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Listing all the minimal separators of a 3-connected planar graph

Frédéric Mazoit

Février 2004

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Listing all the minimal separators of a 3-connected planar graph

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Abstract
I present an efficient algorithm which lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Keywords: 3-connected planar graphs, minimal separator enumeration*

Résumé
Je présente un algorithme d’énnumération des séparateurs minimaux des graphes planaires 3-connexes dont la complexité est $O(n)$ par séparateur.

Mots-clés: graphes planaires 3 connexes, séparateurs minimaux, énumération
1 Introduction

In this paper, we address the problem of finding the minimal separators of a 3-connected planar graph $G$.

In the last ten years, minimal separators have been an increasingly used tool in graph theory with many algorithmic applications (for example [4], [7], [8], [10]). For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [4], Bodlaender and al. conjecture that for a class of graph which a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the notion of potential maximal clique (see [2]) and showed that if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can indeed be solved in polynomial time. They later showed in [3] that if a graph has a polynomial number of minimal separators, then it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Some research has been done to compute the set of the minimal separators of a graph ([1], [5], [6], [9]). In [1], Berry and al. proposed an algorithm of running time $O(n^3)$ per separator which uses the idea of generating a new minimal separator from an older one $S$ by looking at the separator $S \cup N(x)$ for $x \in S$. This separator is not minimal but the neighbourhoods of the connected components it defines are. This simple process can generate all the minimal separators of a graph. The counterpart is that a minimal separator can be generated many times.

In this paper, I adapt this idea to 3-connected planar graphs but to avoid the problem of recalculation, I define the set $S_a(S,O)$ of the $a,b$-minimal separators $S'$ for some $b$ that are such that the connected component of $a$ in $G \setminus S'$ contains the connected component of $a$ in $G \setminus S$ but avoids the set $O$. This way I put restrictions on the minimal separators I compute to ensure I do not compute the same minimal separator over and over.

2 Definitions

Throughout this paper, $G = (V,E)$ will be a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x,y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x,y) \in E\}$.

A set $S \subseteq V$ is an $a,b$ minimal separator if $a$ and $b$ are in two distinct connected components of $G \setminus S$ and no proper subset of $S$ separates them. The connected component of $a$ in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. A set $S$ is a minimal separator if there exists $a$ and $b$ which make it an $a,b$-minimal separator or, which is equivalent, if it has at least two full connected components. An $a,*$-minimal separator of a graph $G = (V,E)$ is a set of vertices $S$ such that there exists $b \in V$ which makes it an $a,b$-minimal separator. The set of the $a,*$-minimal separators is denoted by $S_a$ and the set of the minimal separators of $G$ is denoted by $S(G)$.

We can order the $a,*$-minimal separators in the following way:

$$S_1 \preceq S_2 \quad \text{if} \quad C_a(S_1) \subseteq C_a(S_2).$$

For $S$ an $a,*$-minimal separators and $O \subseteq V$, the set $S_a(S,O)$ is the set of the $a,*$-minimal separator $S'$ such that $S \preceq S'$ and $O \cap C_a(S') = \emptyset$. And if $x \in V$, the set $S^x_a(S,O)$ is the set of $S' \in S_a(S,O)$ such that $x \in C_a(S')$.

Remark 1 If $x \in S$, then $S_a(S,O)$ is the disjoint union

$$S_a(S,O \cup \{x\}) \sqcup S^x_a(S,O).$$

And more precisely, if $(S_i)_{i \in I}$ are the minimal elements of $S^x_a(S,O)$, we have

$$S_a(S,O) = S_a(S,O \cup \{x\}) \sqcup \left( \bigcup_{i \in I} S_a(S_i,O) \right).$$

This gives us the skeleton of an algorithm to compute the set $S_a(S,O)$. 

Listing the minimal separators of a 3-connected planar graph
Remark 2 If $S$ belongs to $S_a^x(S, O)$, then $S_a^x(S, O) = S_a(S, O)$.

The algorithm is based on remarks 1 and 2. To have $S_a$, the algorithm computes the sets $S_a(S, \emptyset)$ for every $S$ minimal in $S_a$. During this calculation, it will have to compute $S_a(S, O)$ with $O \subseteq S$. To do so, it chooses $x \in S \backslash O$ and calculates $S_a^x(S, O)$ and $S_a(S, O \cup \{x\})$. The set $S_a^x(S, O)$ is itself a union of $S_a(S, O)$. But to obtain such a decomposition, one needs to find the minimal elements of $S_a^x(S, O)$, which the following property does.

Property 1 Let $G = (V, E)$ be a graph, $S$ an $a, \ast$-minimal separator, $O \subseteq S$ and $x \in S \backslash O$.

Every minimal element of $S_a^x(S, O)$ is the neighbourhood of a connected component of $G \backslash \{N(C) \cup C\}$ with $C = C_a(S) \cup \{x\}$.

Proof. Let $S_1 \in S_a^x(S, O)$ be an $a,b$-minimal separator.

Let $C' = C_b(N(C))$ and let $S' = N(C')$. $S' \subseteq N(C)$. By construction, $S'$ is an $a,b$-separator. Moreover, $C_a(S')$ and $C_b(S')$ are two full connected components which proves that $S'$ is an $a,b$-minimal separator.

Let $p$ be a path in $C_b(S_1)$ with $b$ as one of its ends. The vertices of $S_1$ are at least at distance 1 of $C'$ so the vertices of $p$ are at least at distance 2 of $C$. Because $S' \subseteq N(C)$, $p \cap S' = \emptyset$. Finally, since $b \in C'$, so does $p$ and $C_b(S_1) \subseteq C_b(S')$. The $a,b$-minimal separators being a lattice for the relation $\leq$, $S_1$ is greater than $S'$. Moreover, since $O \cap C_a(S_1) = \emptyset$, $O \cap C_a(S') = \emptyset$ and $S' \in S_a^x(S, O)$.

If $S_1$ is minimal, then $S_1 = S'$ and $S_1$ is then the neighbourhood of a connected component of $G \backslash \{N(C) \cup C\}$ as required. \hfill \square

The property 1 gives us a good way to find the minimal elements of $S_a^x(S, O)$, using the skeleton of remark 1, we can design an algorithm to compute the set $S_a(S, O)$. It could look like:

**Algorithm: _calc3_**

begin
if $S \backslash O = \emptyset$ then
    return($\{S\}$)
else
    let $x \in S \backslash O$
    $S \leftarrow _\text{calc3}(G, a, S, O \cup \{x\})$

    for each $S_i$ in _find_min_elements($G, a, x, S, O$)_
    $S \leftarrow S \cup _\text{calc3}(G, a, S_i, O)$

    return($S$)
end

But there are several problems to solve.

i. First, we do not know whether the sets $S_a(S_i, O)$ are disjoint or not. If not, we could compute a minimal separator many times which would lead to a bad complexity.

ii. To implement the function _find_min_elements_, property 1 states that we can use a graph search of $G$.

But if $S_a(S, O) = \{S\}$, the algorithm will try to find a minimal element in $S_a^x(S, O)$ for every $x \in S \backslash O$. Each call to _find_min_elements_ costs $O(m)$ and in the end, we would have spent $O(nm)$ to realise that $S_a(S, O) = \{S\}$.

Property 3 ensures that for 3-connected planar graphs, point (i) is true and the section 3.3 shows how to determine that $S_a^x(S, O)$ is empty in an overall $O(n)$.
3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let \( \Sigma \) be the plane. A **plane graph** \( G_\Sigma = (V_\Sigma, E_\Sigma) \) is a graph drawn on the plane, that is \( V_\Sigma \subseteq \Sigma \) and each edge is a simple curve of \( \Sigma \) between two vertices of \( V_\Sigma \) in such a way that the interiors of two distinct edges do not meet. We will denote by \( \tilde{G}_\Sigma \), the drawing of \( G_\Sigma \). A **planar graph** is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of \( G_\Sigma \) is a connected component of \( \Sigma \setminus \tilde{G}_\Sigma \).

3.1 Minimal separators of 3-connected planar graphs

**Property 2** In a 3-connected planar graph, minimal separators are minimal for inclusion.

**Proof.** Suppose that \( S \subseteq S' \) are two minimal separators of a 3-connected planar graph.

Let \( a, b, c \) and \( d \) be vertices such that \( S' \) is an \( a,b \)-minimal separator and \( S \) is a \( c,d \)-minimal separator. Since \( S \) is not an \( a,b \)-minimal separator, either \( C_a(S') \) or \( C_d(S') \) is disjoint with \( C_a(S') \) and \( C_b(S') \). Suppose that \( C_a(S') \) is such a component. \( C_c(S) = C_c(S') \) and \( N(C_c(S)) = \emptyset \).

But then \( G \) admits \( K_{3,3} \) as a minor for if we contract \( C_a(S') \), \( C_b(S') \) and \( C_c(S') \) into the vertices \( a', b' \) and \( c' \), all these vertices have \( S \) in their neighbourhood and since \( G \) is 3-connected \( |S| \geq 3 \). This contradicts that fact that \( G \) is planar. \( \square \)

**Property 3** Let \( G = (V, E) \) be a 3-connected planar graph, \( a \in V, S \) an \( a,- \)-minimal separator, \( O \subseteq S \) and \( x \in S \setminus O \).

If \( S_1 \) and \( S_2 \) are two minimal elements of \( S^*_a(S, O) \), then

\[
S_a(S_1, O) \cap S_a(S_2, O) = \emptyset.
\]

**Proof.** Suppose that \( S_1 \) and \( S_2 \) are two distinct minimal elements of \( S^*_a(S, O) \).

By property 1, \( S_1 \) and \( S_2 \) are subsets of \( S' = N(C_a(S) \cup \{x\}) \).

Let \( b \) be a vertex such that \( S_1 \) is an \( a,b \)-minimal separator. Since \( S_1 \) and \( S_2 \) are not comparable, \( S_2 \) is not an \( a,b \)-separator. Indeed, since the set of all \( a,b \)-minimal separators is a lattice, \( \min(S_1, S_2) \) would be in \( S^*_a(S, O) \) which would contradict the fact that \( S_1 \) and \( S_2 \) are minimal elements of \( S^*_a(S, O) \).

Suppose that \( S_3 \in S_a(S_1, O) \cap S_a(S_2, O) \) is an \( a,c \)-minimal separator.

Since \( S_1 \) and \( S_2 \) are included in \( S' \), \( S'_3 = N(C_c(S')) \) is an \( a,c \)-minimal separator greater than \( S_1 \) and \( S_2 \) and smaller than \( S_3 \) so \( S'_3 \in S^*_a(S, O) \).

But \( S'_3 \) is included in \( S_1 \) and \( S_2 \) which is impossible in a 3-connected graph by property 2. \( \square \)

3.2 The intermediate graph

**Definition 1** Let \( G_\Sigma = (V_\Sigma, E_\Sigma) \) be a plane 3-connected graph. Let \( F \) be the set of its faces. The intermediate graph \( G_I = (V_I, E_I) \) is a plane graph whose vertex set is \( V_I = V_\Sigma \cup F \). We place an edge between a vertex \( v \in V \) and \( f \in F \) if and only if the vertex \( v \) is incident to the face \( f \).

For \( G' \) a subgraph of \( G_I \), the set \( G' \cap V_\Sigma \) will be denoted by \( V(G') \).

**Property 4** Let \( \mu \) be a cycle of \( G_I \) such that the curve \( \tilde{\mu} \) separates at least two vertices \( a \) and \( b \) of \( V_\Sigma \).

The set \( V(\mu) \) is an \( a,b \)-separator of \( G_\Sigma \).

**Proof.** Let \( p \) be a path in \( G_\Sigma \) from \( a \) to \( b \). Since \( a \) and \( b \) are not in the same connected component of \( \Sigma \setminus \tilde{\mu} \), \( \tilde{p} \) intersects \( \tilde{\mu} \). By construction, \( p \cap \mu \subseteq V_\Sigma \). This implies that every path from \( a \) to \( b \) meets \( V(\mu) \) and so \( V(\mu) \) is an \( a,b \)-separator. \( \square \)
Property 5 Let $S$ be an $a,b$-minimal separator of $G$. There exists a simple cycle $\mu$ of $G_I$ such that the Jordan curve it defines separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.

Proof. Let $C$ be the connected component of $a$ in $G \setminus S$. Contract $C$ into a supervertex $v_C$ to build the graph $G/C$. In $G/C$, there is a cycle $\mu/C$ of $(G/C)_I$ such that $V(\mu/C) = N(v_C)$. Therefore, in $G_I$ the neighbourhood of $C$ has the structure of a cycle $\mu$.

Suppose $\tilde{\mu}$ is not a Jordan curve, the border $\mu'$ of the connected component of $b$ in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\mu$ which separates $a$ and $b$. But then property 4 shows that $V(\mu')$ which is a strict subset of $S$ is an $a,b$-separator. This contradicts the fact that $S$ is a $a,b$-minimal separator. □

Property 5 shows that the minimal separators of a 3-connected planar graph are cycles of the intermediate graph which gives a criteria to say when a set is not a minimal separator. It gives nothing more for some cycles of $G_I$ correspond to no minimal separator of $G$.

There are several ways to find an exact criteria for minimal separators. The following section gives one which is well suited for our purpose.

3.3 Ordered separators

Definition 2 An ordered separator of $G$ is a sequence of distinct vertices

$$(v_0, \ldots, v_{p-1}), \ (p > 2)$$

such that

i. there exists a face to which $v_i$ and $v_{i+1}[p]$ are incident;

ii. $v_i$ and $v_j$ are incident to a common face only if $i = j + 1[p]$ or $j = i + 1[p]$;

iii. there is no face incident to $v_i, v_{i+1}[p]$ and $v_{i+2}[p]$:

The notation $i[p]$ means $i$ modulo $p$.

We say that a set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation $\sigma$ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator of $G$, then $S$ is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will either use an ordered separator as a sequence or as the corresponding set.

Remark 3 If $p > 3$, the third condition is a corollary of the second for $v_i$ et $v_{i+2}[p]$ would be too far apart.

Lemma 1 Every minimal separator $S$ of $G$ is ordered.

Proof. Let $S$ be an $a,b$-minimal separator of $G$.

The property 5 states that there exists a simple cycle of $G_I$

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to $S$.

i. The construction of $T$ ensures that $v_i$ and $v_{i+1}$ are incident to a common face $(f_i)$.

ii. Suppose that $v_i$ et $v_j$ are incident to a common face $f$ and that $i + 1 \neq j[p]$ and $j + 1 \neq i[p]$.

$\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)$ are laces of $G_I$. Moreover, since either $\mu_1$ or $\mu_2$ separates $a$ and $b$, there exists an $a,b$-separator strictly included in $S$ which is absurd.

iii. With the remark 3, we can suppose that $p = 3$.

Suppose that $v_0$, $v_1$ et $v_2$ are all incident to a common face $f$. If we add a vertex $f$ to $G$ that we connect to the vertices $v_0$, $v_1$ and $v_2$, the graph remains planar which is absurd for this graph has $K_{3,3}$ as a minor. Indeed, the connected component of $a$, the connected component of $b$ and the vertex $f$ are all incident to $v_0$, $v_1$ and $v_2$ which builds up a $K_{3,3}$. 
The sequence $T$ is an ordered separator corresponding to $S$. \hfill \Box

Conversely,

**Lemma 2** Every ordered separator of $G$ is a minimal separator of $G$.

**Proof.** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of $G$.

First, $S$ is a separator. Otherwise, $G \setminus S$ would be connected or empty. In both cases the vertices of $S$ would be incident to a common face.

Let $S'$ be a minimal separator included in $S$. By lemma 1, $S'$ is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator $S$ is a minimal separator. \hfill \Box

From lemma 1 and 2, we have the following property:

**Property 6** A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of $G$.

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $S^a_\alpha(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \setminus O$).

**Property 7** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered a,*-separator of a 3-connected planar graph $G = (V, E)$.

Let $O = (v_0, \ldots, v_i)$, ($i < p - 1$) be an initial sequence of $S$.

If there exists a face which is incident to both $y \in N(v_{i+1}) \setminus C_\alpha(S)$ and $v_j$ with $0 < j < i$, then $S^a_\alpha(S, O) = \emptyset$.

**Proof.** Suppose that $S'$ is a minimal element of $S^a_\alpha(S, O)$ and $f$ is incident to both $y \in N(v_{i+1}) \setminus C_\alpha(S)$ and $v_j$ with $0 < j < i$.

By property 1, $S' \subseteq N(C_\alpha(S) \cup \{v_{i+1}\})$ and by lemma 1, $S$ is an ordered separator. So $S' = (v_0, \ldots, v_i, y_i, \ldots, y_{i+1})$.

Since $S$ is an ordered separator, no $y_k$ can be incident to $f$.

But since there is a face to which $y_i$ and $y_{i+1}$ are incident and since there is a face to which $v_i$ and $y_i$ are incident, in clockwise order, all the vertices $y_k$ are between $v_i$ and $y_i$. But there is no face to which $y_i$ and $v_0$ are incident and $S'$ is not an ordered separator. \hfill \Box

Conversely,

**Property 8** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered a,*-separator of a 3-connected planar graph $G = (V, E)$.

Let $O = (v_0, \ldots, v_i)$, ($i < p - 1$) be an initial sequence of $S$.

If there is no face incident to both $y \in N(v_{i+1}) \setminus C_\alpha(S)$ and $v_j$ ($0 < j < i$), then there is an ordered separator in $S \cup N(v_{i+1}) \setminus C_\alpha(S)$ which contains $O$.

**Proof.** The neighbours $(y_1, \ldots, y_{i+1})$ of $v_{i+1}$ taken in clockwise order are such that $y_i$ and $y_{i+1}$ are incident to the same face. Moreover, since $v_{i+1}$ and $v_i$ are both incident to a face $f_1$ and since $v_{i+1}$ and $v_{i+2}$ are both incident to a face $f_2$, there is a sequence $P = (v_i, x_1, \ldots, x_k, v_0)$ such that there exists a face incident to any two consecutive vertices of $P$ and such that $P$ uses only vertices of $N(v_{i+1}) \setminus C_\alpha(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is $(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0)$.

Let $P$ be such a sequence between $v_i$ and $v_0$ of minimal length. Together with $(v_1, \ldots, v_{i-1})$, $P$ forms an ordered separator of $G$ as required. \hfill \Box
4 The algorithm

Now we have all we need to build up an algorithm to compute the set $S_a(S, O)$ with $O \subseteq S$.

**ALGORITHM: **calc3_

**input:**
- $G$ a 3-connected planar graph
- $a$ a vertex of $G$
- $S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \not\in S$
- $O = (v_0, \ldots, v_i)$ with $i \leq p-1$ a subset of $S$

The vertices which have an incident face in common with $v_i$ ($i \geq 1$) are tagged $i$ unless they can be tagged $j$ ($1 \leq j \leq i-1$).

These vertices are the forbidden vertices.

The vertices of $C_a(S)$ are also tagged “$C_a(S)$”.

**output:**
- $S_a(S, O)$

**begin**

if $i = p - 1$ then
  return($\{S\}$)
else
  $x \leftarrow v_{i+1}$
  tag if necessary the faces incident to $x$ with $i + 1$
  $S \leftarrow$ calc3$_a(G, a, S, (v_0, \ldots, v_i, x))$
  untag the faces incident to $x$
  for each $y \in N(x)$ not tagged “$C_a(S)$”
    if $y$ is tagged $j < i$ then
      return($S$)
    for each $S'$ in find_min_elements$(G, a, x, S, O)$
      $S \leftarrow S \cup$ calc3$_a(G, a, S', (v_0, \ldots, v_i))$
end

**Property 9** The algorithm calc3$_a$ is correct. It computes the set $S_a(S, O)$ of a 3-connected planar graph.

**Proof.** The algorithm is just an application of remark 1. $\square$

**Property 10** The algorithm can be implemented to compute the set $S_a(S, O)$ in time $O(n|S_a(S, O)|)$.

**Proof.** For each minimal separator $S$, the algorithm does the following:

i. the function find_min_elements produces $S$;

ii. for every $x \in S \setminus O$, there is a recursive call to calc3$_a$ to extend the set $O$;

iii. $S$ is returned.

The function find_min_elements does a graph search to compute the sets $S_i$, and to tag the vertices in $C_a(S_i)$. It orders $S_i$ and tag the forbidden vertices. In a planar graph, the number $m$ of edges satisfies $0 \leq m \leq 3n - 6$, so all this costs $O(n)$.

Each call to calc3$_a$ costs $O(d(x))$ to tag and untag the faces incident to $x$, and $O(d(x))$ to check whether $S_a^*(S, O)$ is empty or not. Since every time a different $x$ is chosen, the recursive calls to calc3$_a$ cost $O(n)$.

The overall complexity of function calc3$_a$ is $O(n|S_a(S, O)|)$. $\square$

The following algorithm uses the function calc3$_a$ to compute the set of all minimal separators of a planar graph $G$. 
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ALGORITHM: all_min_sep3

input: 
G a 3-connected planar graph

output: 
the set of the $a,\ast$-minimal separators of $G$

begin
$S \leftarrow \emptyset$
find $a \in V$ with $d(a) < 6$
for each minimal separator $S \subseteq N(a)$
\hspace{1em} $S \leftarrow S \cup \text{calc3}(G, a, S, \emptyset)$
for each $y \in N(a)$
\hspace{1em} for each minimal separator $S \subseteq N(y)$
\hspace{2em} $S \leftarrow S \cup \text{calc3}(G, y, S, \emptyset)$
return $(S)$
end

Theorem 1 all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time $O(n|S(G)|)$

Proof. Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex $a$, $S \in S(G)$ either belongs to $S_a$ or runs through $a$. In the second case, it is a $b,\ast$-minimal separator for a neighbour $b$ of $a$.

Moreover, there exists a vertex $a$ of degree at most five in a planar graph. Let $b_1, \ldots, b_p$ be its neighbours.

By computing $S_a \cup (\bigcup_{i \in [1..p]} S_{b_i})$, a minimal separator can be calculated at most six times which gives the claimed complexity. $\square$

5 Conclusion

In the conclusion of [1], Berry and al. note that their algorithm may compute a minimal separator up to $n$ times and that this could be improved. This paper confirms this feeling for this is exactly what I have gained for 3-connected planar graphs. However it would be more satisfying to compute the minimal separators of all planar graphs. I feel that a slightly modified version of my algorithm could compute them. I also feel, just like Berry and al., that there could be a better general algorithm to compute the minimal separators of a graph.

This paper gives another proof that planar graphs and their minimal separators in particular are peculiar. I feel that topological properties such as property 5 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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References


