Upper bounds for the span in triangular lattice graphs: application to frequency planning for cellular network

Stephane Ubeda, Janez Zerovnik

Laboratoire de l'Informatique du Parallélisme

Ecole Normale Supérieure de Lyon
Unité de recherche associée au CNRS n°1398

Upper bounds for the span in triangular lattice graphs: application to frequency planning for cellular network

Stéphane Ubéda
Janez Žerovnik

Septembre 1997

Research Report N$^{\text{o}}$ 97-28

Ecole Normale Supérieure de Lyon
46 Allée d'Italie, 69364 Lyon Cedex 07, France
Téléphone : (+33) (0)4.72.72.80.00 Télécopieur : (+33) (0)4.72.72.80.80
Adresse électronique : lip@lip.ens−lyon.fr

Upper bounds for the span in triangular lattice graphs: application to frequency planning for cellular network

Stéphane Ubéda
Janez Žerovnik

Septembre 1997

Abstract

We study a problem coming from the design of wireless cellular radio-communication network. Frequency planning constraints are modelled in terms of graph theory.

For each planning function f let us call $sp(f)$ - or the *span* of the frequency planning f - the difference between the largest and the smallest frequency used. Let the *Order* of the graph be $Or(G) = sp(G) + 1$ and the *maximal local order* of the graph the maximum order of a clique of G, i.e. $Mlo(G) = \max_{X \text{ clique of } G} sp(X)$. We show:

$Mlo(G) \leq sp(G) \leq 8\lceil \frac{Mlo(G)}{6} \rceil$.

Keywords: Graph coloring, Frequency planning

Résumé

Ce rapport explore un problème issue de l'allocation de fréquence dans les réseaux de radiocommunication cellulaire. Le problème de planification est décrit à l'aide de la théorie des graphes.

Pour une fonction donnée f de plannification, on appelle le *span* de f - ou $sp(f)$ - la différence entre la plus grande fréquence employée et la plus petite. Nous définissons aussi l'*ordre* du graphe comme étant $Or(G) = sp(G) + 1$ et l'*ordre local maximum* $Mlo(G)$ comme étant l'ordre maximum d'une clique de G, c'est-à-dire $Mlo(G) = \max_{X \text{ clique of } G} sp(X)$. Nous montrons le résultat suivant :

$Mlo(G) \leq sp(G) \leq 8\lceil \frac{Mlo(G)}{6} \rceil$.

Mots-clés: Coloriage de graphe, plannification de fréquences

Upper bounds for the span of triangular lattice graphs: application to frequency planing for cellular networks

Stéphane Ubéda[1] and Janez Žerovnik[2,3]
ubeda@lip.ens-lyon.fr janez.zerovnik@imfm.uni-lj.si

September 16, 1997

(1)LIP ENS-Lyon, 46, Allée d'Italie, F-69364 Lyon Cedex 07, France;
(2)FS, University of Maribor, Smetanova 17, 2000 Maribor, Slovenia.
(3)IMFM, Jadranska 19, 1111 Ljubljana, Slovenia;

1 Introduction

In this paper, we study a problem derived from the graph coloring problem ; the motivation for such a pseudo-coloring problem comes from the design of wireless cellular radiocommunication network.

Wireless telecommunication systems come from various contexts, i.e. military (command systems) or civil (numerical TV, mobile phone, paging systems...), fixed (TV broadcasting) or Mobile (cellular phone), half-duplex or full-duplex. The next generation of wireless telecommunication systems - refered as Universal Mobile Telecommunication System - will provide a wide variety of services combining a wide variety of telecommunication technologies. All this systems share the spectral congestion problem and user capacity management problem.

The cellular concept was a major breakthrough in solving such problems. It is a system level idea which calls for replacing a single, high power transmitter (large cells) with many low power transmitters (small cells). Each transmitter is allocated a portion of the total number of channels available to the entire system.

1.1 Hexagonal model

The conceptual hexagonal model is a model where each cell has a hexagonal shape with the corresponding transmitter in the center of it. This model is simple but it has been universally adopted since it permits easy and manageable analysis of cellular system.

The cellular planification area is now tilled with hexagons. Radiocomunication parameters are mapped onto this hexagonal grid. The usual parameters are *demands* and *interference constraints*. Each cell of the network receives a demand, i.e. the number of frequencies needed to fulfil the forecasted services in this area.

Interference is a major limiting factor in the performance of cellular radio systems. The two major types of system-generated cellular interference are co-channel interference and adjacent channel interference.

Frequency reuse means that in a given coverage area there are several cells that use the same set of frequencies. To reduce co-channel interference, co-channel cells must be physically separated by a minimum distance to provide sufficient isolation due to propagation.

Interference resulting from signals which are adjacent in frequency to the desired signal is call adjacent channel interference. Adjacent channel interference results from imperfect receiver filters which allow nearby frequencies to leak into the passband.

1.2 Generalized coloring

In the hexagonal model the neighbors of a cell are simply the 6 neighboring hexagons in the grid. The resulting neighboring graph $G(V, E)$ is a triangular lattice. An integer $d(v)$ from $[0, D_{max}]$ is assigned to each node $v \in V$ of graph. Parameters are attached to the graph: the co-site interference constraint K_0 and the adjacent intereference set of constraints K_i, where K_i is the interference constraint for pair of vertices at distance i (usually $K_i = 0$ for i after the "reuse distance").

We assume for the rest of the paper that frequencies are taken from the interval $[1, F_{max}]$. Let us call a *planning function* or a *frequency assignment* of this weighted graph $G(V, E, \{K_i\})$ a function $f : V \to \mathcal{P}([1, F_{max}])$, which assigns a subset of frequencies to each vertex of a graph. Planning function must respect the following constraints:

C0) $\forall v \in G, card(f(v)) = d(v)$

C1) $\forall v \in G,\ \forall f_1, f_2 \in f(v),\quad |f_1 - f_2| \geq K_0$

C2) $\forall v, u \in G, distance(v,u) = i,\ \forall f_1, f_2, f_1 \in f(v), f_2 \in f(u),\quad |f_1 - f_2| \geq K_i$

In this paper we will be interested in the case $K_0 = k \geq 1$, $K_1 = 1$ and $\forall i > 1,\ K_i = 0$.

For each planning function f let us call $sp(f)$ - or the *span* of the frequency planning f - the difference between the largest and the smallest frequency used. The goal of the frequency assignment problem is to find the planning function with the minimal span, i.e to find f^* for which $sp(f^*) = sp(G) = \min_f sp(f)$. We also define $Or(G)$, the *Order* of the graph G as $Or(G) = sp(G) + 1$; and $Mlo(G)$, the *maximal local order* of the graph G as the maximum order of a clique of G, i.e. $Mlo(G) = \max_{X \text{ clique of } G} sp(X)$.

Let us denote by ω and χ respectively the clique number and the chromatic number of the graph G. We define the *weighted clique number* ω_W as the maximal sum of the demand of a clique of G. Note that if $\forall v \in G, d(v) = 1$ then $\omega(G) = \omega_W(G)$. We also define the *weighted chromatic number* $\chi_W(G)$ as the chromatic number of the graph G' obtained by the blowup operation. When the demand is not less or equal to one everywhere, the blowup operation consist of expanding each vertex v with $d(v) > 1$ to a clique of size $d(v)$ (if $d(v) = 0$, then v is deleted). Note that $\omega_W(G) = \omega(G')$. Furthermore, for $k = 1$ we get $Or(G) = \chi_W(G)$.

If $\forall v \in G, d(v) = 1$ and $K_1 = K_0 = 1$ $(\forall i > 1,\ K_i = 0)$ we obtain a graph coloring problem on a triangular lattice. For arbitrary demand and $K_1 = K_0 = 1$ we obtain a graph multicoloring problem on a triangular lattice.

In this paper we give tight upper bounds for χ_W in terms of ω_W for case K_0 arbitrary $k \geq 1$, $K_1 = 1$ and $\forall i > 1,\ K_i = 0$. The main result of our paper is that for arbitrary k:

$$Mlo(G) \leq sp(G) \leq 8\lceil \frac{Mlo(G)}{6} \rceil$$

and in case $k = 1$

$$\omega_W(G) \leq \chi_W(G) \leq \lceil \frac{4}{3}\omega_W \rceil$$

For the later, there is a probabilistic proof of the upper bound [3]. A linear distributed algorithm which guarantees the $4\chi/3$ is reported in [2]. In fact their algorithm guarantees the $4\omega/3$. We are not aware of any work on upper bound for general case.

There was a lot of work done on celebrated Philadelphia examples (see [4] and the references there). In this examples there are constraints at distance 2 and 3 (i.e. K_2 and K_3 are not zero). The results on Philadelphias include tight lower bounds while upper bounds were given by planing functions constructed. Therefore no upper bounds were derived as far as we know. Although instances are relatively small (21 cells) they have proved to be extremely difficult. This anticipates the general problem to be very challenging.

In the next section we start with an example and give some definitions and observations. In section 3 we prove a tight upper bound for χ_W and give a linear time algorithm which finds assignment within this bound. In sections 4 and 5 analogous results are given for cases $k = 2$ and $k \geq 3$.

The problem for $K_1 > K_0 > 1$ is still open!

2 Preliminaries

2.1 A simple case: $k = 1$, $\omega = 3$

We start with an ilustrative example. Although simple, the general idea will be used in some other proofs later.

Proposition 1 *If $k = 1$ and $\omega(G) = 3$ then $\chi(G) \leq 4$.*

Proof: Let G be a triangular lattice graph with $\omega = 3$. First we apply the red-blue-green coloring to the graph G. We will show that it is possible to cover all other cliques by using only one additional color.

Clearly, the red-blue-green coloring reduces all demands by one.

It is easy to see that there are no triangles left in the graph. Even more, the graph induced on vertices, which still have positive demand is a graph of isolated vertices.

If demand of a vertex after red-blue-green coloring is 2, then the vertex was an isolated vertex in the original graph. If not, then there was a clique of demand > 3, which is in contradiction with $\omega = 3$. Hence we can color it by the other two colors.

If demand of a vertex after red-blue-green coloring is 1, we can use the fourth color. This can be done, because these vertices are clearly independent. If not, then there were two adjacent vertices with demand 2 in G, which is in contradiction with $\omega = 3$. □

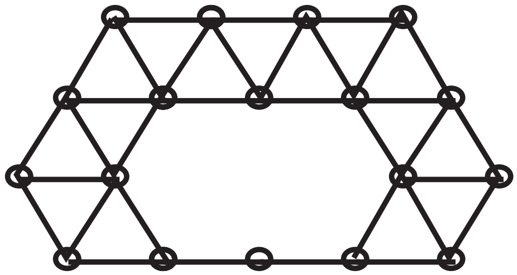


Figure 1: Example of a graph with $\omega = 3$ and $\chi = 4$

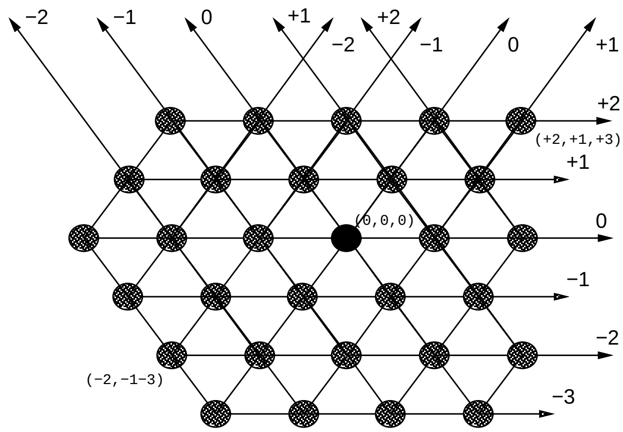


Figure 2: Coordinates

The bound just proved is best possible as the example of Fig. 1 shows.

There is only one vertex with demand 2 in the graph (this is the vertex of degree 2) All other vertices have demand 1. It is easy to see that the maximal clique size of the expanded graph is 3 and that the chromatic number of this graph is 4. □

2.2 Triangular lattice graphs

In this paper we are interested only in graphs which are induced subgraphs of triangular lattice. We therefore can label each vertex of G with coordinates which are defined by embedding in the infinite triangular lattice. (with vertex 0 and orientation in 3 directions given) (see Fig. 2).

Given a graph G we will always assume that there is an embbeding into the infinite triangular lattice given. Equivalently, we will assume that each vertex of G is assigned three coordinates.

(Although 2 coordinates are enough to identify each vertex, we take 3 for symmetry.)

Furthermore, because the infinite triangular lattice has essentialy unique 3 coloring, or in other words its vertex set has a unique partition into three independent sets, we assume this partition is given, and according to this partition each vertex of G is called *red, blue* or *green vertex.*

(Note that it is enough to give each vertex the 3 coordinates and it is possible to derive from this information also in which of the three independent sets the vertex is in.)

Furthermore we call a vertex v of G *odd (even) with respect to coordinate* i, if v's i-th coordinate is an odd (even) number.

2.3 Red-blue-green colorings

For any triple (i, j, k), $i, j, k \geq 0$ we define a (i, j, k) *r-b-g coloring* as follows: assign a set of i colors to each of the red vertices of G, a set of j colors to each of the blue vertices of G and a set of k colors to each of the green vertices of G.

When saying that we have applied a (i, j, k) r-b-g coloring to G, we will assume that we get a new graph with reduced demands. This graph is a subgraph of G, because there may be some vertices, for which the demand was already fulfilled by the r-b-g coloring.

Lemma 1 *Let H be a graph obtained from G after application of (i, j, k) r-b-g coloring, $i + j + k \geq \omega - 2$. There are no triangles in H.*

Proof: if not, then $\omega \geq (i + 1) + (j + 1) + (k + 1) > \omega$, contradiction. $\square$

Definition 1 *A vertex v in H with color $c_1 \in \{r, g, b\}$ is said to be* c-free *($c \neq c_1, c \in \{r, g, b\}$) iff it has no c neighbors in H.*

Lemma 2 *Let H be a graph obtained from G after application of (i, j, k) r-b-g coloring, $i + j + k \geq \omega - 2$. Each vertex of H is either free with respect to two colors, or free with respect to one color or is not free.*

(a) If a vertex is free with respect to two colors, then it is an isolated vertex of H.

(b) The set of c-free vertices is bipartite.

(c) *If a vertex v is not free, then it may have at most two neighbors in H. In this case, the two edges incident to v are on a straight line, i.e. all three vertices differ in the same coordinate.*

(d) *Let K be the graph induced on the vertices which are not free in H. The connected components of K are paths or isolated vertices.*

Proof:

(a) clear.

(b) any such set is colored by the other two colors (as defined at end of 2.2).

(c) Assume v is not free and has two neighbors u and w. We now look at the angle between edges vu and vw. The angle can not be $\frac{\pi}{3}$, because then the vertices u, v, w would induce a triangle in H. The angle can not be $\frac{2\pi}{3}$, as the following argument shows. Assume for a moment the angle is $\frac{2\pi}{3}$. Then u and w are of the same color, say c. Since v is not free with respect to any color this implies that there must be a neighbor of v in H of color different from c. But then this neighbor, the vertex v and u or w induce a triangle in H. Contradiction. The only possibility left is hence angle π.

(d) clear using (c).

□

2.4 Formulas for $Mlo(G)$

Let D_{max} be the maximal demand of a vertex of G.

Denote by Q the set of all cliques of G. These are only triangles, edges or vertices. We assume that every element of Q is a triangle adding "virtual" vertices with demand zero adjacent to isolated vertices and edges.

For $k = 1$, the demand of a clique is simply the sum of demands, since all colors must be distinct and there is no other constraint. Hence

Lemma 3 *If $k = 1$ then,*

$$Mlo(G) = \max\{d_u + d_v + d_w \mid uvw \in Q\}$$

Assume $k = 2$ and let $d_1 \geq d_2 \geq d_3$ be demands on vertices of an arbitrary triangle of Q. Then the order of any planning function for this triangle is at least

$2(d_1 - 1) + 1$ if $d_1 > d_2 + d_3$

or

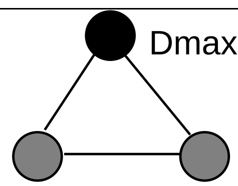
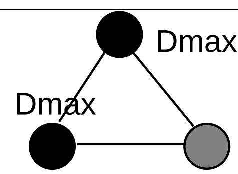
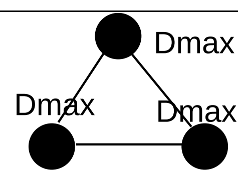
$d_1 + d_2 + d_3$ if $d_1 \leq d_2 + d_3$ (see Fig. 3).

Therefore,

Lemma 4 *If $k = 2$ then,*

$$Mlo(G) = \max\{2D_{max} - 1, \max\{d_u + d_v + d_w \mid uvw \in Q\}\}$$

Let now $k \geq 3$. We compute the maximal order needed to fulfill demand of any triangle. The order of any planning function fullfilling the demand of a triangle only depends on the number of vertices with with demand D_{max} in this triangle:

	$k(D_{max} - 1) + 1$
	$k(D_{max} - 1) + 2$
	$k(D_{max} - 1) + 3$

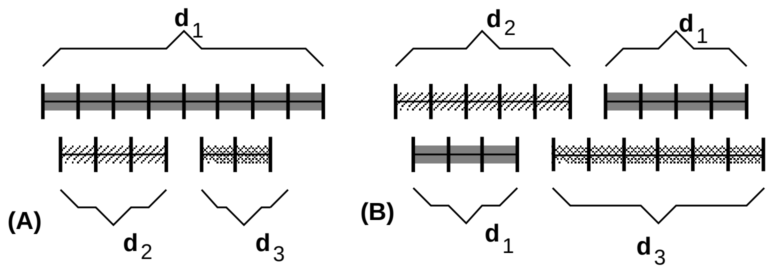


Figure 3: Optimal frequency planning for a triangle, $k=2$.

Lemma 5 *If $k \geq 3$ then,*

$$Mlo(G) = \begin{cases} k(D_{max}-1)+1 & \textit{if all demands } D_{max} \textit{ isolated} \\ k(D_{max}-1)+2 & \textit{if at most two maximal demands adjacent} \\ k(D_{max}-1)+3 & \textit{if there is a triangle with three demands } D_{max} \end{cases}$$

3 Case $k=1$

Recall that for $k=1$, $\omega_W(G) = Mlo(G)$ and $\chi_W(G) = Or(G)$. This is the usual multicoloring of triangular lattice with demand.

3.1 Lower bound

Proposition 2 *If $\omega_W(G) \geq 3$ then $\omega_W(G) \leq \chi_W(G) \leq \lceil \frac{4}{3}\omega_W(G) \rceil$.*

The case $\omega = 1$ is trivial. If $\omega = 2$, then the connected components of G are isolated vertices, paths and cycles. Since odd cycles can be induced subgraphs of triangular lattice, $\chi \leq 3$.

Proof: We give an algorithm which multi-colors any triangular lattice graph with at most $\lceil \frac{4}{3}\omega(G) \rceil$ colors. The algorithm has two steps.

1. step. We apply a $(\omega_1, \omega_2, \omega_3)$ r[ed]-b[lue]-g[reen] coloring to the graph G. where $\omega_1 + \omega_2 + \omega_3 = \omega$. Without loss of generality we can write $\omega_{max} = \omega_1 \geq \omega_2 \geq \omega_3 = \omega_{min}$.

(At first reading, the reader may assume $\omega_1 = \omega_2 = \omega_3$ and $\omega = 3\omega_1$ for simplicity. Or, a little more general: $\lceil \frac{\omega}{3} \rceil = \omega_1 \geq \omega_2 \geq \omega_3 = \lfloor \frac{\omega}{3} \rfloor$.)

We will show that it is possible to cover all other cliques by using only ω_{max} additional colors.

Let H to be the graph obtained after the $(\omega_1, \omega_2, \omega_3)$ r[ed]-b[lue]-g[reen] coloring to the graph G.

By Lemma 1, we know that there is no triangle in H. Furthermore, the vertices of H are either isolated or free with respect to one color or not free. In the second step we show, that in each case the vertex can be colored by at most ω_{max} additional colors.

Step 2.1. Let v be a vertex isolated or with degree one in H. v is free according to at least one color, say c. Let u be a neighbor of v in G with color c and maximal demand (among c-color neighbors), d_u. let us denote the number of colors which were available for u in the r-b-g coloring by ω_u. Then we can color v with $(\omega_u - d_m)$ spare colors of the neighbor u. After that, the demand of the edge uv is at most $\omega - \omega_u - \omega_v \leq \omega_{max}$ and we can fulfill it with ω_{max} additional (white) colors.

Note that since v is isolated in H, there is no possible conflict for using white colors at the rest of H.

Step 2.2. gives coloring of the free vertices which have degre at least 2 in H. It is easy to color these red free vertices by unused red colors, for example as follows. Choose one of the partitions of the set of red vertices, and for any vertex of this partition take as many 'high' red colors as needed. For the other partition of this set, at any vertex, some of the 'low' colors may have been used for red neighbors and some 'high' colors may have been used for red free neighbors.

Fact: there must be enough 'middle' colors to fulfill the demand. (Proof: if not, then we get a contradiction by summing up the demands.)

Step 2.3. Now we color the rest of the graph, i.e. the vertices, which are not free. Denote the graph induced on this set of vertices K.

Since K is a union of paths and isolated vertices, it is bipartite. Take any of the two independent sets and color vertices of it by 'low' white colors. Vertices of other bipartition can clearly be colored by the 'high' white colors. (Proof: if not, sum up the demands and get a contradiction.)

Therefore, all demands were fullfilled by $\omega + \omega_{max}$ colors. $\square$

3.2 Algorithm

All information we need is the following:

-demand of the vertex and of its neighbors

-ω and the "global addresses" of the vertex and its neighbors

The addresses are used for determining which of the sets the vertex is in (in r-b-g coloring of the first step) and second, it can be used to uniquely determine which bipartition of K the vertex is in, i.e. will it receive 'low' or 'high' white colors.

```
if color(v)=red then v receive first I red colors d(v)=d(v)-I endif
if color(v)=green then v receive first J red colors d(v)=d(v)-J endif
if color(v)=blue then v receive first K red colors d(v)=d(v)-K endif
if d(v)<1 then STOP
if v is isolated or c-free with degree 1 then
   let v' be the heaviest neighbor of v in G with color c
   v receives the colors from the c set not used in v' and all
   colors of the w set
else
   if v is a c-free vertex then
        Let c1=color(v) and c2 be the 2 colors of
         the vertex of this c-free componant
        Let c' be the subset of the c set not needed by
         the c neighbor of v
        if c1>c2 then
           v receive the highest color of c' needed to fillful its demand
        else
            v receive the lowest color of c' needed to fillful its deman
        endif
   else
        Let x be the index in which differ from its neighbors
        if x is odd then
           v receive the highest color of w set needed to fillful its demand
        else
           v receive the highest color of w set needed to fillful its demand
        endif
   endif
endif
```

Remark: recall that as soon as $(\omega_1, \omega_2, \omega_3)$ is known frequency planning can be made locally by each cell. If the global ω_W changes it has to be broadcasted to all cells and they can update their frequency plan dynamically. Therefore, if F is the maximal number of available frequencies for the cellular service to be planned, one can set $\omega_1 + \omega_2 + \omega_3$ to $3F/4$. But note that this algorithm guarantees the solution only if $\omega_1 + \omega_2 + \omega_3 \leq 3F/4$.

4 Case $k = 2$

Proposition 3 *Let $k = 2$ and $Mlo(G) \geq 3$. Then $Mlo(G) \leq Or(G) \leq 8\lceil \frac{Mlo(G)}{6} \rceil$*

It will be shown that it is possible to use the same proof as for case $k = 1$, but we must give a more precise definition of color sets. In fact, the construction is even simpler (at step 3) because of

Lemma 6 $D_{max} \leq \lceil \frac{Mlo(G)}{2} \rceil$

The statement follows directly from lemma 4.
Proof: Define the colors in the interval $8\lceil \frac{Mlo}{6} \rceil$ by the following table.

odd numbers	R_1	B_1	G_1	X
even numbers	B_2	G_2	R_2	Y

More precise,

$$
\begin{aligned}
R_1 &= 1, 3, \ldots, 2\lceil \tfrac{Mlo}{6} \rceil - 1 \\
B_2 &= 2, 4, \ldots, 2\lceil \tfrac{Mlo}{6} \rceil \\
B_1 &= 2\lceil \tfrac{Mlo}{6} \rceil + 1, \ldots, 4\lceil \tfrac{Mlo}{6} \rceil - 1 \\
G_2 &= 2\lceil \tfrac{Mlo}{6} \rceil + 2, \ldots, 4\lceil \tfrac{Mlo}{6} \rceil \\
G_1 &= 4\lceil \tfrac{Mlo}{6} \rceil + 1, \ldots, 6\lceil \tfrac{Mlo}{6} \rceil - 1 \\
R_2 &= 4\lceil \tfrac{Mlo}{6} \rceil + 2, \ldots, 6\lceil \tfrac{Mlo}{6} \rceil \\
X &= 6\lceil \tfrac{Mlo}{6} \rceil + 1, \ldots, 8\lceil \tfrac{Mlo}{6} \rceil - 1 \\
Y &= 6\lceil \tfrac{Mlo}{6} \rceil + 2, \ldots, 8\lceil \tfrac{Mlo}{6} \rceil
\end{aligned}
$$

We also define where free vertices will borrow from:

- red borrows from B_1 low and G_2 high
- blue borrows from G_1 low and R_2 high
- green borrows from R_1 low and B_2 high

From the point of view of a vertex lending its colors, we have:

- red reserves R_1 low for greens and R_2 high for blues (and uses from the rest for coloring itself)
- blue reserves B_1 low for reds and B_2 high for greens

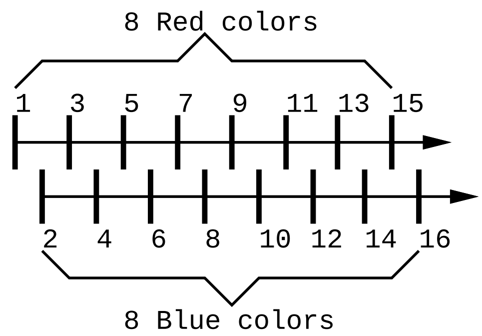


Figure 4: Green vertex v can borrow 8 vertices.

- green reserves G_1 low for blues and G_2 high for reds

Recal from the previous section that every vertex can compute from local information whether it will borrow or lend colors. The vertex also knows how many colors of each type will be given to each neighbor.

Fact: At step 2, after application of r-g-b coloring, there is only demand at most $\lceil \frac{\omega}{6} \rceil$ at any vertex.

(Proof: demand $\leq D_{max} - 2\lceil \frac{\omega}{6} \rceil \leq \lceil \frac{\omega}{2} \rceil - 2\lceil \frac{\omega}{6} \rceil \leq \lceil \frac{\omega}{6} \rceil$, using the lemma 6.)

The last observation implies the following:

(a) colors of X can be assigned to one, and colors of Y to the other partition at step 2.

(b) colors assigned to the same vertex always differ by at least $k = 2$.

This completes the proof of proposition. □

We illustrate case (b) above by the following example. Let v be a green vertex in H. It may borrow from R_1 "low" and B_2 "high". If $\lceil \frac{Mlo}{6} \rceil = 8$, v may want to borrow at most 8 colors. In this case, we see in Fig. 4, that v can borrow without conflict as long as the number of red and blue vertices to be borrowed is not more than 8.

5 Case $k \geq 3$

Assignment

$1, k+1, 2k+1, \ldots, d_{max}k$ to red,

$2, k+2, 2k+2, \ldots, d_{max}k+1$ to blue and
$3, k+3, 2k+3, \ldots, d_{max}k+2$ to green vertices
is always proper.

Therefore, the difference between order $Or(G)$ and $Mlo(G)$ is always 0,1 or 2.

Proposition 4 *Let $k \geq 3$. Then $Mlo(G) \leq Or(G) \leq Mlo(G)+2$*

6 Conclusion

We conclude with a couple of open problems.

Problem I find good upper bounds for $Or(G)$ in term of $Mlo(G)$ for more general K_i. For example:

- $K_0 > K_1 > 1, \forall i \geq 2, K_i = 0$

- $K_0 \geq K_1 \geq K_2 = 1, \forall i \geq 3, K_i = 0$ (some Philadephias fall in this case).

- etc...

Problem II let G be arbitrary k-colorable graph. Is there a bound for $\chi_W(G)$ in terms of $\omega_W(G)$; for example, if $k=3$ we have by simple generalization of our methods $\chi_W(G) \leq \lceil \frac{3}{2}\omega_W(G) \rceil$.

Acknowledgment.
This work was done while the second author was visiting Ecole Normale Supérieure de Lyon, supported by the French ministry of Education and Research. Partially supproted by the Ministry of Science and Technology of Slovenia, grant no. J2-7516-0101-97.

Thanks to Nicolas Schabanel (LIP, ENS-Lyon) for discussion and for suggesting Problem II.

References

[1] J. Heuvel, R.A. Leese, M.A. Shepherd, *Graph Labeling and Radio Channel Assignement*, Manuscript, 1997. (http://www.maths.ox.ac.uk/users/gowerr/preprints.html)

[2] L.Narayanan, S.Shende, *Static Frequency Assignment in Cellular Networks*, manuscript 1997. (presented at SIROCCO 97, Ascona, Switzerland.)

[3] B.Reed, personal communication.

[4] D.H. Smith, S. Hurley, *Bounds for the Frequency Assignment Problem*, Discrete Mathematics 167/168, (1997) 571-582.