}^{+}, \text{such that}\ \forall m \in \mathbb{N}, (m \equiv n \bmod p) \Rightarrow (c(m) = c(n)).$$

Trivially we note that a spatial periodic configuration is Toeplitz, but in general the converse is not true.

Remark 1 *From Theorem 4 we can give an alternative proof of the fact that the set of spatial periodic configurations $\mathcal{SP}$ is not dense in $S^{\mathbb{Z}}/_{\doteq}$ (since $\mathcal{SP}$ is a countable set).*

Proposition 1 *Every two Toeplitz configurations are in distinct classes of $\doteq$.*

As a byproduct of the previous proposition we prove that, if a dynamical system preserves a particular subset A of "Toeplitz sequences" (a detailed definition of A is given in Section 2) then it has at least a fixed point. This fact has some analogies with the set continuous functions on the interval $[0, 1]$ (Fixed point Theorem).

Theorem 5 *Every dynamical system on $S^{\mathbb{Z}}/_{\doteq}$ which preserves the set A has at least a fixed point. Moreover this fixed point contains a Toeplitz configuration.*

Proposition 2 *There is an equivalence class of $\doteq$ (i.e. a point of $S^{\mathbb{Z}}/_{\doteq}$) that does not contain any Toeplitz configuration.*

We think that Proposition 2 can be furtherly extended to the following.

Conjecture 1 *Toeplitz configurations are not dense in $S^{\mathbb{Z}}/_{\doteq}$.*

2 Proofs of results

In order to prove the results we are going to use an alternative definition of Toeplitz sequences. We take the definition as presented in [1].

Let us consider a finite alphabet S and a special symbol '.', called *hole*, not belonging to S. Let $b \in (S \cup \{.\})^{\mathbb{N}}$. For any sequence $c \in (S \cup \{.\})^{\mathbb{N}}$, let $t_0, t_1, \ldots, t_j, \ldots$ be a strictly increasing sequence of integers such that $c(t_i) =$ '.'. Let us define a transformation T_b: $(S \cup \{.\})^{\mathbb{N}} \to (S \cup \{.\})^{\mathbb{N}}$ in the following way:

$$T_b(c_i) = \begin{cases} c_i & \text{if } c_i \neq \text{`.'} \\ b(j) & \text{if } i = t_j \text{ for some } j \end{cases}$$

Now if we take a sequence $\{b_j\}_{j\in\mathbb{N}} \subset (S \cup \{.\})^{\mathbb{N}}$ of spatial periodic sequences and we denote T_{b_j} by T_j, it is not difficult to prove that the limit $\lim_{j\to\infty} T_j(c)$ always exists. Moreover if $c(0) \neq$ '.' then the limit does not contain any hole.

Notation 1 *We use the under-bar notation for spatial periodic configurations. For example* $\underline{b_0, \ldots, b_k}$ *denotes a spatial periodic configuration* c *of period* $k+1$ *such that* $\forall i \in \mathbb{Z},\ c(i) = b(i \bmod k+1)$.

For any $x \in [0,1]$ let us consider its binary expansion: $\sum_{i=1}^{\infty} \frac{x_i}{2^i}$. We build the Toeplitz sequence *associated with* x in this way:

$x_1 = 0$	:	0.0.0.0.0.0.0.0 ...
$x_1 = 1$	:	.1.1.1.1.1.1.1.1 ...
$x_1x_2 = 00$	:	000.000.000.000.000 ...
$x_1x_2 = 01$	:	0.010.010.010.010 ...
$x_1x_2 = 10$	:	01.101.101.101 ...
$x_1x_2 = 11$	:	.111.111.111.111 ...

and so on. Note that the construction process works for all reals such that $x \in [0,1)$. The point $x = 1$ is associated with the configuration $\underline{1}$.

We remark that the above construction always admits a limit and the limit has no holes. In fact, consider $c_0 = \underline{0.}$ and $c_1 = \underline{.1}$. Let $b_j = c_0$ if j is even, c_1 otherwise. By the above remarks we have that for any binary expansion x, $\lim_{j\to\infty} T_j(x)$ exists. Note that if the binary expansion contains a 0, say at position k, then $T_k(x)_0 \neq$ '.' and so the limit does not contain any hole.

For the sake of simplicity we will prove the results for $\{0,1\}^{\mathbb{N}}$. The generalizations to $S^{\mathbb{Z}}$ is straightforward.

Proof of Theorem 2. Let $f : [0,1] \to \{0,1\}^{\mathbb{N}} /_{\doteq}$ be the mapping that for any $x \in [0,1]$ it gives the Toeplitz sequence associated with x. Let us define f by induction on finite strings:

$$\forall x \in [0,1], \quad f(x_0, \ldots, x_{n+1}) = \begin{cases} T_{\underline{0}}(f(x_0, \ldots, x_n) & \text{if } x_{n+1} = 0 \\ T_{\underline{1}}(f(x_0, \ldots, x_n) & \text{if } x_{n+1} = 1. \end{cases}$$

finally $f(x) = \lim_{n\to\infty} f(x_0, \ldots, x_n)$. We underline that f is a mapping. In fact even if a point $x \in [0,1]$ can have more than one binary expansion, it has a unique associated Toeplitz sequence. We prove that f is continuous by using the classical ϵ-δ definition. For any $\epsilon > 0$ and for any $x \in [0,1]$, fix $m \in \mathbb{N}$ such that $2m > \lfloor \frac{1}{\epsilon} \rfloor$. Let $\delta = \frac{1}{2^m}$. The binary expansion of $y \in [0,1]$ such that $|x - y| \leq \delta$ has a certain number of digits in common with the one of x, more precisely $n \geq -\lfloor \log_2(\delta) \rfloor$. Therefore the Toeplitz sequences associated with x and y differ only in one cell per block of $2n$ consecutive cells. Hence $d(f(x), f(y)) \leq \frac{1}{2n} \leq \frac{1}{2m} < \epsilon$. Let us note that $f(0) = \underline{0}$ and $f(1) = \underline{1}$, therefore f is a walk from $\underline{0}$ to $\underline{1}$. For any $w, z \in \{0,1\}^{\mathbb{N}} /_{\doteq}$ let $g_w : [0,1] \to \{0,1\}^{\mathbb{N}} /_{\doteq}$ be such that $\forall x \in [0,1]$ $g_w(x)_i = f(x)_i \cdot w_i$. As before, for any $x \in \{0,1\}^{\mathbb{N}}$ and for any $\epsilon > 0$, let $2m > \lfloor \frac{1}{\epsilon} \rfloor$, and assume $\delta = \frac{1}{2^m}$. It is not difficult to see that g_w is continuous. In fact, for any $y \in \{0,1\}^{\mathbb{N}}$ such that $|x - y| \leq \delta$ we have that there exists $m \in \mathbb{N}$ such that $m \geq -\lfloor \log_2 \delta \rfloor$ (continuity of f). This means that $g_w(x)$ and $g_w(y)$ will be possibly different only in one cell per blocks of $2m$ cells. Note that this difference is maintained if and only if the corresponding value in w is 1. Therefore $d(g_w(x), g_w(y)) \leq \frac{1}{2m} < \epsilon$. It is a matter of thought to verify that $g_w(0) = 0$ and $g_w(1) = w$ and hence g_w is a walk from $\underline{0}$ to w. In the same way we can define a walk g_z from $\underline{0}$ to z. The mapping $g_{w,z} : [0,1] \to \{0,1\}^{\mathbb{N}} /_{\doteq}$ defined as follows

$$\forall x \in [0,1] \quad g_{w,z}(x) = \begin{cases} g_w(2x) & 0 \leq x \leq \frac{1}{2} \\ g_z(2x-1) & \frac{1}{2} < x \leq 1 \end{cases}$$

is a walk from z to w. ⋄

Remark 2 *The map f defined in the proof of Theorem 4 has other interesting properties. Let us define the map $h : \{0,1\}^{\mathbb{N}} /_{\doteq} \to [0,1]$ as follows:*

$$\forall x \in \{0,1\}^{\mathbb{N}}, \ h(x) = d(\underline{0}, x) = \limsup_{k\to\infty} \frac{\sum_{i=0}^{k} x_i}{k+1}.$$

It is not difficult to prove that $h \circ f(x) = x$. In literature h is called upper density*, see for example [11]. Moreover it is not difficult to see that*

$$\forall x, y \in \{0,1\}^{\mathbb{N}}, \ h(x - y) = d(\underline{0}, x - y) = d(x, y).$$

Proof of Theorem 3. We are going to prove the thesis by induction on the dimension n, that is to say that for all $n \in \mathbb{N}$, the space $[0,1]^n$ can be embedded in $\{0,1\}^{\mathbb{N}} / \doteq$.

The mapping f proves the theorem for $n = 1$. Let us suppose that the thesis holds for $n = t$, we prove the thesis for $n = t+1$. Let us consider the mapping $f_{t+1} \colon [0,1]^{t+1} \to \{0,1\}^{\mathbb{N}} / \doteq$ defined as follows:

$$\forall x \in [0,1]^{t+1},\ x = (x_0, \ldots, x_{t+1})$$
$$f_{t+1}(x) = f(x_0)_0 f(x_1)_0 \ldots f(x_{t+1})_0 f(x_0)_1 f(x_1)_1 \ldots f(x_{t+1})_1 \ldots .$$

We claim that f_{t+1} is continuous. Let $\epsilon > 0$ and $m \in \mathbb{N}$ such that $2m > \left\lfloor \frac{1}{\epsilon} \right\rfloor$. Let $\delta = \frac{1}{2^m}$. We consider the cube $[0,1]^{t+1}$ endowed with the following metric

$$\forall x = (x_0, \ldots, x_{t+1}) \in [0,1]^{t+1},\ y = (y_0, \ldots, y_{t+1}) \in [0,1]^{t+1}$$
$$d_p(x,y) = \max \{|x_0 - y_0|, \ldots, |x_{t+1} - y_{t+1}|\}.$$

For any $y \in [0,1]^{t+1}$ such that $d_p(x,y) < \delta$ let us consider the binary expansions of $(x_0, y_0), \ldots, (x_{t+1}, y_{t+1})$. Any x_i will coincide with y_i on the first $n_i \geq -\lfloor \log_2 \delta \rfloor$. We build the image of f_{t+1} with the extraction process depicted in Figure 1. Hence,

$$\begin{aligned} d(f_{t+1}(x), f_{t+1}(y)) = \ & \limsup_{k \to \infty} \frac{\#_{-k,k}(f_{t+1}(x), f_{t+1}(y)}{2k+1} \\ & \leq \limsup_{k \to \infty} \frac{\left\lfloor \frac{2k+1}{2(t+1)n_1} \right\rfloor + \ldots + \left\lfloor \frac{2k+1}{2(t+1)n_{t+1}} \right\rfloor}{2k+1} \\ & \leq \limsup_{k \to \infty} \frac{(t+1) \cdot \left\lfloor \frac{2k+1}{2(t+1)m} \right\rfloor}{2k+1} \\ & \leq \frac{1}{2m} \leq \epsilon. \end{aligned}$$

◇

Let us define a mapping $g \colon (0,1) \to \{0,1\}^{\mathbb{N}}$ as follows:

$$\forall x \in (0,1),\ \forall n \in \mathbb{N} \quad g(x)_n = \begin{cases} 0, & \text{if } 0 < nx - k < 1 - x \text{ for some } k \in \mathbb{N} \\ 1, & \text{otherwise.} \end{cases}$$

For all $x \in (0,1)$, $g(x)$ is the *Sturmian sequence* with density x, i.e. $h \circ g(x) = x$.

A Sturmian sequence can be defined by considering the sequence of intersections with a squared lattice of a semi-line having a slope which is an irrational number. Horizontal intersections are denoted by a 1, vertical intersections by a 0 (intersections with corners are denoted by 01 or 10, but

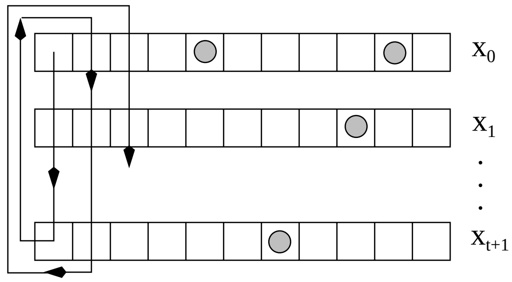

Figure 1: Extraction process of the image of f_{t+1} from the Toeplitz sequences associated with $x = (x_0, \ldots, x_{t+1})$. Spotted cells represent "holes".

this new happen if the slope is an irrational). Sturmian sequences represented by a semi-line starting at the origin are called *standard*. Let **Stand** be the set of all such sequences. (For more on Sturmian sequences see for example [7, 4]).

Remark 3 *Let us consider the set $\mathcal{A} \subseteq \{0,1\}^{\mathbb{N}}$ of words*

$$w = 0^{q_0} 1^{q_1} 0^{q_2} 1^{q_3} 0^{q_4} \ldots$$

such that $q_0 \geq 0$ and $q_i > 0$ for $i \geq 1$. Clearly $\Pi(\mathcal{A}) = \{0,1\}^{\mathbb{N}} /_{\doteq}$. In fact, if w is such that $\exists i \in \mathbb{N}, q_t = 0$ for $t \geq i$ then $w \in \{\underline{0}_{\doteq}, \underline{1}_{\doteq}\}$. It is not difficult to prove that also in $\underline{0}_{\doteq}$ and $\underline{1}_{\doteq}$ there are elements of $\mathcal{A}$ (see Remark 6 in [5]).

By a cardinality argument one can prove the existence of a bijection Ψ between $\mathcal{A}$ and **Stand**. *In [7], de Luca presents an interesting example of such a mapping. We give its definition by induction:*

$$\forall x \in \mathcal{A}, \text{ let } \psi(x)_0 = \varepsilon, \quad \psi(x)_{n+1} = (x_n \psi(x)_n)^-,$$

where $(u)^-$ is the smallest palindrome word having u as suffix. Then $\Psi(x) = \lim_{n\to\infty} \psi(x)_n$.
Therefore we have the following situation:

$$\textbf{Stand} \xleftrightarrow{\Psi} \mathcal{A} \xrightarrow{\Pi} \{0,1\}^{\mathbb{N}} /_{\doteq}$$

It would be interesting to prove that if $x \doteq y$ then $\Psi(x) \doteq \Psi(y)$.

Moreover note that if we extend the definition of $\mathcal{A}$ to $S^{\mathbb{Z}}$ then $\Pi(\mathcal{A}) \subset S^{\mathbb{Z}}/_{\doteq}$, but $\Pi(\mathcal{A})$ is dense in $S^{\mathbb{Z}}/_{\doteq}$.

Lemma 1 *If $x, y, \frac{x}{y} \in [0,1]$ are irrationals then $d(g(x), g(y)) = x(1-y) + y(1-x)$.*

Proof. Let $x, y, \frac{x}{y} \in [0,1]$ be all irrationals and consider the following dynamical system (rotation of the torus):

$$T(a,b) = (a+x, b+y) \text{ mod } 1$$

defined on the torus $\mathbb{R}^2/_{\mathbb{Z}^2}$. The system T is uniquely ergodic [8, 13] and the Lebesgue measure can be assumed as the unique invariant measure. We note that $g(x)_n \neq g(y)_n$ if and only if $T^n(0,0) \in [0, 1-x] \times [1-y, 1] \cup [1-x, 1] \times [0, 1-y]$. This set has Lebesgue measure $x(1-y) + y(1-x)$. $\diamond$

Proof of Theorem 4. If we consider $x, y, \frac{x}{y}$ all irrationals and $\frac{1}{3} < x, y < \frac{2}{3}$ then by Lemma 1 we have $d(g(x), g(y)) > \frac{2}{9}$. Therefore there exists a continuum of points which are at distance greater than $\frac{2}{9}$ but this implies that the space cannot have a countable dense set. $\diamond$

Proof of Proposition 1. Let x, y be two distinct Toeplitz configurations. It follows that there exists an integer i such that $x(i) \neq y(i)$. From the definition of Toeplitz sequence it holds that there exists $p_x \in \mathbb{N}^+$ such that $\forall n \equiv i \text{ mod } p_x$, $x(n) = x(i)$. In the same way there exists p_y such that $\forall m \equiv i \text{ mod } p_y$, $y(m) = y(i)$. Therefore if we take $p = p_x \cdot p_y$ we have that $\forall h \equiv i \text{ mod } p$, $x(h) = x(i) \neq y(i) = y(h)$, this implies $d(x,y) \geq \frac{1}{p}$. $\diamond$

Proof of Theorem 5. From Proposition 1 it follows that the mapping f is injective. Let us consider a generic dynamical system over $S^{\mathbb{Z}}/_{\doteq}$. i.e. a continuous map of $S^{\mathbb{Z}}/_{\doteq}$ on itself, such that $F(\text{Im}(f)) \subseteq \text{Im}(f)$. Let us consider the following diagram

$$\begin{array}{ccc} \text{Im}(f) & \xrightarrow{F} & \text{Im}(f) \\ \uparrow f & & \downarrow f^{-1} \\ [0,1] & \xrightarrow{fFf^{-1}} & [0,1] \end{array}$$

We remark that $f\colon [0,1] \to \text{Im}(f)$ is injective and therefore it is a homeomorphism from $[0,1]$ to $\text{Im}(f) \subset S^{\mathbb{Z}}/_{\doteq}$. Therefore fFf^{-1} and F are topologically conjugated (here we consider the restriction of F to $\text{Im}(f)$). It is well known that every continuous mapping from $[0,1]$ to itself has a fixed point, therefore F has a fixed point, and it is, clearly, a Toeplitz configuration. $\diamond$

Proof of Proposition 2. The idea of the proof is to find a configuration which is at non-null distance from any Toeplitz configuration. Let us consider the configuration b as defined in Example 1. Let c be an arbitrary Toeplitz configuration. Let i be an integer such that $b(i) \neq c(i)$ and p_i its period. Without loss of generality we can suppose that $b(i) = 1$. Under this assumption it is straightforward to prove that $d(b, c) \geq \frac{1}{p_i} \cdot \frac{q}{1+q}$. ◇

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