Listing all the minimal separators of a 3-connected planar graph.

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Abstract

I present an efficient algorithm which lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Keywords: 3-connected planar graphs, minimal separator enumeration*

Résumé

Je présente un algorithme d'énumération des séparateurs minimaux des graphes planaires 3-connexes dont la complexité est $O(n)$ par séparateur.

Mots-clés: graphes planaires 3 connexes, séparateurs minimaux, énumération

1 Introduction

In this paper, we address the problem of finding the minimal separators of a 3-connected planar graph G.

In the last ten years, minimal separators have been an increasingly used tool in graph theory with many algorithmic applications (for example [4], [7], [8], [10]).

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [4], Bodlaender and al. conjecture that for a class of graph which a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the notion of potential maximal clique (see [2]) and showed that if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can indeed be solved in polynomial time. They later showed in [3] that if a graph has a polynomial number of minimal separators, then it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Some research has been done to compute the set of the minimal separators of a graph ([1], [5], [6],[9]). In [1], Berry and al. proposed an algorithm of running time $O(n^3)$ per separator which uses the idea of generating a new minimal separator from an older one S by looking at the separator $S \cup N(x)$ for $x \in S$. This separator is not minimal but the neighbourhoods of the connected components it defines are. This simple process can generate all the minimal separators of a graph. The counterpart is that a minimal separator can be generated many times.

In this paper, I adapt this idea to 3-connected planar graphs but to avoid the problem of recalculation, I define the set $\mathcal{S}_a(S,O)$ of the a,b-minimal separators S' for some b that are such that the connected component of a in $G\backslash S'$ contains the connected component of a in $G\backslash S$ but avoids the set O. This way I put restrictions on the minimal separators I compute to ensure I do not compute the same minimal separator over and over.

2 Definitions

Throughout this paper, $G = (V, E)$ will be a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$.

A set $S \subseteq V$ is an a,b *minimal separator* if a and b are in two distinct connected components of $G\backslash S$ and no proper subset of S separates them. The connected component of a in $G\backslash S$ is $C_a(S)$. The component $C_a(S)$ *is a full connected component* if $N(C_a(S)) = S$. A set S is a *minimal separator* if there exists a and b which make it an a,b-minimal separator or, which is equivalent, if it has at least two full connected components. An a,$*$ *-minimal separator* of a graph $G = (V, E)$ is a set of vertices S such that there exists $b \in V$ which makes it an a,b-minimal separator. The set of the a,$*$-minimal separators is denoted by $\mathcal{S}_a$ and the set of the minimal separators of G is denoted by $\mathcal{S}(G)$.

We can order the a,$*$-minimal separators in the following way:

$$S_1 \preccurlyeq S_2 \quad \text{if} \quad C_a(S_1) \subseteq C_a(S_2).$$

For S an a,$*$-minimal separators and $O \subseteq V$, the set $\mathcal{S}_a(S, O)$ is *the set of the* a,$*$ *-minimal separator* S' *such that* $S \preccurlyeq S'$ *and* $O \cap C_a(S') = \emptyset$. And if $x \in V$, the set $\mathcal{S}_a^x(S, O)$ is *the set of* $S' \in \mathcal{S}_a(S, O)$ such that $x \in C_a(S')$.

Remark 1 *If* $x \in S$*, then* $\mathcal{S}_a(S, O)$ *is the disjoint union*

$$\mathcal{S}_a\big(S, O \cup \{x\}\big) \bigsqcup \mathcal{S}_a^x(S, O).$$

And more precisely, if $(S_i)_{i \in I}$ *are the minimal elements of* $\mathcal{S}_a^x(S, O)$*, we have*

$$\mathcal{S}_a(S, O) = \mathcal{S}_a\big(S, O \cup \{x\}\big) \bigsqcup \Big(\bigcup_{i \in I} \mathcal{S}_a(S_i, O)\Big).$$

This gives us the skeleton of an algorithm to compute the set $\mathcal{S}_a(S, O)$*.*

Remark 2 *If S belongs to $\mathcal{S}_a^x(S,O)$, then $\mathcal{S}_a^x(S,O) = \mathcal{S}_a(S,O)$.*

The algorithm is based on remarks 1 and 2. To have $\mathcal{S}_a$, the algorithm computes the sets $\mathcal{S}_a(S,\emptyset)$ for every S minimal in $\mathcal{S}_a$. During this calculation, it will have to compute $\mathcal{S}_a(S,O)$ with $O \subseteq S$. To do so, it chooses $x \in S \backslash O$ and calculates $\mathcal{S}_a^x(S,O)$ and $\mathcal{S}_a(S, O \cup \{x\})$. The set $\mathcal{S}_a^x(S,O)$ is itself a union of $\mathcal{S}_a(S_i,O)$. But to obtain such a decomposition, one needs to find the minimal elements of $\mathcal{S}_a^x(S,O)$, which the following property does.

Property 1 *Let $G = (V,E)$ be a graph, S an $a,*$-minimal separator, $O \subset S$ and $x \in S \backslash O$.*

Every minimal element of $\mathcal{S}_a^x(S,O)$ is the neighbourhood of a connected component of $G \backslash \{N(C) \cup C\}$ with $C = C_a(S) \cup \{x\}$.

Proof. Let $S_1 \in \mathcal{S}_a^x(S,O)$ be an a,b-minimal separator.

Let $C' = C_b(N(C))$ and let $S' = N(C')$. $S' \subseteq N(C)$. By construction, S' is an a,b-separator. Moreover, $C_a(S')$ and $C_b(S')$ are two full connected components which proves that S' is an a,b-minimal separator.

Let p be a path in $C_b(S_1)$ with b as one of its ends. The vertices of S_1 are at least at distance 1 of C so the vertices of p are at least at distance 2 of C. Because $S' \subseteq N(C)$, $p \cap S' = \emptyset$. Finally, since $b \in C'$, so does p and $C^b(S_1) \subseteq C^b(S')$. The a,b-minimal separators being a lattice for the relation $\preccurlyeq$, S_1 is greater than S'. Moreover, since $O \cap C_a(S_1) = \emptyset$, $O \cap C_a(S') = \emptyset$ and $S' \in \mathcal{S}_a^x(S,O)$.

If S_1 is minimal, then $S_1 = S'$ and S_1 is then the neighbourhood of a connected component of $G \backslash \{N(C) \cup C\}$ as required. □

The property 1 gives us a good way to find the minimal elements of $\mathcal{S}_a^x(S,O)$, using the skeleton of remark 1, we can design an algorithm to compute the set $\mathcal{S}_a(S,O)$. It could look like:

ALGORITHM: `_calc3_`
begin
  if $S \backslash O = \emptyset$ **then**
    return$(\{S\})$
  else
    let $x \in S \backslash O$
    $\mathcal{S} \leftarrow$`_calc3_`$(G, a, S, O \cup \{x\})$

    for each S_i in `find_min_elements`(G, a, x, S, O)
      $\mathcal{S} \leftarrow \mathcal{S} \cup$`_calc3_`$(G, a, S_i, O)$
  return$(\mathcal{S})$
end

But there are several problems to solve.

i. First, we do not know whether the sets $\mathcal{S}_a(S_i,O)$ are disjoint or not. If not, we could compute a minimal separator many times which would lead to a bad complexity.

ii. To implement the function `find_min_elements`, property 1 states that we can use a graph search of G.

    But if $\mathcal{S}_a(S,O) = \{S\}$, the algorithm will try to find a minimal element in $\mathcal{S}_a^x(S,O)$ for every $x \in S \backslash O$. Each call to `find_min_elements` costs $O(m)$ and in the end, we would have spent $O(nm)$ to realise that $\mathcal{S}_a(S,O) = \{S\}$.

Property 3 ensures that for 3-connected planar graphs, point (i) is true and the section 3.3 shows how to determine that $\mathcal{S}_a^x(S,O)$ is empty in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let Σ be the plane. A *plane graph* $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of Σ between two vertices of V_Σ in such a way that the interiors of two distinct edges do not meet. We will denote by $\widetilde{G}_\Sigma$ the drawing of G_Σ. A *planar graph* is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of G_Σ is a connected component of $\Sigma \backslash \widetilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Property 2 *In a 3-connected planar graph, minimal separators are minimal for inclusion.*

Proof. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let a, b, c and d be vertices such that S' is an a,b-minimal separator and S is a c,d-minimal separator. Since S is not an a,b-minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. $C_c(S) = C_c(S')$ and $N(C_c(S) = S$.

But then G admits $K_{3,3}$ as a minor for if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices a', b' and c', all these vertices have S in their neighbourhood and since G is 3-connected $|S| \geq 3$. This contradicts that fact that G is planar. □

Property 3 *Let $G = (V, E)$ be a 3-connected planar graph, $a \in V$, S an a,$*$-minimal separator, $O \subseteq S$ and $x \in S \backslash O$.*

If S_1 and S_2 are two minimal elements of $\mathcal{S}_a^x(S, O)$, then

$$\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O) = \emptyset.$$

Proof. Suppose that S_1 and S_2 are two distinct minimal elements of $\mathcal{S}_a^x(S, O)$.

By property 1, S_1 and S_2 are subsets of $S' = N(C_a(S) \cup \{x\})$.

Let b be a vertex such that S_1 is an a,b-minimal separator. Since S_1 and S_2 are not comparable, S_2 is not an a,b-separator. Indeed, since the set of all a,b-minimal separators is a lattice, $\min(S_1, S_2)$ would be in $\mathcal{S}_a^x(S, O)$ which would contradict the fact that S_1 and S_2 are minimal elements of $\mathcal{S}_a^x(S, O)$.

Suppose that $S_3 \in \mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O)$ is an a,c-minimal separator.

Since S_1 and S_2 are included in S', $S_3' = N(C_c(S'))$ is an a,c-minimal separator greater than S_1 and S_2 and smaller than S_3 so $S_3' \in \mathcal{S}_a^x(S, O)$.

But S_3' is included in S_1 and S_2 which is impossible in a 3-connected graph by property 2. □

3.2 The intermediate graph

Definition 1 *Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a plane 3-connected graph. Let F be the set of its faces. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup F$. We place an edge between a vertex $v \in V$ and $f \in F$ if and only if the vertex v is incident to the face f.*

For G' a subgraph of G_I, the set $\widetilde{G'} \cap V_\Sigma$ will be denoted by $V(G')$.

Property 4 *Let μ be a cycle of G_I such that the curve $\widetilde{\mu}$ separates at least two vertices a and b of V_Σ.*

The set $V(\mu)$ is an a,b-separator of G_Σ.

Proof. Let p be a path in G_Σ from a to b. Since a and b are not in the same connected component of $\Sigma \backslash \widetilde{\mu}$, $\widetilde{p}$ intersects $\widetilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from a to b meets $V(\mu)$ and so $V(\mu)$ is an a,b-separator. □

Property 5 *Let S be an a,b-minimal separator of G. There exists a simple cycle μ of G_I such that the Jordan curve it defines separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.*

Proof. Let C be the connected component of a in $G\backslash S$. Contract C into a supervertex v_C to build the graph $G_{/C}$. In $G_{/C}$, there is a cycle $\mu_{/C}$ of $(G_{/C})_I$ such that $V(\mu_{/C}) = N(v_C)$. Therefore, in G_I the neighbourhood of C has the structure of a cycle μ.

Suppose $\widetilde{\mu}$ is not a Jordan curve, the border μ' of the connected component of b in $\Sigma\backslash\widetilde{\mu}$ is a strict sub-lace of $\widetilde{\mu}$ which separates a and b. But then property 4 shows that $V(\mu')$ which is a strict subset of S is an a,b-separator. This contradicts the fact that S is a a,b-minimal separator. □

Property 5 shows that the minimal separators of a 3-connected planar graph are cycles of the intermediate graph which gives a criteria to say when a set is not a minimal separator. It gives nothing more for some cycles of G_I correspond to no minimal separator of G.

There are several ways to find an exact criteria for minimal separators. The following section gives one which is well suited for our purpose.

3.3 Ordered separators

Definition 2 *An ordered separator of G is a sequence of distinct vertices*

$$(v_0, \dots, v_{p-1}),\ (p > 2)$$

such that

i. there exists a face to which v_i and $v_{i+1\,[p]}$ are incident;

ii. v_i and v_j are incident to a common face only if $i = j + 1\,[p]$ or $j = i + 1\,[p]$;

iii. there is no face incident to v_i, $v_{i+1\,[p]}$ and $v_{i+2\,[p]}$.

The notation $i\,[p]$ means i modulo p.

We say that a set $S = \{v_0, \dots, v_{p-1}\}$ is an ordered separator if there exists a permutation σ such that $(v_{\sigma(0)}, \dots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = (v_0, \dots, v_{p-1})$ is an ordered separator of G, then S is naturally associated to the set $\{v_0, \dots, v_{p-1}\}$. We will either use an ordered separator as a sequence or as the corresponding set.

Remark 3 *If $p > 3$, the third condition is a corollary of the second for v_i et $v_{i+2\,[p]}$ would be too far apart.*

Lemma 1 *Every minimal separator S of G is ordered.*

Proof. Let S be an a,b-minimal separator of G.

The property 5 states that there exists a simple cycle of G_I

$$\mu = (v_0, f_0, \dots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \dots, v_{p-1})$ is an ordered separator corresponding to S.

i. The construction of T ensures that v_i and v_{i+1} are incident to a common face (f_i).

ii. Suppose that v_i et v_j are incident to a common face f and that $i+1 \neq j\,[p]$ and $j+1 \neq i\,[p]$.

   $\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \dots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \dots, v_i, f)$ are laces of G_I. Moreover, since either μ_1 or μ_2 separates a and b, there exists an a,b-separator strictly included in S which is absurd.

iii. With the remark 3, we can suppose that $p = 3$.

   Suppose that v_0, v_1 et v_2 are all incident to a common face f. If we add a vertex f to G that we connect to the vertices v_0, v_1 and v_2, the graph remains planar which is absurd for this graph has $K_{3,3}$ as a minor. Indeed, the connected component of a, the connected component of b and the vertex f are all incident to v_0, v_1 and v_2 which builds up a $K_{3,3}$.

The sequence T is an ordered separator corresponding to S. □

Conversely,

Lemma 2 *Every ordered separator of G is a minimal separator of G.*

Proof. Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of G.

First, S is a separator. Otherwise, $G \backslash S$ would be connected or empty. In both cases all the vertices of S would be incident to a common face.

Let S' be a minimal separator included in S. By lemma 1, S' is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator S is a minimal separator. □

From lemma 1 and 2, we have he following property:

Property 6 *A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of G.*

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $\mathcal{S}_a^x(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \backslash O$).

Property 7 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered a,$*$-separator of a 3-connected planar graph $G = (V, E)$.*

Let $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of S.

If there exists a face which is incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j with $0 < j < i$, then $\mathcal{S}_a^{v_{i+1}}(S, O) = \emptyset$.

Proof. Suppose that S' is a minimal element of $\mathcal{S}_a^{v_{i+1}}(S, O)$ and f is incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j with $0 < j < i$.

By property 1, $S' \subseteq N\big(C_a(S) \cup \{v_{i+1}\}\big)$ and by lemma 1, S is an ordered separator. So $S' = (v_0, \ldots, v_i, y_1, \ldots, y_l)$.

Since S is an ordered separator, no y_k can be incident to f.

But since there is a face to which y_k and y_{k+1} are incident and since there is a face to which v_i and y_1 are incident, in clockwise order, all the vertices y_k are between v_i and y. But there is no face to which y_l and v_0 are incident and S' is not an ordered separator. □

Conversely,

Property 8 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered a,$*$-separator of a 3-connected planar graph $G = (V, E)$.*

Let $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of S.

If there is no face incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j $(0 < j < i)$, then there is an ordered separator in $S \cup N(v_{i+1}) \backslash C_a(S)$ which contains O.

Proof. The neighbours $(y_1, \ldots, y_l)$ of v_{i+1} taken in clockwise order are such that y_i and y_{i+1} are incident to the same face. Moreover, since v_{i+1} and v_i are both incident to a face f_1 and since v_{i+1} and v_{i+2} are both incident to a face f_2, there is a sequence $P = (v_i, x_1, \ldots, x_k, v_0)$ such that there exists a face incident to any two consecutive vertices of P and such that P uses only vertices of $N(v_{i+1}) \backslash C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is $(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0)$.

Let P be such a sequence between v_i and v_0 of minimal length. Together with $(v_1, \ldots, v_{i-1})$, P forms an ordered separator of G as required. □

4 The algorithm

Now we have all we need to build up an algorithm to compute the set $\mathcal{S}_a(S, O)$ with $O \subseteq S$.

ALGORITHM: `_calc3_`
input:
G a 3-connected planar graph
a a vertex of G
$S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
$O = (v_0, \ldots, v_i)$ with $i \leq p-1$ a subset of S
The vertices which have an incident face in common with v_i $(i \geq 1)$ are tagged i
unless they can be tagged j $(1 \leq j \leq i-1)$.
Theses vertices are the forbidden vertices.
The vertices of $C_a(S)$ are also tagged "$C_a(S)$".
output:
$\mathcal{S}_a(S, O)$

begin
  if $i = p-1$ **then**
    return($\{S\}$)
  else
    $x \leftarrow v_{i+1}$
    tag if necessary the faces incident to x with $i+1$
    $\mathcal{S} \leftarrow$`_calc3_`$(G, a, S, (v_0, \ldots, v_i, x))$
    untag the faces incident to x

    for each $y \in N(x)$ not tagged "$C_a(S)$"
      if y is tagged $j < i$ **then**
        return($\mathcal{S}$)
    for each S' in `find_min_elements`(G, a, x, S, O)
      $\mathcal{S} \leftarrow \mathcal{S} \cup$ `_calc3_`$(G, a, S', (v_0, \ldots, v_i))$
end

Property 9 *The algorithm* `_calc3_` *is correct. It computes the set $\mathcal{S}_a(S, O)$ of a 3-connected planar graph.*

Proof. The algorithm is just an application of remark 1. □

Property 10 *The algorithm can be implemented to compute the set $\mathcal{S}_a(S, O)$ in time $O(n|\mathcal{S}_a(S, O)|)$.*

Proof. For each minimal separator S, the algorithm does the following:

i. the function `find_min_elements` produces S;

ii. for every $x \in S \backslash O$, there is a recursive call to `_calc3_` to extend the set O;

iii. S is returned.

The function `find_min_elements` does a graph search to compute the sets S_i, and to tag the vertices in $C_a(S_i)$. It orders S_i and tag the forbidden vertices. In a planar graph, the number m of edges satisfies $0 \leq m \leq 3n-6$, so all this costs $O(n)$.

Each call to `_calc3_` costs $O(d(x))$ to tag and untag the faces incident to x, and $O(d(x))$ to check whether $\mathcal{S}_a^x(S, O)$ is empty or not. Since every time a different x is chosen, the recursive calls to `_calc3_` cost $O(n)$.

The overall complexity of function `_calc3_` is $O(n|\mathcal{S}_a(S, O)|)$. □

The following algorithm uses the function `_calc3_` to compute the set of all minimal separators of a planar graph G.

ALGORITHM: `all_min_sep3`
input:
G a 3-connected planar graph
output:
the set of the $a,*$-minimal separators of G

begin
$\mathcal{S} \leftarrow \emptyset$
find $a \in V$ **with** $d(a) < 6$
for each minimal separator $S \subseteq N(a)$
$\mathcal{S} \leftarrow \mathcal{S} \cup$ `_calc3_`$(G, a, S, \emptyset)$
for each $y \in N(a)$
for each minimal separator $S \subseteq N(y)$
$\mathcal{S} \leftarrow \mathcal{S} \cup$ `_calc3_`$(G, y, S, \emptyset)$
return$(\mathcal{S})$
end

Theorem 1 `all_min_sep3` *computes the set of the minimal separators of a 3-connected planar graph in time* $O(n|\mathcal{S}(G)|)$

Proof. Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex a, $S \in \mathcal{S}(G)$ either belongs to $\mathcal{S}_a$ or runs through a. In the second case, it is a $b,*$-minimal separator for a neighbour b of a.

Moreover, there exists a vertex a of degree at most five in a planar graph. Let $b_1, \ldots, b_p$ be its neighbours.

By computing $\mathcal{S}_a \bigcup \left(\bigcup_{i \in [1..p]} \mathcal{S}_{b_i}\right)$, a minimal separator can be calculated at most six times which gives the claimed complexity. □

5 Conclusion

In the conclusion of [1], Berry and al. note that their algorithm may compute a minimal separator up to n times and that this could be improved. This paper confirms this feeling for this is exactly what I have gained for 3-connected planar graphs. However it would be more satisfying to compute the minimal separators of all planar graphs. I feel that a slightly modified version of my algorithm could compute them. I also feel, just like Berry and al., that there could be a better general algorithm to compute the minimal separators of a graph.

This paper gives another proof that planar graphs and their minimal separators in particular are peculiar. I feel that topological properties such as property 5 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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References

[1] A. Berry, J.P. Bordat, and O. Cogis. Generating all the minimal separators of a graph. In *Workshop on Graphs (WG'99)*, volume 1665 of *Lecture Notes in Computer Science*. Springer-Verlag, 1999.

[2] V. Bouchitté and I. Todinca. Minimal triangulations for graphs with "few" minimal separators. In *Proceedings 6th Annual European Symposium on Algorithms (ESA'98)*, volume 1461 of *Lecture Notes in Computer Science*, pages 344–355. Springer-Verlag, 1998.

[3] V. Bouchitté and I. Todinca. Listing all potential maximal cliques of a graph. In *Proceedings 17th Annual Symposium on Theoretical Aspects of Computer Science (STACS 2000)*, volume 1770 of *Lecture Notes in Computer Science*, pages 503–515. Springer-Verlag, 2000.

[4] T. Kloks, H.L. Bodlaender, H. Müller, and D. Kratsch. Computing treewidth and minimum fill-in: all you need are the minimal separators. In *Proceedings First Annual European Symposium on Algorithms (ESA'93)*, volume 726 of *Lecture Notes in Computer Science*, pages 260–271. Springer-Verlag, 1993.

[5] T. Kloks and D. Kratsch. Finding all minimal separators of a graph. In *Proceedings 11th Annual Symposium on Theoretical Aspects of Computer Science (STACS'94)*, volume 775 of *Lecture Notes in Computer Science*, pages 759–768. Springer-Verlag, 1994.

[6] T. Kloks and D. Kratsch. Listing all minimal separators of a graph. *SIAM J. Comput.*, 27(3):605–613, 1998.

[7] A. Parra. *Structural and Algorithmic Aspects of Chordal Graph Embeddings.* PhD thesis, Technische Universität Berlin, 1996.

[8] A. Parra and P. Scheffler. Characterizations and algorithmic applications of chordal graph embeddings. *Discrete Appl. Math.*, 79(1-3):171–188, 1997.

[9] H. Shen and W. Liang. Efficient enumeration of all minimal separators in a graph. *Theoretical Computer Science*, 180:169–180, 1997.

[10] I. Todinca. *Aspects algorithmiques des triangulations minimales des graphes.* PhD thesis, École Normale Supérieure de Lyon, 1999.