Length and Number of Buses for Gossiping in Networks

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Gossiping is an information dissemination problem in which each node of a communication network has a unique piece of information that must be transmitted to all the other nodes. A *bus network* is a network of processing elements that communicate by sending messages along buses in a sequence of calls. We assume that (i) each node can participate to at most one call at a time, (ii) a node can either read or write on a bus, (iii) no more than one node can write on a given bus at a given time, and (iv) communicating a message on a bus takes a unit of time. This model extends the *telegraph* model in allowing the number of nodes connected to each bus to be as large as needed, instead on being bounded by 2.

In this paper, we are interested in minimizing the "hardware" of a bus network in keeping optimal the communication performances for solving the gossiping problem. More precisely, we compute the minimum *number* of buses required for a gossiping to be optimal. Similarly, we give upper bounds on the minimum *length* of buses required for a gossiping to be optimal. Finally, we combine the two approaches in trying to minimize both parameters: length and number of buses.

Keywords: Hypergraph, bus, network, gossiping, communication

Résumé

L'*échange total* consiste en des échanges de données entre les processeurs d'un réseau d'interconnexion où chaque processeur possède une information qu'il doit diffuser à l'ensemble des autres processeurs. Un *réseau par bus* est un réseau d'interconnexion dont les éléments échangent leurs informations au moyen de bus. Dans cet article, nous supposons que (i) chaque processeur ne peut participer qu'à une communication à un instant donné, (ii) chaque processeur peut soit lire ou écrire sur un bus, mais pas les deux à la fois, (iii) au plus un processeur peut écrire sur un bus donné à un instant donné, et (iv) la communication d'une information sur un bus coûte une unité de temps. Ce modèle peut être vu comme une extension du modèle *télégraphe* en supposant que le nombre de processeurs connectés à un même bus peut être aussi grand que souhaité au lieu d'être limité à deux.

Dans cet article, nous nous sommes intéressés à minimiser le "matériel" d'un réseau par bus tout en permettant de résoudre de façon optimale le problème de l'échange total. Plus précisément, nous calculons le nombre minimum de bus que requiert un réseau par bus afin de réaliser l'échange total de façon optimale. De la même manière, nous donnons des bornes supérieures sur la *longueur* des bus que requiert un réseau par bus afin de réaliser l'échange total de façon optimale. Finalement, nous combinons les deux approches en essayant de minimiser ces deux paramètres : longueur et nombre de bus.

Mots-clés: hypergraphe, bus, réseaux, échange total, communication

Length and Number of Buses for Gossiping in Networks *

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1 Introduction

Let us consider the following problem: how to compute on a distributed memory parallel computer with n processors the sequence $Z^{(k+1)} = H(Z^{(k)})$ where $Z^{(k)} \in \mathbb{C}^n$ for any $k \geq 0$, and H is a map from $\mathbb{C}^n$ to $\mathbb{C}^n$? A simple solution consists in evenly distributing the components of the vectors among the processors and, for $i = 1, \ldots, n$, processor i computes $Z_i^{(k+1)} = H_i(Z^{(k)})$. Of course, after each iteration, all the processors need to exchange their new data to be able to complete the next iteration. This is the *gossip* problem [3].

More precisely, *gossiping* is an information dissemination problem in which each node of a communication network has a unique piece of information that must be transmitted to all the other nodes. The example above is studied in [4] and there are many other examples of problems where the key to obtain good performances depends on an appropriate solution for the gossip problem (see for instance [17]).

The gossip problem has been intensively studied in the literature (see the surveys in [8] and [10]). In particular, many solutions have been proposed depending on the communication model and the network topology. There exists another approach that consists in minimizing the "hardware" available in keeping optimal the complexity of the algorithm. For instance, under the telephone model (constant time, 1-port, full-duplex), it is not possible to broadcast a message in less than $\lceil \log_2 n \rceil$ steps, and a *minimum broadcast graph* (MBG) is a graph of n vertices that allows to broadcast in $\lceil \log_2 n \rceil$ steps with a minimum number of edges [7]. Similarly, a *minimum gossip graph* (MGG) is a graph of n vertices that allows to gossip in an optimal time with a minimum number of edges [14].

Finding MBG's or MGG's is known to be hard, but numerous methods have been given to construct sparse broadcast graphs or sparse gossip graphs (see [2, 14]). Much more difficult seems to be the gossip problem under the telegraph model (constant time, 1-port, half-duplex). Indeed, the lower bound for gossiping turns to be $2 + \lceil \log_\rho \frac{n}{2} \rceil$ where $\rho = \frac{1+\sqrt{5}}{2}$, and optimal gossip algorithms are very tricky [5, 6, 15, 18].

In this paper, we loosen the constraint of the telegraph model that allows a node to either send or receive at most one message from one of its neighbors, in letting a node that wants to send a message to send it simultaneously to many other nodes (all these nodes can be supposed to be connected by a bus). Hypergraphs are the natural underlying structure for that model, however, since we will not use the mathematical properties of hypergraph for solving our problems, we prefer to use the terminology of *bus networks*.

Again, bus networks have been studied in the literature (see for instance some references in [11]) but the topology is always fixed in advance. In this paper we look for the sparsest topology with the best communication performances. Indeed, we investigate *minimum gossip bus network* with a double goal. First, trying to understand why communication problems are more difficult under the half-duplex model than under the full-duplex model, and also trying to derive solutions that may apply in practical situations (for instance our model applies for real distributed memory parallel machines where communications between processors are supported by buses, and applies also for communication between nodes that exchange messages by radio).

In the next section, we describe our model in detail and give the first definitions. Then, in Section 3, we compute the exact number of buses that interconnect a minimum gossip bus network.

This result is surprising in comparison with the same problem for graphs where one even does not know if the minimum number of edges increases with the order of the graph [9]: for bus networks, one can find the exact value of the *gossip function* that returns the minimum number of buses as a function of the number of vertices. However, our bus networks which match the lower bounds have a bus of *length* n which might be not realistic. Therefore, in Section 4, we are interested in minimizing also the length of the buses. It appears that, in general, a minimum gossip bus network possesses at least a bus of length n. Thus, constructions of Section 3 can be considered as optimal both regarding the number of buses and the maximum length of these buses. We finally focus in the reverse problem, that is finding the minimum length of the buses of a gossip bus network. This question is studied in Section 5 and appears to be much more complicate. Nevertheless, we derive an upper bound of $\sqrt{n}+1$ for $n \neq 2^d$ and $n \neq 2^d - 1$ (the exact values are known in these two cases). We conjecture that this order is also a lower bound but we were unable to prove that fact. Section 6 resumes our results and contains some concluding remarks.

2 Statement of the problem

As said before, the underlying structure for our problem is the structure of hypergraph [1]. A *bus network* is a network of *processing nodes* interconnected by *buses*. A processing node is modeled by a vertex of an hypergraph G, and buses are modeled by the hyper-edges of G. See Figure 1 for examples of bus networks. In this section, we describe the communication model, we define what is a gossip bus network and we give the main definitions related to that concept. We refer to [13] for a generalization of the gossiping problem in hypergraphs.

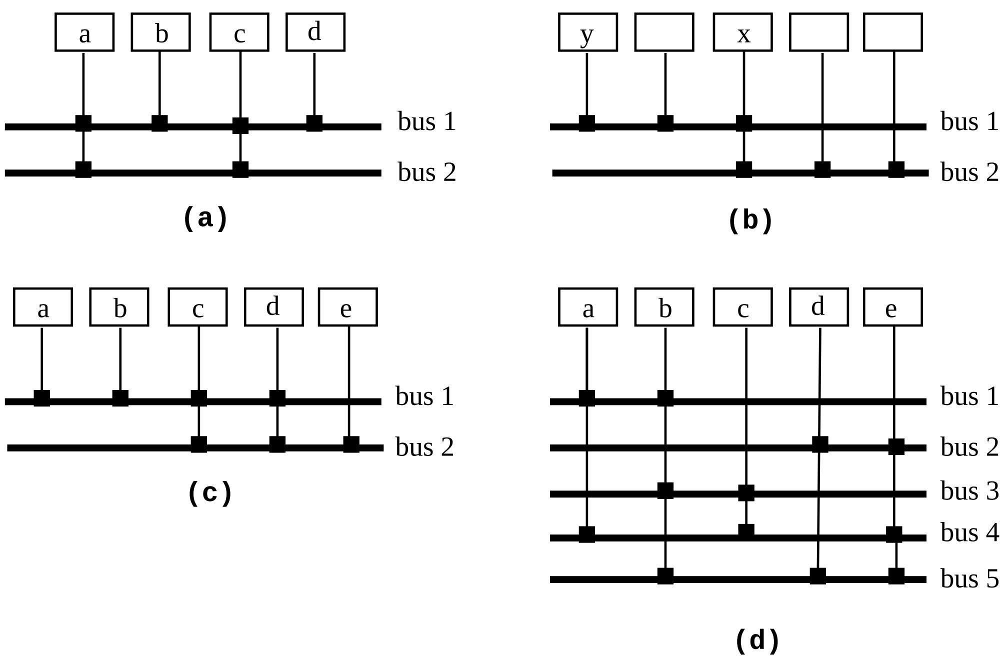

Figure 1: Exemples of particular (gossip) bus networks

2.1 The communication model

Our communication model is the following (see [8, 11] for an overview of many communication models):

Constant time: communications proceed step by step, and communicating a message along a bus takes a unit of time (that is 1 step) whatever the length of the message and the number of nodes connected to the bus are;

Whispering or 1-port: a node can either send or receive at most one message per step, that is can read or write on only one bus at a time;

Width 1: given a bus, only one processor can send a message on it at a given step;

Unbounded length: the number of nodes connected to a same bus is not bounded.

We denote this model the $\mathcal{B}$ model (for *basic* model). Note that if one bounds by 2 the length of the buses, this model turns to be the telegraph model (1-port, constant time, half-duplex). On the other hand, if one bounds by $k > 2$ the length of the buses, our model is distinct from a k-port telegraph model (k-port, constant time, half-duplex) [8, 16] because for instance a vertex can receive at most one message per step (whispering constraint).

Definition 1 *The* gossiping time *of a bus network is the minimum number of steps required to perform a gossiping on that network under the $\mathcal{B}$ model.*

2.2 Gossip bus networks

Theorem 1 *The time to complete a gossip in any bus network of order n under the $\mathcal{B}$ model is at least $\lceil \log_2 n \rceil + 1$. For any positive integer n, there exists a bus network of order n whose gossiping time under the $\mathcal{B}$ model is $\lceil \log_2 n \rceil + 1$.*

Proof. Let G be any bus network of order n. Let $\alpha(t)$ be the maximum number of messages that a vertex of G can know at time t of a gossiping in G. Of course $\alpha(0) = 1$ and $\alpha(1) \leq 2$. Let us show by induction that $\alpha(t) \leq 2^t$ for any $t \geq 0$. If $\alpha(t-1) \leq 2^{t-1}$ then, since during step t a vertex can receive informations from at most one other vertex, $\alpha(t) \leq 2\alpha(t-1) \leq 2^t$. Thus, at time $\lceil \log_2 n \rceil - 1$, no vertex of G is aware of all the informations, and therefore at least two more steps are necessary to complete the gossiping.

Let us now consider the following gossiping algorithm where the network is supposed to support as many buses as necessary:

1. Concentrate all the information in a vertex r (this can be done in $\lceil \log_2 n \rceil$ steps under the $\mathcal{B}$ model – for instance using the broadcast algorithm under the telephone model in the reverse order);
2. Broadcast all the informations from r to all the other vertices (this can be done in one step under the $\mathcal{B}$ model by a bus of length n).

$\square$

Definition 2 *A* gossip bus network *of order n is a bus network whose gossiping time is $\lceil \log_2 n \rceil + 1$ under the $\mathcal{B}$ model. An* optimal *gossiping algorithm on a gossip bus network of order n is a gossiping algorithm that performs in time $\lceil \log_2 n \rceil + 1$ under the $\mathcal{B}$ model.*

2.3 Notations and definitions

For a bus network G, let $K(G)$ be the number of buses of G, and $L(G)$ be the maximum length of the buses of G. We denote by $K(n)$ the minimum number of buses of a gossip bus network of order n, that is

$$K(n) = \min_G K(G), G \text{ gossip bus network of order } n.$$

Definition 3 *A* minimum gossip bus network *of order n is a gossip bus network with $K(n)$ buses.*

We denote by $L(n)$ the minimum over all the gossip bus networks G of order n of $L(G)$. We also denote $LK(n)$ the minimum over all the minimum gossip bus networks G of order n of $L(G)$.

$$\begin{aligned} L(n) &= \min_G L(G), G \text{ gossip bus network of order } n \\ LK(n) &= \min_G L(G), G \text{ minimum gossip bus network of order } n \end{aligned}$$

In the rest of this paper, we will concentrate on these three functions. Mainly, we will give analytic expressions for K and LK, and give an upper bound for L.

3 Minimum gossip bus networks

In this section, we are interested in minimizing the number of buses. A gossip bus network of two vertices has at least one bus, and this is enough to gossip in two steps: $K(2) = 1$. The gossiping time of a gossip bus network of order 3 or 4 is 3. A gossip bus network of 3 vertices has at least one bus, and this is enough to gossip in three steps: $K(3) = 1$. It is not possible to gossip in 3 steps in a bus network of order 4 with one bus because every vertex has to write on the bus, and this will take at least 4 steps. Now, the bus network of Figure 1(a) has two buses labeled 1 and 2 and it is possible to gossip in 3 steps in this network: (1) d sends its message to b on bus 1 while c sends its message to a on bus 2, (2) b sends to a the two messages of b and d on bus 1, (3) a broadcasts the four messages to the three other vertices on bus 1. Thus $K(4) = 2$.

The next theorem gives the value of $K(n)$ for any integer n.

Theorem 2 *Let n be any integer, $n > 1$, and let $d = \lceil \log_2 n \rceil$, we get*

$$K(n) = \begin{cases} \lceil \frac{n}{2} \rceil - 2^{d-3} & \text{if } 2^{d-1} < n < 2^{d-2} + 2^{d-1} \\ n - 2^{d-1} & \text{if } 2^{d-1} + 2^{d-2} \leq n \leq 2^d \end{cases}$$

The proof of this theorem is based on the following definitions and lemmas.

Definition 4 *A* gathering *is an information dissemination problem that consists for all vertices of a bus network to send a message to a same vertex.*

From Theorem 1, under the $\mathcal{B}$ model, the gathering time of any bus network of order n is at least $\lceil \log_2 n \rceil$ and there exists a bus network of order n whose gathering time is $\lceil \log_2 n \rceil$.

Definition 5 *During a gossiping algorithm in a bus network G, a vertex x of G is called* expert at time t *if it is aware of all the informations of all the other vertices at step t.*

Remark 1. From Theorem 1, under the $\mathcal{B}$ model, during any optimal gossiping algorithm on a gossip bus network of order n, there exists at least one expert at time $\lceil \log_2 n \rceil$ and there cannot exist any expert before time $\lceil \log_2 n \rceil$.

Lemma 1 *Let n be any integer, $n > 1$, and let $d = \lceil \log_2 n \rceil$, we get*

$$K(n) \le \begin{cases} \lceil \frac{n}{2} \rceil - 2^{d-3} & \text{if } 2^{d-1} < n < 2^{d-2} + 2^{d-1} \\ n - 2^{d-1} & \text{if } 2^{d-1} + 2^{d-2} \le n \le 2^d \end{cases}$$

Proof. Let n be any positive integer greater than 1. The proof is based on the gossiping algorithm described in the proof of Theorem 1. The total number of buses is the number of buses used during the gathering phase of this algorithm. In this proof, we will consider only gathering algorithms based on *broadcast trees* (see [2]) where the labeling of the edges is reversed: label i is replaced by the label $\lceil \log_2 n \rceil - i + 1$. We call a *gathering tree* a broadcast tree labeled in the reverse order. See Figure 2(a) for a broadcast tree, and Figure 2(b) for a gathering tree.

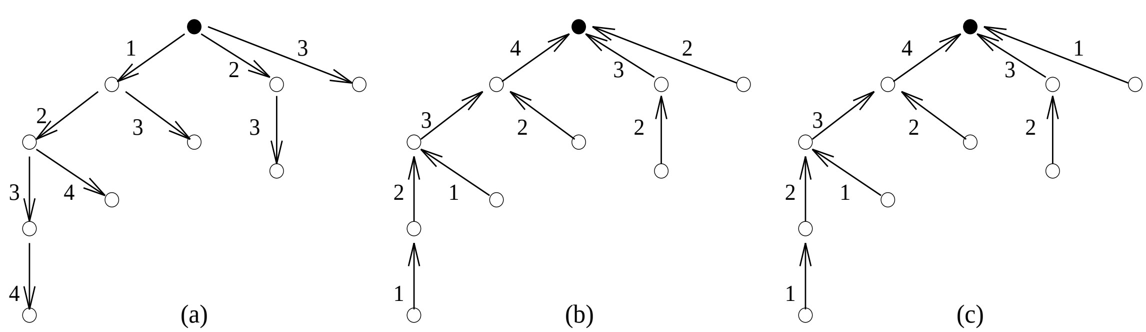

Figure 2: A broadcast tree, a gathering tree and a minimum gathering tree for 10 vertices.

A gathering algorithm based on a gathering tree will require as many buses as the maximum number of edges having the same label in the tree. For any integer n, let us find a gathering tree of n vertices that minimizes this number, and denote $b(n)$ the corresponding minimum. For instance, on Figure 2, the gathering tree (b) uses 4 buses, while the gathering tree (c) uses only 3 buses.

A gathering tree of n vertices with $\lceil \log_2 n \rceil$ different labels and at most k edges having the same label interconnects at most $2^{\lceil \log_2 k \rceil} + k(\lceil \log_2 n \rceil - \lceil \log_2 k \rceil)$ vertices because, by considering the corresponding broadcast tree, during the $\lceil \log_2 k \rceil$ first steps of a broadcast in this tree, no more than $2^{\lceil \log_2 k \rceil}$ vertices can be informed, and during the $\lceil \log_2 n \rceil - \lceil \log_2 k \rceil$ next steps, no more than k vertices can be informed at each step.

Thus $b(n)$ satisfies

$$2^{\lceil \log_2 (b(n)-1) \rceil} + (b(n)-1)(\lceil \log_2 n \rceil - \lceil \log_2 (b(n)-1) \rceil) < n \leq 2^{\lceil \log_2 b(n) \rceil} + b(n)(\lceil \log_2 n \rceil - \lceil \log_2 b(n) \rceil). \tag{1}$$

Conversely, for any integer b that satisfies the inequalities (1) for $b(n) = b$, there exists a gathering tree of n vertices with $\lceil \log_2 n \rceil$ different labels and with no more than b edges having the same label, and b is minimum for this property. The parameter b is in fact the minimum number of buses that a gathering algorithm performing in time $\lceil \log_2 n \rceil$ and based on a gathering tree can use. This is an upper bound of $K(n)$ because, as said before, under the $\mathcal{B}$ model one can gossip in time $\lceil \log_2 n \rceil + 1$ in first gathering all the messages in time $\lceil \log_2 n \rceil$ and then broadcasting them to all the vertices in one step.

We let the reader check that the values given in the statement of this lemma satisfy the inequalities (1). $\square$

Lemma 2 *Under the $\mathcal{B}$ model, from any gathering algorithm performing in time $\lceil \log_2 n \rceil$ on a bus network G of order n, one can extract a gathering algorithm on the same bus network G performing in the same time and based on a gathering tree.*

Proof. Let $\mathcal{A}$ be any gathering algorithm on a bus network $G = (V, E)$ that gathers the informations of all vertices in a vertex r in time $\lceil \log_2 n \rceil$. During $\mathcal{A}$, from any vertex $x \neq r$, the message m_x of x follows a path

$$C_x = \{(x_0, x_1) = (x, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k) = (x_{k-1}, r)\} \tag{2}$$

from x to r: vertex x_i receives m_x from x_{i-1} and forwards it later to x_{i+1}.

Let us consider the directed graph $H = (V, X)$ [1] where $X = \cup_{x \neq r} C_x$. We construct $\lceil \log_2 n \rceil + 1$ anti-arborescences $T_i, i = 0, \ldots, \lceil \log_2 n \rceil$ as follows. Set $T_0 = (\{r\}, \emptyset)$. There is only one arc of the form $(x, r) \in X$ that represents a communication at time $\lceil \log_2 n \rceil$ of $\mathcal{A}$. Set $T_1 = T_0 \cup (\{x\}, \{(x, r)\})$ where, in all this proof, if $G_1 = (V_1, X_1)$ is an induced subgraph of $G = (V, X)$, and G_2 is a couple (V_2, X_2) satisfying $V_2 \subset V$ and $X_2 \subset (V_1 \cup V_2) \times (V_1 \cup V_2)$ (G_2 is not necessarily a graph), then $G_1 \cup G_2$ is defined by $G_1 \cup G_2 = (V_1 \cup V_2, X_1 \cup X_2)$.

In general, given T_{i-1}, let S be the set of vertices $x \in V$ such that

1. $x \notin V(T_{i-1})$;
2. $\exists y \in V(T_{i-1})$ such that $(x, y) \in X$ and (x, y) represents a communication in $\mathcal{A}$ at time $\lceil \log_2 n \rceil - i + 1$.

Let $\tilde{U}$ be a set of arcs (x, y), $x \in S, y \in V(T_{i-1})$ obtained from condition 2. Consider the set $U \subset \tilde{U}$ such that for any vertex $x \in S$, there is one and only one arc $(x, y) \in U$. The $\mathcal{B}$ model implies that such a set U exists. Then, set $T_i = T_{i-1} \cup (S, U)$.

Clearly, each T_i is an anti-arborescence. Moreover, $T_{\lceil \log_2 n \rceil}$ is a spanning anti-arborescence of H. Indeed, in any path C_x from x to r as presented in (2), for any $i, i = 0, \ldots, k-1$, we get that if the arc (x_i, x_{i+1}) represents a communication in $\mathcal{A}$ at time t_i, then $x_i \in T_{t_i}$, and in particular $x \in T_{t_0}$.

Let us label the arcs of $T_{\lceil \log_2 n \rceil}$ such that an arc (x, y) of $T_{\lceil \log_2 n \rceil}$ is labeled $\lceil \log_2 n \rceil - i + 1$ where i satisfies $(x, y) \notin T_{i-1}$ and $(x, y) \in T_i$. By construction, for any $x \in V, x \neq r$, the label of its output arc is strictly greater than the label of any of its input arcs. Furthermore, due to the communication model $\mathcal{B}$, two input arcs in the same vertex cannot have the same label.

Therefore $T_{\lceil \log_2 n \rceil}$ is a gathering tree and clearly uses at most as many buses as $\mathcal{A}$. □

Proof of Theorem 2. Lemma 1 gives an upper bound on $K(n)$. Lemma 2 shows that this upper bound is the best possible. Therefore this upper bound is also a lower bound because any optimal gossiping algorithm contains a gathering algorithm in $\lceil \log_2 n \rceil$ steps. Indeed, from Remark 1, any optimal gossiping algorithm consists in two phases: first phase is a gathering of all the information in one or more experts in $\lceil \log_2 n \rceil$ steps, and the second phase is a broadcasting from all the experts to all the other vertices. □

From Theorem 2, we know the minimum number of bus that a gossip bus network can have. However, even if we know how to construct gossip bus networks that match this value, one can be embarrassed by the fact that the bus networks that we constructed have a bus of length n (even if the other buses have a length that can be reduced to two). In fact, the next section shows that one cannot do better in general, that is there is necessarily a bus of length n in a minimum gossip bus network.

4 Length of buses of minimum gossip bus networks

In this section, we will compute the minimum $LK(n)$ of the maximum buses length of a minimum gossip bus network of order n. Clearly, $LK(2) = 2$. Since $K(3) = 1$, we get $LK(3) = 3$. It is easy to see that it is not possible to get more than one expert after two steps in a bus network of 4 vertices. Thus $LK(4) = 4$ and the bus network of Figure 1 is therefore optimal for that property. As it is shown below, the relation $LK(n) = n$ generalizes for many other values of n.

Lemma 3 *If $n = 2^d$, then during any optimal gossiping algorithm in a bus network of order n, there is exactly one expert at time d, and this expert does not send its information before step $d+1$.*

Proof. From the proof of Theorem 1, if a vertex is aware of 2^t informations (including its own) after t steps, then this vertex has never sent its message. Therefore, being an expert after d steps implies that nobody else knows your information. Since there is necessarily an expert at time d, there is one and only one expert at time d. □

Corollary 1 *Let n be any integer such that $2^{d-1} + 2^{d-2} \leq n \leq 2^d$, $LK(n) = n$.*

Proof. For such an integer n, $K(n) = n - 2^{d-1}$ and thus after one step of a gossiping algorithm in a minimum gossip bus network, at least 2^{d-1} vertices have not sent their message to anybody. From Lemma 3, in $d-1$ steps there will be only one vertex among these 2^{d-1} vertices that will be aware of all their messages and this vertex cannot have sent its message. Therefore a bus of length n is required for the last step of the gossiping. $\square$

Corollary 2 *Let n be an even integer such that $2^{d-1} < n < 2^{d-1} + 2^{d-2}$, $LK(n) = n$.*

Proof. For such an integer n, $K(n) = \frac{n}{2} - 2^{d-3}$ and thus after two steps of a gossiping algorithm in a minimum gossip bus network, at least $n - 2(\frac{n}{2} - 2^{d-3}) = 2^{d-2}$ vertices have not sent their message to anybody. The rest of the proof is similar to the one of Corollary 1. $\square$

Now, for orders n that do not fall into the ranges of the two previous corollaries, the problem turns to be more tricky. For instance, let us compute $LK(5)$. From Theorem 2, $K(5) = 2$. This implies a bus of length at least 3, otherwise the network would not be connected. If the two buses have length no more than 3, then there are 4 vertices connected to only one bus, and a vertex x connected to two buses (see Figure 1(b)). From remark 1, one needs at least two experts for the second phase which is a broadcast from expert vertices to the other (one expert only would imply a bus of length 5). It is not possible that all the experts be different from x and be connected to the same bus. Therefore there must be one expert y connected to bus 1 and another z connected to bus 2. In fact, either y or z must be x. Assume $z = x$. Now, to gather the messages of vertices of bus 2 and send them to y in no more than 3 steps, x must receive on bus 2 at step 2 and must send on bus 1 at step 3. This implies that x is only free at step 1 to gather the two messages of the vertices connected only to bus 1: this is not possible. Therefore $LK(5) = 3$ leads to a contradiction, thus $LK(5) > 3$. Figure 1(c) is a gossip bus network of 5 vertices, with two buses of length at most 4. The gossip scheme is: (1) $b \to c$ on bus 1, and $e \to d$ on bus 2, (2) $a \to c$ on bus 1, and $d \to e$ on bus 2. (3) $c \to d$ and e on bus 2, (4) $d \to a, b$ and c on bus 1. Thus $LK(5) = 4$. This small example shows that when the order n is an odd integer satisfying $2^{d-1} < n < 2^{d-1} + 2^{d-2}$, the behavior of $LK(n)$ is difficult to handle. In the following, we derive a lower bound for $LK(n)$.

Lemma 4 *If $n = 2^d - 1$, then during any optimal gossiping algorithm in a bus network of order n, there is exactly one expert at time d, and this expert does not send its information before step $d+1$.*

Proof. Since $n = 2^d - 1$, there is at most $2^{d-1} - 1$ vertices that can send their information at the first step. Thus, at least 2^{d-1} vertices has not sent their information at the first step. The rest of the proof is similar to the one of Lemma 3. $\square$

Corollary 3 *Let n be an odd integer such that $2^{d-1} < n < 2^{d-1} + 2^{d-2}$, $LK(n) \geq \sqrt{n}$.*

Proof. For such an integer n, $K(n) = \frac{n+1}{2} - 2^{d-3}$ and thus after two steps of an optimal gossiping algorithm in a minimum gossip bus network G, at least $n - 2(\frac{n+1}{2} - 2^{d-3}) = 2^{d-2} - 1$ vertices have not sent their message to anybody. Let S be the set of these vertices. From Lemma 4, at most one vertex of S can be an expert at time d.

Assume no vertex of S becomes an expert. If there are e experts in G at time d, the set Γ of the e experts behaves like the single expert of a bus network of 2^{d-2} vertices, that is at each step from 3 to d, a vertex in S sends a message to vertices of Γ. This implies a bus of length at least $e+1$.

Assume there is a vertex x of S that becomes an expert at time d. By definition of S, x has not sent its information at step 1 and 2. Now, since x is an expert and therefore must know all the messages of the vertices of S, x cannot send any information after step 3. It means that the only "free time" of x is at step 3. If there are e experts in G at time d, x must have sent its message to them in one step (at step 3), and this implies a bus of length at least e.

In both cases, e experts implies a bus of a length at least e. Moreover, these e experts will require buses of length at least $\lceil \frac{n}{e} \rceil$ to inform the other vertices at the last step (step $d+1$). Altogether, $LK(n) \geq \min_e \max(e, \lceil \frac{n}{e} \rceil) \geq \sqrt{n}$. □

This corollary provides a lower bound on $LK(n)$, but we guess that it can be reached. However, the best upper bound that we get is still far from this bound as it is stated below.

Proposition 1 *Let n be an odd integer such that $2^{d-1} < n < 2^{d-1} + 2^{d-2}$, $LK(n) \leq K(n) + \lceil \log_2 n \rceil = \lceil \frac{n}{2} \rceil - 2^{d-3} + \lceil \log_2 n \rceil$.*

Proof. As in section 3 for Lemma 1, consider the following gossip algorithm (see Figure 3). At the first step, $K(n)$ vertices inform $K(n)$ other vertices. We call the $K(n)$ senders, the *initiators*. At the second step, these last $K(n)$ vertices inform $K(n)$ new vertices. Vertices that never sent their information form a set S of $2^{d-2}-1$ vertices (in grey on Figure 3). The $d-2$ next steps consist in a gathering in S performed following a binomial tree $\mathrm{BT}(d-2)$ with a *virtual root* where, instead of informing this virtual root, the vertices of the first level of $\mathrm{BT}(d-2)$ inform the $K(n)$ initiators. The $K(n)$ initiators are experts at time d. These experts inform the rest of the vertices at the last step.

Let us compute the length of the buses needed for such an algorithm.

The gathering phase uses a bus of length at most $K(n)+d$ that connect the $K(n)$ initiator, the $d-2$ vertices at level 1 of $\mathrm{BT}(d-2)$ and two other vertices that receive at the first and the second steps. The other buses are of length at most $2d$ (d steps, each step involving two vertices in each of these buses).

Since there is one expert on each bus (each of the $K(n)$ initiators is connected to a different bus from the first step), the broadcasting phase from the experts does not require additional material. □

Note that in the proof of the previous proposition, creating more than $K(n)$ experts implies a longer bus. On the other hand, creating less that $K(n)$ experts might allow to decrease the length of the bus. However, the broadcasting phase might then require more material since there will no longer be one expert on each bus. This is the general problem for the determination of $LK(n)$: the gathering phase and the broadcasting phase are here strongly related.

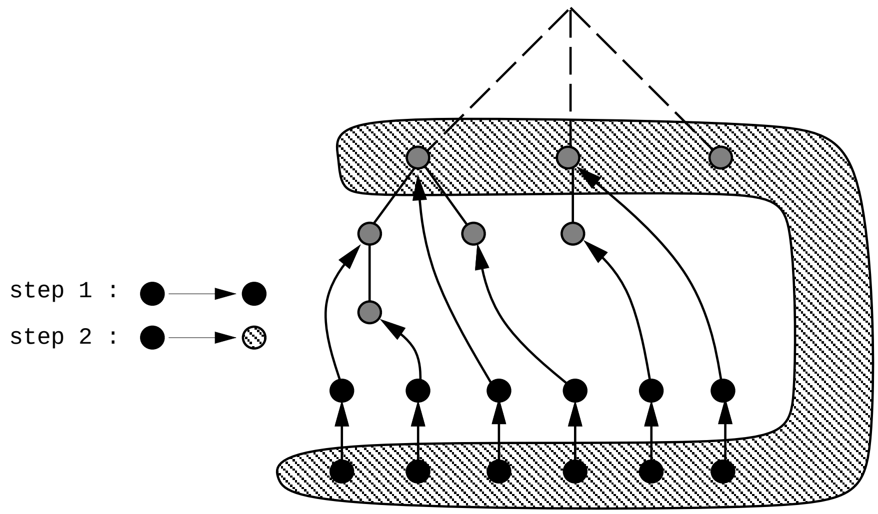

Figure 3: Gossiping in a bus network of 19 vertices with $K(19) = 6$ buses of length at most $K(n) + \lceil \log_2 n \rceil$.

The following theorem summarizes the results obtained in this section:

Theorem 3

$$\begin{aligned} LK(n) &= n \textit{ if } 2^{d-1} + 2^{d-2} \leq n \leq 2^d; \\ LK(n) &= n \textit{ if } 2^{d-1} < n < 2^{d-1} + 2^{d-2} \textit{ and } n \textit{ even;} \\ LK(n) &\geq \sqrt{n} \textit{ if } 2^{d-1} < n < 2^{d-1} + 2^{d-2} \textit{ and } n \textit{ odd.} \end{aligned}$$

Therefore, minimizing the number of bus might be not a good approach to minimize the "hardware" because the length of the bus will be high. So, in the next section, we are interested in minimizing the length, whatever is the number of buses of the obtained gossip bus networks. This approach seems promising because we will prove that $L(n) \ll LK(n)$ for almost any value of n.

5 Minimizing the length of the buses

We are no more interested in the number of buses (this number can be as large as needed), we only want to minimize the length of the buses. Clearly, $L(2) = 2$. Now, from Lemmas 3 and 4, $L(3) = 3$ and $L(4) = 4$ because, in both cases, one needs a bus of length n to broadcast from the unique expert to all the other vertices. If $L(5) = 2$ then each communication step corresponds to at most two matching. In particular, the third step can create at most two experts. Now, two experts implies a bus of length at least 3 at step 4. Thus $L(5) \geq 3$. Now, Figure 1(d) is a gossip bus network of 5 vertices with buses of length no more than 3. The gossip scheme is: (1) $a \to b$ on bus 1, and $d \to e$ on bus 2, (2) $c \to b$ on bus 3, and $e \to d$ on bus 2, (3) $b \to d$ and e on bus 5, (4) $d \to b$ on bus 5, and $e \to a$ and c on bus 4.

As we will see, $L(n)$ is strongly related to the number of experts. Let $e_{\max}(n)$ be the maximum number of experts at time $\lceil \log_2 n \rceil$ that we can obtain during a gossiping algorithm in any bus network of order n. For any positive integers n and e, $0 \leq e \leq e_{\max}(n)$, we denote $\mathcal{L}(n, e)$ the minimum of $L(G)$ over all the bus networks G of order n for which there exists a "multi-gathering" algorithm that allows to obtain at least e experts at time $\lceil \log_2 n \rceil$.

Proposition 2 *For any positive integer n, $L(n) = \min_{e, 1 \leq e \leq e_{\max}(n)} \max(\mathcal{L}(n, e), \lceil \frac{n}{e} \rceil)$.*

Proof. At the last step of a gossiping algorithm in $\lceil \log_2 n \rceil + 1$ steps, experts inform non expert vertices. Assume there are e experts at the end of step $\lceil \log_2 n \rceil$. This can be done using buses of length $\mathcal{L}(n, e)$, and the last step will need buses of length at most $\lceil \frac{n}{e} \rceil$. Thus for any $e, 1 \leq e \leq e_{\max}(n)$, $L(n) \leq \max(\mathcal{L}(n, e), \lceil \frac{n}{e} \rceil)$.

Consider now a gossiping algorithm that performs using buses of length at most $L(n)$. Necessarily some vertices are experts after step $\lceil \log_2 n \rceil$, say there are e experts. Thus $L(n) \geq \mathcal{L}(n, e)$. Moreover, during the last step, these e experts inform the non expert vertices using buses of length at least $\lceil \frac{n}{e} \rceil$). Thus there exists $e, 1 \leq e \leq e_{\max}(n)$, $L(n) \geq \max(\mathcal{L}(n, e), \lceil \frac{n}{e} \rceil)$, and the proof is completed. □

Proposition 3 *For any positive integer n different from a power of 2, let $d = \lceil \log_2 n \rceil$ and p such that $n = 2^d - p$, and let $d' = \lfloor \log_2 p \rfloor$, we get $e_{\max}(n) \geq n - 2^{d-d'} + 2$.*

Proof. We give an algorithm that allows to obtain $n - 2^{d-d'} + 2$ experts in time $\lceil \log_2 n \rceil$.

Let us consider the *binomial tree* (see [12]) of dimension d. Recall that the binomial tree of dimension i, that we denote $\mathrm{BT}(i)$, can be recursively defined by: $\mathrm{BT}(0)$ is reduced to one vertex and $\mathrm{BT}(i)$ is obtained from two $\mathrm{BT}(i-1)$'s by adding an edge between the two roots of the trees, one of the roots becoming the new root. One can label the edges of a binomial tree as follows: the label of $\mathrm{BT}(1)$ is one and, given a labeling of $\mathrm{BT}(i-1)$, the edge between the two roots is labeled i and the two $\mathrm{BT}(i-1)$'s keep the same labeling. See Figure 4.

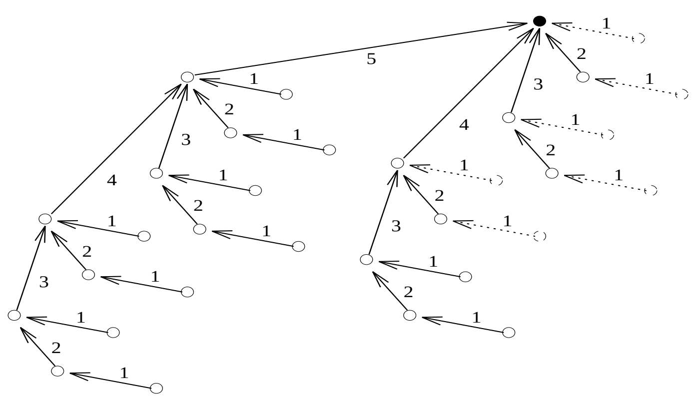

Figure 4: A Binomial Tree of dimension 5 (the dotted lines indicate some particular arcs of BT(5) that we will consider later).

$\mathrm{BT}(d)$ consists in fact of a root r having d disjoint subtrees $\mathrm{BT}(i), i = 0, \ldots, d-1$ attached to r. From $\mathrm{BT}(d)$, we construct a gathering tree of n vertices by deleting all the leaves of the subtrees $\mathrm{BT}(i), i = 0, \ldots, d'$ and $p - 2^{d'}$ leaves of the subtree $\mathrm{BT}(d'+1)$. The tree obtained after these deletions is a gathering tree on n vertices. It consists of a root r having d disjoint subtrees PBT$(i), i = 0, \ldots, d$ (for partial BT) where any PBT$(i), i = 0, \ldots, d'$ has no more edges labeled 1, PBT$(d'+1)$ may have some leaves removed, and PBT(i) =BT(i) for $i = d'+2, \ldots, d$. We modify the labeling of the subtrees PBT$(i), i = 0, \ldots, d'$ by subtracting 1 from all the labels (note that PBT(0) is reduced to $\emptyset$). The tree obtained by this second transformation is still a gathering tree (the labeling is consistent with a gathering). See Figure 5: the 6 right most vertices are removed from BT(5), then one subtracts 1 to all the labels of the 2 left most trees (PBT(0) is reduced to $\emptyset$).

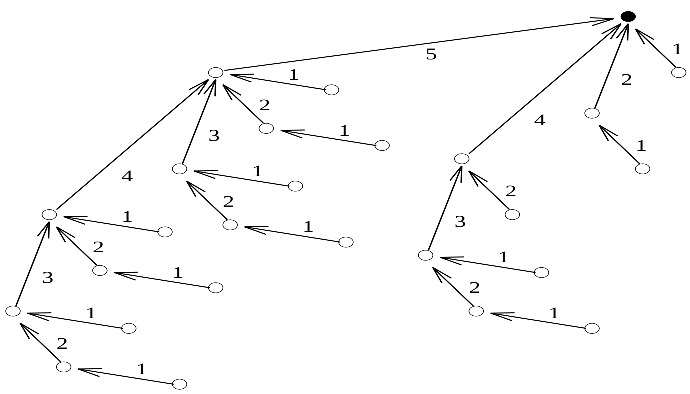

Figure 5: A gathering tree of 26 vertices obtained from BT(5).

Let us show that from this gathering tree, one can deduce a gathering algorithm that allows to obtain $n - 2^{d-d'} + 2$ experts at time $\lceil \log_2 n \rceil$. Steps 1 to d' are directly induced by the gathering tree. After step d' the root is aware of all the messages from vertices of the subtree PBT$(i), i = 0, \ldots, d'$. Moreover, in the gathering tree, the root is idle at step $d'+1$. The root can therefore use this idle time to inform all the vertices that will be idle forever in the gathering tree after step d'. These vertices are composed of two subsets:

- S_1: vertices of the subtrees PBT$(i), i = 0, \ldots, d'$;
- S_2: vertices of the subtrees PBT$(i), i = d'+1, \ldots, d$ that are extremities of edges of label at most d'.

After step $d'+1$, every time that a vertex informs the root r of some information, it also informs the vertices of $S_1 \cup S_2$ by a bus of length at most $|S_1| + |S_2| + 2$. Thus, after d steps, there are $|S_1| + |S_2| + 1$ experts.

There are $2^{d'} - 1$ vertices in S_1. In S_2, there are $2^{d-1} - p$ vertices that are both leaves and extremities of edges labeled 1. Now, for $i \geq d'+1$, if we remove edges labeled by 1, each subtree PBT$(i), i = d'+1, \ldots, d$ becomes respectively a BT$(i-1)$. Thus the number of vertices of S_2 that are extremities of an edge of label 2 and some other edges of label less than 2 is $\frac{1}{2}(2^{d-1} - 2^{d'})$. Similarly, the number of vertices of S_2 that are extremities of an edge of label i and some other edges of label less than i is $\frac{1}{2^{i-1}}(2^{d-1} - 2^{d'})$ for $2 \leq i \leq d'$.

Altogether S_2 contains $n - 2^{d'} - 2^{d-d'} + 2$ vertices, and our gathering allows to obtain $n - 2^{d-d'} + 2$ experts at time $\lceil \log_2 n \rceil$. $\square$

Corollary 4 *For any integer* $n, 2^{d-1} < n \leq 2^{d-1} + 2^{d-2}$, $e_{\max}(n) = n - 2$.

Proof. From Proposition 3, $e_{\max}(n) \geq n-2$, and of course $e_{\max}(n) \leq n-1$. Let us show that $e_{\max}(n) \neq n-1$. Otherwise, there exists a unique non expert vertex x after step d. During step d all the experts must have received because no vertex can be an expert before step d. Thus at step d, x has sent some messages to all the experts.

If x has received informations from other vertices before step d, let y be the vertex that performs the last send to x, and assume this last send happens at step t. Before step d, y is aware of at most $2^{d-1}-2^{t-1}$ messages, and x knows at most 2^{t-1} messages that y does not know. Therefore, after step d, y can only be aware of 2^{d-1} messages and cannot be an expert.

Therefore, x has never received any information from another vertex before step d. Let y and z be two different vertices, both distinct from x and such that y sent some information to z at step $d-1$. Vertex y only knows at most 2^{d-2} messages before step d and can only receive a single message from x during this step. Thus y cannot be an expert after step d, a contradiction. □

Corollary 5 *For any integers $n = 2^d - p$ and e, $1 \leq e \leq n - 2^{d-d'} + 2$ where $d' = \lfloor \log_2 p \rfloor$, we have $\mathcal{L}(n,e) \leq e+1$.*

Proof. Directly follows the construction given in the proof of Proposition 3. □

Theorem 4 *$L(2^d) = 2^d$, $L(2^d-1) = 2^d-1$ and for any integer n different from a power of 2, $L(n) \leq \sqrt{n}+1$.*

Proof. It is clear from Proposition 2 and Lemmas 3 and 4 that $L(2^d) = 2^d$ and $L(2^d-1) = 2^d-1$ respectively. For the other values, we get that

$$\begin{array}{rcl} L(n) & = & \min_{e, 1 \leq e \leq e_{\max}(n)} \max(\mathcal{L}(n,e), \lceil \frac{n}{e} \rceil) \\ & \leq & \min_{e, 1 \leq e \leq e_{\max}(n)} \max(e+1, \frac{n+1}{e}) \\ & \leq & \min_{e, 1 \leq e \leq f_{\max}(n)} \max(e+1, \frac{n+1}{e}) \end{array}$$

where $f_{\max}(n) = n - 2^{d-d'} + 2$ with $n = 2^d - p, 1 \leq p < 2^{d-1}$ and $d' = \lfloor \log_2 p \rfloor$. The value $\max(e+1, \frac{n+1}{e})$ is minimum when $e+1 = \frac{n+1}{e}$, that is $e_{\min}(n) = \frac{-1+\sqrt{1+4(n+1)}}{2}$ and this is acceptable if $e_{\min}(n) \leq f_{\max}(n)$.

Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ such that $\phi(x) = 2^d - 2^{d-x} - 2^{x+1} - 2^{\frac{d}{2}}$. It is easy to see that on the interval $[1, d-2]$, ϕ is minimum in 1 or $d-2$. Now

$$\phi(1) = \phi(d-2) = 2^{d-1} - 2^{\frac{d}{2}} - 4.$$

Therefore, on $[1, d-2]$, $\phi(x) \geq 0$ as soon as $d \geq 4$, and thus

$$\begin{array}{lll} & 2^d - 2^{d-d'} - 2^{d'+1} - 2^{\frac{d}{2}} \geq 0 & \forall d \geq 4 \\ \Rightarrow & n - 2^{d-d'} \geq \sqrt{n} & \forall n > 8 \\ \Rightarrow & f_{\max}(n) \geq \sqrt{n}+2 & \forall n > 8 \\ \Rightarrow & e_{\min}(n) \leq f_{\max}(n) & \forall n > 8 \\ \Rightarrow & L(n) \leq e_{\min}(n)+1 & \forall n > 8 \\ \Rightarrow & L(n) \leq \sqrt{n}+1 & \forall n > 8 \end{array}$$

We have seen at the beginning of the section that that $L(5) = 3$. Let us show that $L(6) = 3$ which will complete the proof. Assume $L(6) = 2$. It means that we can obtain three experts at time 3. Step 3 consists in three matching so that non expert vertices informs expert vertices. Two cases may happen for each matching: either both vertices have 3 messages, or one has 4 messages and the other has 2 messages. None of these combination can happen after 2 steps under the $\mathcal{B}$ model. We let the reader check that it is possible to construct a gossip bus network with 6 vertices and bus of length at most 3. $\square$

6 Conclusion

Table 1 summarizes the general results of this paper, and Table 2 gives the values of K, LK and L for small values of n.

	$2^{d-1} < n < 2^{d-1} + 2^{d-2}$	$2^{d-1} + 2^{d-2} \leq n \leq 2^d - 2$	$n = 2^d - 1$	$n = 2^d$
$K(n)$	$\lceil \frac{n}{2} \rceil - 2^{d-3}$	$n - 2^{d-1}$	$2^{d-1} - 1$	2^{d-1}
$LK(n)$	$\begin{cases} n & \text{if } n \text{ even;} \\ \geq \sqrt{n} & \text{if } n \text{ odd;} \end{cases}$	n	$2^d - 1$	2^d
$L(n)$	$\leq \sqrt{n} + 1$	$\leq \sqrt{n} + 1$	$2^d - 1$	2^d

Table 1: General expressions of K, LK and L for $n \geq 8$.

	2	3	4	5	6	7	8
$K(n)$	1	1	2	2	2	3	4
$LK(n)$	2	3	4	4	6	7	8
$L(n)$	2	3	4	3	3	7	8

Table 2: K, LK and L for $2 \leq n \leq 8$.

This table lets many problems open, in particular concerning $L(n)$. Note that since it is shown in [6, 15, 18] that a lower bound for gossiping under the k-port telegraph model (half-duplex, constant time) is

$$\lceil \log_{\lambda(k)} n \rceil \text{ where } \lambda(k) = \frac{k + \sqrt{k^2 + 4}}{2},$$

we get that $L(n) \neq \Theta(1)$ because the $\mathcal{B}$ model is weaker than the k-port telegraph model.

Finally, we would be interested in knowing the value of

$$KL(n) = \min_{G | L(G) = L(n)} K(G), G \text{ gossip bus network of order } n.$$

As far as we know, these problems are still unsolved (excepted for few cases as for instance $KL(n) = K(n)$ for $n = 2^d$ or $n = 2^d - 1$).

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