Listing all the minimal separators of a planar graph.

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Abstract

I present an efficient algorithm which lists the minimal separators of a planar graph in $O(n)$ per separator.

Keywords: planar graphs, minimal separator enumeration*

Résumé

Je présente un algorithme d'énumération des séparateurs minimaux des graphes planaires dont la complexité est $O(n)$ par séparateur.

Mots-clés: graphes planaires, séparateurs minimaux, énumération

1 Introduction

In this paper, we address the problem of finding the minimal separators of a connected planar graph G.

In the last ten years, minimal separators have been an increasingly used tool in graph theory with many algorithmic applications (for example [4], [7], [8], [10]).

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [4], Bodlaender and al. conjecture that for a class of graphs which have a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the notion of potential maximal clique (see [2]) and showed that if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can indeed be solved in polynomial time. They later showed in [3] that if a graph has a polynomial number of minimal separators, then it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Some research has been done to compute the set of the minimal separators of a graph ([1], [5], [6],[9]). In [1], Berry and al. proposed an algorithm of running time $O(n^3)$ per separator which uses the idea of generating a new minimal separator from an older one S by looking at the separator $S \cup N(x)$ for $x \in S$. This separator is not minimal but the neighbourhoods of the connected components it defines are. This simple process can generate all the minimal separators of a graph. The counterpart is that a minimal separator can be generated many times.

In this paper, we adapt this idea to connected planar graphs but to avoid the problem of recalculation, we define the set $\mathcal{S}_{a,B}(S,O)$ of the a,b-minimal separators S' for some $b \in B$ that are such that the connected component of a in $G \backslash S'$ contains the connected component of a in $G \backslash S$ but avoids the set O. This way we put restrictions on the minimal separators we compute to ensure we do not compute the same minimal separator over and over.

2 Definitions

Throughout this paper, $G = (V,E)$ will be a connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x,y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x,y) \in E\}$.

A set $S \subseteq V$ is an a,b *minimal separator* if a and b are in two distinct connected components of $G \backslash S$ and no proper subset of S separates them. $\mathcal{S}_{a,b}$ is the set of the a,b-minimal separators and $\mathcal{S}_{a,B}$ the set $\cup_{b \in B} \mathcal{S}_{a,b}$. An $a,*$-*minimal separator* is an element of $\mathcal{S}_{a,V}$. We will abbreviate $\mathcal{S}_{a,V}$ in $\mathcal{S}_a$. The set $\mathcal{S}_a^\sharp$ is the set of the minimal separators that run through a and $\mathcal{S}_a^{\neg\sharp}$ is the set of the minimal separators that does not. $C_a(S)$ is the connected component of a in $G \backslash S$. The component $C_a(S)$ *is a full connected component* if $N(C_a(S)) = S$. A set S is a *minimal separator* if there exists a and b which make it an a,b-minimal separator or, which is equivalent, if it has at least two full connected components. $\mathcal{S}$ is the set of the minimal separators of G.

We can order the $a,*$-minimal separators in the following way:

$$S_1 \preccurlyeq S_2 \quad \text{if} \quad C_a(S_1) \subseteq C_a(S_2).$$

For S an $a,*$-minimal separator $B \subseteq V$ and $O \subseteq V$, the set $\mathcal{S}_{a,B}(S,O)$ is the set of the a,B-minimal separators S' such that $S \preccurlyeq S'$ and $O \cap C_a(S') = \emptyset$. And if $x \in V$, the set $\mathcal{S}^x_{a,B}(S,O)$ is *the set of* $S' \in \mathcal{S}_{a,B}(S,O)$ such that $x \in C_a(S')$.

The set of the vertices b such that an $a,*$-minimal separator S is a a,b-minimal separator is the *identity of* S denoted by B_S.

We want to compute the set $\mathcal{S}_a$. To do so, we will decompose it recursively in a union of sets $\mathcal{S}_{a,B}(S,O)$. The following remarks and lemmas give us an obvious way to do it.

Property 1 *Let S be an $a,*$-minimal separator, $x \in S$, $O \subseteq V$ and S_i be the neighbourhood of the connected components of $G \backslash \big(N(C_a(S) \cup \{x\})\big)$.*

The sets S_i are $a,*$-minimal separators. We have the disjoint unions:

$$\mathcal{S}_{a,B_S}(S,O) = \mathcal{S}_{a,B_S}(S,O\cup\{x\}) \bigsqcup \mathcal{S}^x_{a,B_S}(S,O) \text{ and}$$
$$\mathcal{S}^x_{a,B_S}(S,O) = \bigsqcup_{i\in I} \mathcal{S}_{a,B_{S_i}}(S_i,O).$$

Proof. The first equality is obvious.

Let us prove the second one.
The set $T = \bigcup_{i\in I} \mathcal{S}_{a,B_{S_i}}(S_i,O)$ is clearly a subset of $\mathcal{S}^x_{a,B_S}(S,O)$.

Let $S_1 \in \mathcal{S}^x_{a,B_S}(S,O)$ and $b \in B_S$ such that S_1 is an a,b-minimal separator. Let C be the connected component of b in $G\backslash\big(N(C_a(S)\cup\{x\})\big)$. The neighbour S_2 of C is a minimal separator. Indeed, C is a full component for S and $C_a(S_2)$ contains $C_a(S)$ and x which implies that $C_a(S_2)$ is also a full component. Clearly, $C_b(S_1) \subseteq C_b(S_2)$ so $S_2 \preccurlyeq S_1$. We have $S_1 \in \mathcal{S}_{a,B_{S_2}}(S_2,O)$ and $S_1 \in T$ which proves that $\mathcal{S}^x_{a,B_S}(S,O)$ is a subset of T.

Now let us prove that the union is disjoint.
The sets B_i and B_j are disjoint ($i \neq j$). Otherwise, let $b \in B_i \cap B_j$ and C be the connected component of b in $G\backslash\big(N(C_a(S)\cup\{x\})\big)$. By definition $S_i = N(C) = S_j$ which is absurd.
Let $S' \in \mathcal{S}_{a,B_{S_i}}(S_i,O)$. $B_{S'} \subseteq B_{S_i}$ which proves that $\mathcal{S}_{a,B_{S_i}}(S_i,O)$ and $\mathcal{S}_{a,B_{S_j}}(S_j,O)$ are disjoint for otherwise $B_i \cap B_j \neq \emptyset$.
This proves that the second union is disjoint. □
Property 1 proves that the following algorithm is correct.

ALGORITHM: `_calc2_`
input:
- $G=(V,E)$ a graph
- a a vertex of G
- S an $a,*$-minimal separator
- O a subset of S

output:
- $\mathcal{S}_{a,B}(S,O)$

```
begin
    if S\O = ∅ then
        return({S})
    else
        B ←_calc_B(G, a, S)
        let x ∈ S\O
        𝒮 ←_calc2_(G, a, B, S, O ∪ {x})

        for each S_i in find_min_elements(G, a, x, S, O)
            B_i ←_calc_B(G, a, S_i)
            𝒮 ← 𝒮∪_calc2_(G, a, B_i, S_i, O)
    return(𝒮)
end
```

To compute `find_min_elements` and `_calc_B`, we can use a graph search but if $\mathcal{S}^x_{a,B}(S,O)$ is empty, we still need the graph search. And in the worst case, all these sets are empty which leads to a running time of $O(nm/\text{separator})$.

3 Planar graphs

In a planar graph, $m = O(n)$. The running time of this algorithm is $O(n^2/\text{separator})$.

We will now prove that the complexity of finding the $a, *$-minimal separators of a planar graph is $O(n/\text{separator})$ and that the complexity of finding all the minimal separators of a planar graph is $O(n/\text{separator})$.

Let Σ be the plane. A *plane graph* $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of Σ between two vertices of V_Σ in such a way that the interiors of two distinct edges do not meet. We will denote by $\widetilde{G}_\Sigma$ the drawing of G_Σ. A *planar graph* is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of G_Σ is a connected component of $\Sigma \backslash \widetilde{G}_\Sigma$.

3.1 Minimal separators of 2-connected planar graphs

Property 2 *In a planar graph, if S and S' are minimal separators and $S \subset S'$, then $|S| \leq 2$.*

Proof. Suppose that $S \subset S'$ are two minimal separators of a planar graph and $|S| > 2$.

Let a, b, c and d be vertices such that S' is an a, b-minimal separator and S is a c, d-minimal separator. Since S is not an a, b-minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. $C_c(S) = C_c(S')$ and $N(C_c(S)) = S$.

But then G admits $K_{3,3}$ as a minor for if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices a', b' and c', all these vertices have S in their neighbourhood and $|S| \geq 3$. This contradicts the fact that G is planar. □

We say that a curve μ of Σ is G_Σ *nice* if $\mu \cap G_\Sigma \subseteq V_\Sigma$.

Property 3 *Let μ be a G_Σ nice lace that separates at least two vertices a and b of V_Σ.*
The set $V(\mu)$ is an a, b-separator of G_Σ.

Proof. Let p be a path in G_Σ from a to b. Since a and b are not in the same connected component of $\Sigma \backslash \mu$, $\widetilde{p}$ intersects μ. By construction, $\widetilde{p} \cap \mu \subseteq V_\Sigma$. This implies that every path from a to b meets $V(\mu)$ and so $V(\mu)$ is an a, b-separator. □

Property 4 *Let S be an a, b-minimal separator of G. There exists a G_Σ nice Jordan lace μ that separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.*

Proof. Let C be the connected component of a in $G \backslash S$. Contract C into a super-vertex v_C to build the graph $G_{/C}$. There is a $(G_{/C})_\Sigma$ nice lace $\mu_{/C}$ which separates v_C and the other vertices of $(G_{/C})_\Sigma$ and such that $V(\mu_{/C}) = N(v_C)$.

Suppose that μ is not a Jordan lace. Let μ' be the border of the connected component of b in $\Sigma \backslash \mu$. The curve μ' is a sub-lace of μ and is the border of two simply-connected components of $\Sigma \backslash \mu'$ (the one containing v_C and b) so μ' is a Jordan lace.

In the graph G_Σ, μ' corresponds to a Jordan lace that separates a and b and such that $V(\mu) = S$. □

Property 4 shows that the minimal separators of a planar graph G can be seen as a G_Σ nice Jordan laces. We can obtain from this point of view an exact criteria for the minimal separators of a 2-connected planar graph.

3.2 Ordered separators

Definition 1 *An ordered separator of G is a sequence of distinct vertices*

$$(v_0, \ldots, v_{p-1}), \qquad (p > 1) \text{ such that}$$

i. there exists a face to which v_i and $v_{i+1\,[p]}$ are both incident;

ii. v_i and v_j are incident to a common face only if $i = j + 1\,[p]$ or $j = i + 1\,[p]$;

iii. *there is no face incident to v_i, $v_{i+1\,[\,p\,]}$ and $v_{i+2\,[\,p\,]}$.*

iv. *if $p = 2$ then there exists 2 distinct faces f_1 and f_2 incident to both v_0 and v_1 such that if $(v_0, v_1) \in E$, either f_1 or f_2 is not incident to (v_0, v_1).*

The notation $i\,[\,p\,]$ means i modulo p.

We say that a set $S = \{v_0, \dots, v_{p-1}\}$ is an ordered separator if there exists a permutation σ such that $(v_{\sigma(0)}, \dots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = (v_0, \dots, v_{p-1})$ is an ordered separator of G, then S is naturally associated to the set $\{v_0, \dots, v_{p-1}\}$. We will either use an ordered separator as a sequence or as the corresponding set.

Remark 1 *If $p > 3$, the third condition is a corollary of the second for v_i et $v_{i+2\,[\,p\,]}$ would be too far apart.*

Remark 2 *If S is an ordered separator and S' is a sub-ordered separator of S, then $|S'| = 2$.*

Lemma 1 *Every minimal separator S of G is ordered.*

Proof. Let S be an a, b-minimal separator of G.

The property 4 states that there exists a G_Σ nice Jordan lace μ that separates a and b and such that $V(\mu) = S$. Let $v_0, \dots, v_{p-1}$ be the vertices through which μ goes. We know that $S = \{v_0, \dots, v_{p-1}\}$.

Let us prove that $T = (v_0, \dots, v_{p-1})$ is an ordered separator corresponding to S.

i. Since μ goes from v_i to v_{i+1}, without going through another vertex, v_i and v_{i+1} are incident to a common face.

ii. Suppose that v_i et v_j are incident to a common face f and that $i+1 \neq j\,[\,p\,]$ and $j+1 \neq i\,[\,p\,]$. There is a curve ν from v_j and v_i. Let μ_1 and μ_2 be the two sub-laces of μ from v_i and v_j.

   $\mu_1.\nu$ and $\mu_2.\nu$ are G_Σ nice laces. Moreover, since either μ_1 or μ_2 separates a and b, property 3 states that there exists an a, b-separator strictly included in S which is absurd.

iii. With the remark 1, we can suppose that $p = 3$.

   Suppose that v_0, v_1 et v_2 are all incident to a common face f. If we add a vertex f to G that we connect to the vertices v_0, v_1 and v_2, the graph remains planar which is absurd for this graph has $K_{3,3}$ as a minor. Indeed, the connected component of a, the connected component of b and the vertex f are all incident to v_0, v_1 and v_2 which builds up a $K_{3,3}$.

iv. Suppose that $|S| = 2$ and (v_0, v_1) is an edge of G. Since μ separates a and b, μ cannot go through the faces incident to (v_0, v_1).

The sequence T is an ordered separator as required. □

Conversely,

Lemma 2 *Every ordered separator of G is a minimal separator of G.*

Proof. Let $S = (v_0, \dots, v_{p-1})$ be an ordered separator of G.

First, S is a separator. Otherwise

- if $p > 2$, $G \backslash S$ would be connected or empty. In both cases all the vertices of S would be incident to a common face;
- if $p = 2$, and v_0 and v_1 are both incident to two distinct faces f_1 and f_2 then (v_0, v_1) is an edge of G and f_1 and f_2 are incident to (v_0, v_1) which contradicts the definition of S.

By induction on the number k of connected components of $G \backslash S$.

- if $k = 2$. Suppose that S is not a minimal separator. Then at least one of the connected components of $G\backslash S$ (C) has a neighbourhood which is not S. If $|N(C)| = 2$, then S is also an ordered separator of $G\backslash C$ which is absurd for S must be a separator of $G\backslash C$. If $|N(C)| > 2$, the neighbourhood of C is a sub-ordered separator of S which is also impossible.

- if $k > 2$, let S' be a minimal separator included in S. Either $S' = S$ and we are done, or S' is a sub-ordered separator of S which implies that $|S'| = 2$. Let C be a connected component of $G\backslash S$ with a $|N(C)| = 2$. Since $|N(C)| = 2$, S is also an ordered separator of $G\backslash C$ and by induction, S is a minimal separator of $G\backslash C$ and thus, a minimal separator of G.

□

From lemma 1 and 2, we have he following property:

Property 5 *A set $S \subseteq V$ is a minimal separator of a 2-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of G.*

4 Listing the $a, *$-minimal separators of a 2-connected planar graph

At this point, we have a characterisation of the minimal separators of a 2-connected planar graph. Let us see how it enables us to find out whether $\mathcal{S}_a^x(S, O)$ is empty or not when $O \subseteq S$ and $x \in S\backslash O$.

The *landing site* of an element x of an ordered separator S with $O \subseteq S$ and $x \notin O$ is the subsequence $l_x(S, O) = (v_i, \ldots, v_j)$ of S containing x and such that $v_k \in O$ $(i \le k \le j)$ if and only if $k = i$ or $k = j$.

The following lemma gives a necessary condition for $\mathcal{S}_{a,B_S}^x(S, O)$ to be non-empty.

Lemma 3 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of a 2-connected planar graph $G = (V, E)$.*

Let O be a subset of S and $v_i \notin O$.

If there exists a face which is incident to both $y \in N(v_i)\backslash C_a(S)$ and $v_j \notin l_{v_i}(S, O)$, then $\mathcal{S}_{a,B_S}^{v_i}(S, O) = \emptyset$.

Proof. Let μ be a G_Σ nice Jordan lace that corresponds to S.

Suppose that y and $v_l \notin l_{v_i}(S, O)$ are incident to a common face f. This hypothesis implies that there exists a G_Σ nice curve ν such that $V(\nu) = \{v_i, y, v_l\}$.

Suppose for a contradiction that S' is a minimal element of $\mathcal{S}_{a,B_S}^{v_i}(S, O)$. Let b be such that S' is an a, b-minimal separator and μ' be the G_Σ nice Jordan lace corresponding to S'.

Since S' is a subset of $(S\backslash\{v_i\}) \cup (N(v_i)\backslash C_a(S))$, we can suppose that μ does not intersect the connected component of a in $\Sigma\backslash\mu$. But then, since v_l is not in the landing site of v_i, μ' must cross ν and there is a G_Σ nice Jordan lace μ'' in $\mu' \cup \nu$ that separates a and b. By construction $V(\mu'') \subset V(\mu')$ which contradicts the fact that μ' is an a, b-minimal separator. □

We can now prove the theorem

Theorem 1 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of a 2-connected planar graph $G = (V, E)$, O be a subset of S and $v_i \notin O$.*

The set $\mathcal{S}_{a,B_S}^{v_i}(S, O)$ is not empty if and only if

i. there is no face incident to both $y \in N(v_i) \cap B_S$ and $v_j \notin l_{v_i}(S, O)$;

ii. there exists $v \in N(O) \cap B_S$ which is not a neighbour of v_i.

Proof. Suppose that $\mathcal{S}_{a,B_S}^{v_i}(S, O)$ is not empty and that S' is an a, b-minimal separator of $\mathcal{S}_{a,B_S}^{v_i}(S, O)$ which if minimal.

Lemma 3 proves that condition i is satisfied.

Since S' is the neighbour of the connected component of b in $G\backslash\big(S \cup N(v_i)\big)$, and $O \subseteq S'$, condition ii is also satisfied.

Suppose now that i and ii are true.

Number the neighbours $(y_1, \dots, y_l)$ of v_i in B_S in clockwise order. Suppose that S is numbered in such a way that y_1 and v_{i-1} (resp. y_l and v_{i+1}) are incident to a common face. Let v_m and v_n be the vertices of $l_{v_i}(S, O)$ which are also in O (v_n and v_m can be equal).

Since the vertices y_i and y_{i+1} are incident to a common face. There exists a sequence $P = (v_n, x_0, \dots, x_k, v_m)$ in $\big(S\backslash\{v_i\}\big) \cup \{y_1, \dots, y_l\}$ such that x_i and x_{i+1} are incident to a common face. Take for example $(v_n, \dots, v_{i-1}, y_1, \dots, y_l, v_{i+1}, \dots, v_m)$.

Let P be such a sequence between v_n and v_m of minimal length. Together with $(v_m, \dots, v_n)$, we claim that P forms an ordered separator T of G.

- By construction, the first condition of an ordered separator is satisfied;
- Since no face is incident to both y_k and $l_{v_i}(S, O)$ and since P is minimal, the second condition of an ordered separator is satisfied;
- Suppose that $|T| = 3$ and there exists a face which is incident to all the elements of P. Then all the vertices of $N(O) \cap B_S$ are also neighbours of v_i which is absurd;
- If $|T| = 2$, then $P = (x_0, x_1)$ with $O = \{x_0\}$ and $x_1 \in N(v_i)$.

  Since there exists a vertex $z \in N(O) \cap B_S$ which is not a neighbour of v_i, P is not an edge of G which proves that the fourth condition of an ordered separator is satisfied.

The minimal separator T is clearly an a, B_S-minimal separator. □

4.1 An algorithm

Now we have all we need to build up an algorithm to compute the set $\mathcal{S}_{a,B_S}(S, O)$ with $O \subseteq S$.

ALGORITHM: `_calc2_`
input:
G a 2-connected planar graph
a a vertex of G
$S = (v_0, \dots, v_{p-1})$ an ordered $a, *$-minimal separator
O a subset of S
The landing sites of S are tagged i
The faces incident to a vertex not in the landing site i are tagged i
The vertices of $C_a(S)$ are also tagged "$C_a(S)$".
output:
$\mathcal{S}_{a,B_S}(S, O)$

begin
  if $O = S$ **then**
    return($\{S\}$)
  else
    let $v \in O$ be in a landing site i
    let $x \in S\backslash O$ be next to v on S
    tag if necessary the faces incident to v with i and v
    $\mathcal{S} \leftarrow$`_calc2_`$(G, a, S, O \cup \{x\})$
    untag if necessary the faces incident to v

    for each $y \in N(x)$ not tagged "$C_a(S)$"
      if y is tagged i **then**
        return($\mathcal{S}$)

if $O = \{v\}$ and there is no neighbour of v in $B_S \backslash N(x)$ **then**
  return($\mathcal{S}$)
for each S' in `find_min_elements`(G, a, x, S, O)
  $\mathcal{S} \leftarrow \mathcal{S} \cup$ `_calc2_`$(G, a, S', (v_0, \ldots, v_i))$
end

Property 6 *The algorithm* `_calc2_` *is correct. It computes the set $\mathcal{S}_{a,B_S}(S, O)$ of a 2-connected planar graph.*

Proof. The algorithm is just an application of property 1. □

Property 7 *The algorithm can be implemented to compute the set $\mathcal{S}_{a,B_S}(S, O)$ in time $O(n|\mathcal{S}_{a,B_S}(S, O)|)$.*

Proof. For each minimal separator S, the algorithm does the following:

i. the function `find_min_elements` produces S;

ii. for every $x \in S \backslash O$, there is a recursive call to `_calc2_` to extend the set O;

iii. S is returned.

The function `find_min_elements` does a graph search to compute the sets S_i, and to tag the vertices in $C_a(S_i)$. It orders S_i, tags the landings sites and the faces incident to S_i. In a planar graph, the number m of edges satisfies $0 \leq m \leq 3n - 6$, so all this costs $O(n)$.

Each call to `_calc2_` costs $O(d(x))$ to tag and untag the faces incident to x, and $O(d(x))$ to check whether $\mathcal{S}^x_a(S, O)$ is empty or not. Since every time a different x is chosen, the recursive calls to `_calc2_` cost $O(n)$.

The overall complexity of function `_calc2_` is $O(n|\mathcal{S}_a(S, O)|)$. □

The algorithm `_calc2_` does a kind of depth first search. We can use a variant that does a breadth first search which can be implemented using a queue.

The set of all the $a, *$-minimal separators of G is equal to $\cup_{i \in I} \mathcal{S}_{a,B_{S_i}}(S_i, \emptyset)$ for S_i, the minimal separators included in $N(a)$. The running time of an algorithm `calc_a` using `_calc2_` to list all the $a, *$-minimal separator of a 2-connected planar graph is $O(n/\text{separator})$.

From now on, a will be a vertex of degree at most five of G.

4.2 Listing the minimal separators that run through a

A minimal separator that runs through a is a $b, *$-minimal separator for $b \in N(a)$. So the set $\mathcal{S}^\sharp_a$ of the minimal separators of G that run through a is equal to $\cup_{b \in N(a)} \mathcal{S}_{b,V}$. With at most five run of `calc_a`, we can list the elements of $\mathcal{S}^\sharp_a$. And since a separator of $\mathcal{S}^\sharp_a$ can be computed at most five times, the running time of this algorithm `calc_cross` is $O(n/\text{separator})$.

5 Listing the minimal separators of a planar graph

It is easy to see that if $(G_i)_{i \in I}$ are the 2-connected components of a graph $G = (V, E)$, then

$$\mathcal{S}(G) = \{v \mid v \text{ is a cut-vertex of } G\} \cup \bigcup_{i \in I} \mathcal{S}(G_i).$$

Since all the cut-vertices and the 2-connected components of a graph can be computed in $O(n + m)$, we can consider 2-connected planar graphs.

Property 8 *If S is an $a, *$-minimal separator of size two which is minimal for $\preccurlyeq$, and S' is a minimal separator such that $a \notin S'$ and such that S' intersects both $C_a(S)$ and another connected component of $G \backslash S$. Such a minimal separator* cuts *S.*

Then S' is an $a, *$-minimal separator.

Proof. If S' is not an $a,*$-minimal separator, then the neighbourhood of $C_a(S')$ is a minimal separator S'' included in S'.

Since S is minimal, S'' is not included in $C_a(S)$ but since S' intersects $C_a(S)$, $S'' \cap C_a(S) \neq \emptyset$. Let b be the vertex in $S'' \cap C_a(S)$. b is a cut-vertex of $C_a(S)$ and the neighbourhood of $C_a(S \cup \{b\})$ is a separator of G of size two. Since G is 2-connected, it is an $a,*$-minimal separator of size two which is smaller than S. This is absurd. So S' is an $a,*$-minimal separator. □

An $a,*$-minimal separator of size two which is minimal for $\preccurlyeq$ is an *a-critical separator.*

Let S be an a-critical separator, G^a_S is the graph $G[C_a(S)]$ with an edge between the vertices of S, $G^{\neg a}_S$ is the graph G with the connected component $C_a(S)$ replaced by an edge between the vertices of S and $G^\sharp_S$ is the graph G with a new vertex v_S connected to the vertices of S.

Property 9 *Let S be an a-critical separator of G. We have the disjoint union:*

$$\mathcal{S}^{\neg\sharp}_a(G) = \{S\} \sqcup \mathcal{S}^{\neg\sharp}_a(G^a_S) \sqcup \mathcal{S}(G^{\neg a}_S) \sqcup \{S' \; a,*\textit{-minimal separator} \mid S' \textit{ cuts } S\}.$$

Proof. Let S' be a minimal separator that avoids a.

By construction, the union is clearly disjoint.

Since G^a_S (resp. $G^{\neg a}_S$) is obtained from G by contracting a connected component into a super-vertex (of size two), any minimal separator of G^a_S (resp. $G^{\neg a}_S$) is a minimal separator of G. So we have:

$$\{S\} \sqcup \mathcal{S}^{\neg\sharp}_a(G^a_S) \sqcup \mathcal{S}(G^{\neg a}_S) \sqcup \{S' \; a,*\text{-minimal separator} \mid S' \text{ cuts } S\} \subseteq \mathcal{S}^\sharp_a(G).$$

Conversely,

i. if $S' \subset C_a(S) \cup S$ and $S' \neq S$, let b be such that S' is an a,b-minimal separator. If $b \in C_a(S)$, then S' is a minimal separator of G^a_S. Otherwise, since $S' \neq S$, b is in the same connected component of $G \backslash S'$ as one of the vertices of S and S' is a minimal separator of G^a_S;

ii. if S' cuts S, then property 8 proves that it is an $a,*$-minimal separator;

iii. if $S' \cap C_a(S) = \emptyset$, then S' is a minimal separator of $G^{\neg a}_S$. □

Before we describe the algorithm that lists the minimal separators of G, we can remark that the set of the $a,*$-minimal separators that cut S is the set $\{S' \backslash \{v_S\} \mid S' \in \mathcal{S}_a(G^\sharp_S) \text{ and } v_S \in S'\}$.

The algorithm that lists the minimal separators of a 2-connected planar graph does the following:

- Find a of degree at most 5;
- Run the algorithm `calc_cross`(G, a);
- Run the breadth first search variant of the algorithm `calc_2` on a and each time a new minimal separator S is found
  - Check if $|S| = 2$;
  - Check if S cuts a minimal separator S' of size two.

  The first time a minimal separator of size two is found, it is a critical separator
- compute $G^{\neg a}_S$ and run `calc`$(G^{\neg a}_S)$;
- for each couple (S', O) still in the queue,
  - if S' cuts S, then run `_calc2_`$(G^\sharp_S, S \cup \{v_S\}, O \cup \{v_S\})$
  - if not, then continue the breadth first search but on the graph G^a_S.

Property 10 *The algorithm `calc` is correct.*

Proof. It lists the elements of $\mathcal{S}_a^\sharp(G)$ and $\mathcal{S}_a^{\neg\sharp}$.

To list the elements of $\mathcal{S}_a^{\neg\sharp}$, it uses property 9.

All the minimal separators produced by the breadth first search variant of `_calc2_` are minimal separators of $\mathcal{S}_a^{\neg\sharp}(G_S^a)$ and once a critical separator is found, it goes on with the listing of the elements of $\mathcal{S}_a^{\neg\sharp}(G_S^a)$, $\mathcal{S}(G_S^{\neg a})$ and the $a, *$-minimal separators that cut S.

Property 11 *The running time of* `calc` *is* $O(n/\mathit{separator})$.

Proof. The running time of `calc_cross` is $O(n/\text{separator})$.

By induction on the size of G, the graphs G_S^a and $G_S^{\neg a}$ are smaller than G so by induction hypothesis, the listing of the elements of $\mathcal{S}_a^{\neg\sharp}(G_S^a)$ and $\mathcal{S}(G_S^{\neg a})$ takes $O(n/\text{separator})$. The graph $G_S^\sharp$ is bigger than G but there are at most n critical separators in G for a so the total running time is $O(n/\text{separator})$.

6 Conclusion

In the conclusion of [1], Berry and al. note that their algorithm may compute a minimal separator up to n times and that this could be improved. This paper confirms this feeling for this is exactly what we have gained for planar graphs. We feel, just like Berry and al., that there could be a better general algorithm to compute the minimal separators of a graph.

This paper gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as property 4 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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