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Listing all the minimal separators of a 3-connected planar graph

Frédéric Mazoit  Février 2004
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Abstract
I present an efficient algorithm which lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Keywords: 3-connected planar graphs, minimal separator enumeration*

Résumé
Je présente un algorithme d’Énumération des séparateurs minimaux des graphes planaires 3-connexes dont la complexité est $O(n)$ par séparateur.

Mots-clés: graphes planaires 3 connexes, séparateurs minimaux, énumération
1 Introduction

In this paper, we address the problem of finding the minimal separators of a 3-connected planar graph $G$.

In the last ten years, minimal separators have been an increasingly used tool in graph theory with many algorithmic applications (for example [4], [7], [8], [10]).

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [4], Bodlaender and al. conjecture that for a class of graph which a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the notion of potential maximal clique (see [2]) and showed that if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can indeed be solved in polynomial time. They later showed in [3] that if a graph has a polynomial number of minimal separators, then it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Some research has been done to compute the set of the minimal separators of a graph ([1], [5], [6],[9]). In [1], Berry and al. proposed an algorithm of running time $O(n^3)$ per separator which uses the idea of generating a new minimal separator from an older one $S$ by looking at the separator $S \cup N(x)$ for $x \in S$. This separator is not minimal but the neighbourhoods of the connected components it defines are. This simple process can generate all the minimal separators of a graph. The counterpart is that a minimal separator can be generated many times.

In this paper, I adapt this idea to 3-connected planar graphs but to avoid the problem of recalculation, I define the set $S_a(S,O)$ of the $a$-$b$-minimal separators $S'$ for some $b$ that are such that the connected component of $a$ in $G\setminus S'$ contains the connected component of $a$ in $G\setminus S$ but avoids the set $O$. This way I put restrictions on the minimal separators I compute to ensure I do not compute the same minimal separator over and over.

2 Definitions

Throughout this paper, $G = (V,E)$ will be a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x,y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x,y) \in E\}$.

A set $S \subseteq V$ is an $a,b$ minimal separator if $a$ and $b$ are in two distinct connected components of $G\setminus S$ and no proper subset of $S$ separates them. The connected component of $a$ in $G\setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. A set $S$ is a minimal separator if there exists $a$ and $b$ which make it an $a,b$-minimal separator or, which is equivalent, if it has at least two full connected components. An $a,*$-minimal separator of a graph $G = (V,E)$ is a set of vertices $S$ such that there exists $b \in V$ which makes it an $a,b$-minimal separator. The set of the $a,*$-minimal separators is denoted by $S_a$ and the set of the minimal separators of $G$ is denoted by $S(G)$.

We can order the $a,*$-minimal separators in the following way:

$$S_1 \preceq S_2 \quad \text{if} \quad C_a(S_1) \subseteq C_a(S_2).$$

For $S$ an $a,*$-minimal separators and $O \subseteq V$, the set $S_a(S,O)$ is the set of the $a,*$-minimal separator $S'$ such that $S \preceq S'$ and $O \cap C_a(S') = \emptyset$. And if $x \in V$, the set $S^x_a(S,O)$ is the set of all $S'$ in $S_a(S,O)$ such that $x \in C_a(S')$.

Remark 1 If $x \in S$, then $S_a(S,O)$ is the disjoint union

$$S_a(S,O \cup \{x\}) \cup S^x_a(S,O).$$

And more precisely, if $(S_i)_{i \in I}$ are the minimal elements of $S^x_a(S,O)$, we have

$$S_a(S,O) = S_a(S,O \cup \{x\}) \bigcup_{i \in I} S_a(S_i,O).$$

This gives us the skeleton of an algorithm to compute the set $S_a(S,O)$. 
Remark 2 If \( S \) belongs to \( S_a^x(S,O) \), then \( S_a^x(S,O) = S_a(S,O) \).

The algorithm is based on remarks 1 and 2. To have \( S_a \), the algorithm computes the sets \( S_a(S,\emptyset) \) for every \( S \) minimal in \( S_a \). During this calculation, it will have to compute \( S_a(S,O) \) with \( O \subseteq S \). To do so, it chooses \( x \in S\setminus O \) and calculates \( S_a^x(S,O) \) and \( S_a(S,O \cup \{x\}) \). The set \( S_a^x(S,O) \) is itself a union of \( S_a(S,O) \). But to obtain such a decomposition, one needs to find the minimal elements of \( S_a^x(S,O) \), which the following property does.

Property 1 Let \( G = (V,E) \) be a graph, \( S \) an \( a,* \)-minimal separator, \( O \subseteq S \) and \( x \in S \setminus O \).

Every minimal element of \( S_a^x(S,O) \) is the neighbourhood of a connected component of \( G \setminus \{N(C) \cup C\} \) with \( C = C_a(S) \cup \{x\} \).

Proof. Let \( S_1 \in S_a^x(S,O) \) be an \( a,b \)-minimal separator.

Let \( C' = C_b(N(C)) \) and let \( S' = N(C') \). \( S' \subseteq N(C) \). By construction, \( S' \) is an \( a,b \)-separator. Moreover, \( C_a(S') \) and \( C_b(S') \) are two full connected components which proves that \( S' \) is an \( a,b \)-minimal separator.

Let \( p \) be a path in \( C_b(S_1) \) with \( b \) as one of its ends. The vertices of \( S_1 \) are at least at distance 1 of \( C \) so the vertices of \( p \) are at least at distance 2 of \( C \). Because \( S' \subseteq N(C) \), \( p \cap S' = \emptyset \). Finally, since \( b \in C' \), so does \( p \) and \( C_b(S_1) \subseteq C_b(S') \). The \( a,b \)-minimal separators being a lattice for the relation \( \preceq \), \( S_1 \) is greater than \( S' \). Moreover, since \( O \cap C_a(S_1) = \emptyset \), \( O \cap C_a(S') = \emptyset \) and \( S' \in S_a^x(S,O) \).

If \( S_1 \) is minimal, then \( S_1 = S' \) and \( S_1 \) is then the neighbourhood of a connected component of \( G \setminus \{N(C) \cup C\} \) as required.

The property 1 gives us a good way to find the minimal elements of \( S_a^x(S,O) \), using the skeleton of remark 1, we can design an algorithm to compute the set \( S_a(S,O) \). It could look like:

**ALGORITHM: _calc3_

begin
  if \( S \setminus O = \emptyset \) then
    return(\{\}
  else
    let \( x \in S \setminus O \)
    \( S \leftarrow \_	ext{calc3}_\_ (G,a,S,O \cup \{x\}) \)
    for each \( S_1 \) in \( \text{find\_min\_elements}(G,a,x,S,O) \)
      \( S \leftarrow S \cup \_	ext{calc3}_\_ (G,a,S_1,O) \)
    return(\( S \))
end

But there are several problems to solve.

i. First, we do not know whether the sets \( S_a(S_1,O) \) are disjoint or not. If not, we could compute a minimal separator many times which would lead to a bad complexity.

ii. To implement the function \( \text{find\_min\_elements} \), property 1 states that we can use a graph search of \( G \).

But if \( S_a(S,O) = \{S\} \), the algorithm will try to find a minimal element in \( S_a^x(S,O) \) for every \( x \in S \setminus O \). Each call to \( \text{find\_min\_elements} \) costs \( O(m) \) and in the end, we would have spent \( O(nm) \) to realise that \( S_a(S,O) = \{S\} \).

Property 3 ensures that for 3-connected planar graphs, point (i) is true and the section 3.3 shows how to determine that \( S_a^x(S,O) \) is empty in an overall \( O(n) \).
3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let $\Sigma$ be the plane. A plane graph $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of $\Sigma$ between two vertices of $V_\Sigma$ in such a way that the interiors of two distinct edges do not meet. We will denote by $\bar{G}_\Sigma$ the drawing of $G_\Sigma$. A planar graph is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of $G_\Sigma$ is a connected component of $\Sigma \setminus \bar{G}_\Sigma$.

### 3.1 Minimal separators of 3-connected planar graphs

**Property 2** In a 3-connected planar graph, minimal separators are minimal for inclusion.

**Proof.** Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let $a$, $b$, $c$ and $d$ be vertices such that $S'$ is an $a,b$-minimal separator and $S$ is a $c,d$-minimal separator. Since $S$ is not an $a,b$-minimal separator, either $C_a(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_a(S')$ is such a component. $C_a(S) = C_a(S')$ and $N(C_a(S)) = S$.

But then $G$ admits $K_{3,3}$ as a minor if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices $a'$, $b'$ and $c'$, all these vertices have $S$ in their neighbourhood and since $G$ is 3-connected $|S| \geq 3$. This contradicts that fact that $G$ is planar. \qed

**Property 3** Let $G = (V, E)$ be a 3-connected planar graph, $a \in V$, $S$ an $a,*$-minimal separator, $O \subseteq S$ and $x \in S \setminus O$.

If $S_1$ and $S_2$ are two minimal elements of $S^*_a(S,O)$, then

$$S_a(S_1,O) \cap S_a(S_2,O) = \emptyset.$$ 

**Proof.** Suppose that $S_1$ and $S_2$ are two distinct minimal elements of $S^*_a(S,O)$.

By property 1, $S_1$ and $S_2$ are subsets of $S' = N(C_a(S) \cup \{x\})$.

Let $b$ be a vertex such that $S_1$ is an $a,b$-minimal separator. Since $S_1$ and $S_2$ are not comparable, $S_2$ is not an $a,b$-separator. Indeed, since the set of all $a,b$-minimal separators is a lattice, $\text{min}(S_1, S_2)$ would be in $S^*_a(S,O)$ which would contradict the fact that $S_1$ and $S_2$ are minimal elements of $S^*_a(S,O)$.

Suppose that $S_3 \in S_a(S_1,O) \cap S_a(S_2,O)$ is an $a,c$-minimal separator.

Since $S_1$ and $S_2$ are included in $S'$, $S'_3 = N(C_c(S'))$ is an $a,c$-minimal separator greater than $S_1$ and $S_2$ and smaller than $S_3$ so $S'_3 \in S^*_a(S,O)$.

But $S'_3$ is included in $S_1$ and $S_2$ which is impossible in a 3-connected graph by property 2. \qed

### 3.2 The intermediate graph

**Definition 1** Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a plane 3-connected graph. Let $F$ be the set of its faces. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup F$. We place an edge between a vertex $v \in V$ and $f \in F$ if and only if the vertex $v$ is incident to the face $f$.

For $G'$ a subgraph of $G_I$, the set $G' \cap V_\Sigma$ will be denoted by $V(G')$.

**Property 4** Let $\mu$ be a cycle of $G_I$ such that the curve $\tilde{\mu}$ separates at least two vertices $a$ and $b$ of $V_\Sigma$.

The set $V(\mu)$ is an $a,b$-separator of $G_\Sigma$.

**Proof.** Let $p$ be a path in $G_\Sigma$ from $a$ to $b$. Since $a$ and $b$ are not in the same connected component of $\Sigma \setminus \tilde{\mu}$, $\tilde{p}$ intersects $\tilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from $a$ to $b$ meets $V(\mu)$ and so $V(\mu)$ is an $a,b$-separator. \qed
Property 5 Let $S$ be an $a,b$-minimal separator of $G$. There exists a simple cycle $\mu$ of $G_I$ such that the Jordan curve it defines separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.

Proof. Let $C$ be the connected component of $a$ in $G \setminus S$. Contract $C$ into a supervertex $v_C$ to build the graph $G_{C}$. In $G_{C}$, there is a cycle $\mu_{C}$ of $(G_{C})_{I}$ such that $V(\mu_{C}) = N(v_{C})$. Therefore, in $G_{I}$, the neighbourhood of $C$ has the structure of a cycle $\mu$.

Suppose $\tilde{\mu}$ is not a Jordan curve, the border $\mu'$ of the connected component of $b$ in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates $a$ and $b$. But then property 4 shows that $V(\mu')$ which is a strict subset of $S$ is an $a,b$-separator. This contradicts the fact that $S$ is a $a,b$-minimal separator. \qed

Property 5 shows that the minimal separators of a 3-connected planar graph are cycles of the intermediate graph which gives a criteria to say when a set is not a minimal separator. It gives nothing more for some cycles of $G_I$ correspond to no minimal separator of $G$.

There are several ways to find an exact criteria for minimal separators. The following section gives one which is well suited for our purpose.

3.3 Ordered separators

Definition 2 An ordered separator of $G$ is a sequence of distinct vertices

$$(v_0, \ldots, v_{p-1}), \ (p > 2)$$

such that

i. there exists a face to which $v_i$ and $v_{i+1}[p]$ are incident;

ii. $v_i$ and $v_j$ are incident to a common face only if $i = j + 1[p]$ or $j = i + 1[p]$;

iii. there is no face incident to $v_i$, $v_{i+1}[p]$ and $v_{i+2}[p]$;

The notation $i[p]$ means $i$ modulo $p$.

We say that a set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation $\sigma$ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator of $G$, then $S$ is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will either use an ordered separator as a sequence or as the corresponding set.

Remark 3 If $p > 3$, the third condition is a corollary of the second for $v_i$ et $v_{i+2}[p]$ would be too far apart.

Lemma 1 Every minimal separator $S$ of $G$ is ordered.

Proof. Let $S$ be an $a,b$-minimal separator of $G$.

The property 5 states that there exists a simple cycle of $G_I$

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to $S$.

i. The construction of $T$ ensures that $v_i$ and $v_{i+1}$ are incident to a common face $(f_i)$.

ii. Suppose that $v_{i} \text{ et } v_{j}$ are incident to a common face $f$ and that $i+1 \neq j[p]$ and $j+1 \neq i[p]$.

$$\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$$

and $\mu_2 = (v_i, f_j, v_{j+1}, f_{j+1}, \ldots, v_j, f)$ are laces of $G_I$. Moreover, since either $\mu_1$ or $\mu_2$ separates $a$ and $b$, there exists an $a,b$-separator strictly included in $S$ which is absurd.

iii. With the remark 3, we can suppose that $p = 3$.

Suppose that $v_0, v_1 \text{ et } v_2$ are all incident to a common face $f$. If we add a vertex $f$ to $G$ that we connect to the vertices $v_0, v_1$ et $v_2$, the graph remains planar which is absurd for this graph has $K_{3,3}$ as a minor. Indeed, the connected component of $a$, the connected component of $b$ and the vertex $f$ are all incident to $v_0, v_1$ et $v_2$ which builds up a $K_{3,3}$.
The sequence \( T \) is an ordered separator corresponding to \( S \).

Conversely,

**Lemma 2** Every ordered separator of \( G \) is a minimal separator of \( G \).

**Proof.** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered separator of \( G \).

First, \( S \) is a separator. Otherwise, \( G \setminus S \) would be connected or empty. In both cases all the vertices of \( S \) would be incident to a common face.

Let \( S' \) be a minimal separator included in \( S \). By lemma 1, \( S' \) is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, \( S' = S \). The ordered separator \( S \) is a minimal separator.

From lemma 1 and 2, we have the following property:

**Property 6** A set \( S \subseteq V \) is a minimal separator of a 3-connected planar graph \( G = (V, E) \) if and only if it corresponds to an ordered separator of \( G \).

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether \( S_n(S, O) \) is empty or not (\( O \subseteq S \) and \( x \in S \setminus O \)).

**Property 7** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered a,\(-\)-separator of a 3-connected planar graph \( G = (V, E) \).

Let \( O = (v_0, \ldots, v_i), (i < p - 1) \) be an initial sequence of \( S \).

If there exists a face which is incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) with \( 0 < j < i \), then 
\[ S_n^{i+1}(S, O) = \emptyset. \]

**Proof.** Suppose that \( S' \) is a minimal element of \( S_n^{i+1}(S, O) \) and \( f \) is incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) with \( 0 < j < i \).

By property 1, \( S' \subseteq N(C_a(S) \cup \{v_{i+1}\}) \) and by lemma 1, \( S \) is an ordered separator. So \( S' = (v_0, \ldots, v_i, y_1, \ldots, y_l) \).

Since \( S \) is a minimal separator, no \( y_k \) can be incident to \( f \).

But since there is a face to which \( y_k \) and \( y_k+1 \) are incident and since there is a face to which \( v_i \) and \( y_i \) are incident, in clockwise order, all the vertices \( y_k \) are between \( v_i \) and \( y_i \). But there is no face to which \( y_k \) and \( v_0 \) are incident and \( S' \) is not an ordered separator.

Conversely,

**Property 8** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered a,\(-\)-separator of a 3-connected planar graph \( G = (V, E) \).

Let \( O = (v_0, \ldots, v_i), (i < p - 1) \) be an initial sequence of \( S \).

If there is no face incident to both \( y \in N(v_{i+1}) \setminus C_a(S) \) and \( v_j \) (\( 0 < j < i \)), then there is an ordered separator in \( S \cup N(v_{i+1}) \setminus C_a(S) \) which contains \( O \).

**Proof.** The neighbours \( (y_1, \ldots, y_l) \) of \( v_{i+1} \) taken in clockwise order are such that \( y_i \) and \( y_{i+1} \) are incident to the same face. Moreover, since \( v_i+1 \) and \( v_i \) are both incident to a face \( f_1 \) and since \( v_{i+1} \) and \( v_{i+2} \) are both incident to a face \( f_2 \), there is a sequence \( P = (v_i, x_1, \ldots, x_k, v_0) \) such that there exists a face incident to any two consecutive vertices of \( P \) and such that \( P \) uses only vertices of \( N(v_{i+1}) \setminus C_a(S) \) and \( v_{i+2}, \ldots, v_{p-1} \). One such sequence is \( (v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0) \).

Let \( P \) be such a sequence between \( v_i \) and \( v_0 \) of minimal length. Together with \((v_1, \ldots, v_{i-1})\), \( P \) forms an ordered separator of \( G \) as required.
4 The algorithm

Now we have all we need to build up an algorithm to compute the set $S_a(S, O)$ with $O \subseteq S$.

**ALGORITHM: _calc3_**

**input:**
- $G$ a 3-connected planar graph
- $a$ a vertex of $G$
- $S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
- $O = (v_0, \ldots, v_i)$ with $i \leq p - 1$ a subset of $S$

The vertices which have an incident face in common with $v_i$ ($i \geq 1$) are tagged $i$
unless they can be tagged $j$ ($1 \leq j \leq i - 1$).
Theses vertices are the forbidden vertices.
The vertices of $C_a(S)$ are also tagged "$C_a(S)$".

**output:**
$S_a(S, O)$

**begin**

if $i = p - 1$ then

return($\{S\}$)

else

$x \leftarrow v_{i+1}$
tag if necessary the faces incident to $x$ with $i + 1$

$S \leftarrow _{calc3_}(G, a, S, (v_0, \ldots, v_i, x))$
untag the faces incident to $x$

for each $y \in N(x)$ not tagged "$C_a(S)$"

if $y$ is tagged $j < i$ then

return($S$)

for each $S'$ in $\text{find\_min\_elements}(G, a, x, S, O)$

$S \leftarrow S \cup _{calc3_}(G, a, S', (v_0, \ldots, v_i))$
end

**Property 9** The algorithm _calc3_ is correct. It computes the set $S_a(S, O)$ of a 3-connected planar graph.

*Proof.* The algorithm is just an application of remark 1. \hfill $\square$

**Property 10** The algorithm can be implemented to compute the set $S_a(S, O)$ in time $O(n |S_a(S, O)|)$.

*Proof.* For each minimal separator $S$, the algorithm does the following:

i. the function $\text{find\_min\_elements}$ produces $S$;

ii. for every $x \in S\setminus O$, there is a recursive call to _calc3_ to extend the set $O$;

iii. $S$ is returned.

The function $\text{find\_min\_elements}$ does a graph search to compute the sets $S_i$, and to tag the vertices in $C_a(S_i)$. It orders $S_i$ and tag the forbidden vertices. In a planar graph, the number $m$ of edges satisfies $0 \leq m \leq 3n - 6$, so all this costs $O(n)$.

Each call to _calc3_ costs $O(d(x))$ to tag and untag the faces incident to $x$, and $O(d(x))$ to check whether $S^x_a(S, O)$ is empty or not. Since every time a different $x$ is chosen, the recursive calls to _calc3_ cost $O(n)$.

The overall complexity of function _calc3_ is $O(n |S_a(S, O)|)$. \hfill $\square$

The following algorithm uses the function _calc3_ to compute the set of all minimal separators of a planar graph $G$.
Listing the minimal separators of a 3-connected planar graph

**ALGORITHM:** all_min_sep3

**input:**
- $G$ a 3-connected planar graph

**output:**
- the set of the $a,*$-minimal separators of $G$

**begin**

$S \leftarrow \emptyset$

**find** $a \in V$ **with** $d(a) < 6$

**for each** minimal separator $S \subseteq N(a)$

$S \leftarrow S \cup \text{calc3}_*(G, a, S, \emptyset)$

**for each** $y \in N(a)$

**for each** minimal separator $S \subseteq N(y)$

$S \leftarrow S \cup \text{calc3}_*(G, y, S, \emptyset)$

**return** $(S)$

**end**

**Theorem 1** all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time $O(n|S(G)|)$

**Proof.** Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex $a$, $S \in S(G)$ either belongs to $S_a$ or runs through $a$. In the second case, it is a $b,*$-minimal separator for a neighbour $b$ of $a$.

Moreover, there exists a vertex $a$ of degree at most five in a planar graph. Let $b_1, \ldots, b_p$ be its neighbours.

By computing $S_a \cup (\bigcup_{i \in [1..p]} S_{b_i})$, a minimal separator can be calculated at most six times which gives the claimed complexity. \qed

5 **Conclusion**

In the conclusion of [1], Berry and al. note that their algorithm may compute a minimal separator up to $n$ times and that this could be improved. This paper confirms this feeling for this is exactly what I have gained for 3-connected planar graphs. However it would be more satisfying to compute the minimal separators of all planar graphs. I feel that a slightly modified version of my algorithm could compute them. I also feel, just like Berry and al., that there could be a better general algorithm to compute the minimal separators of a graph.

This paper gives another proof that planar graphs and their minimal separators in particular are peculiar. I feel that topological properties such as property 5 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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**References**


