

Listing all the minimal separators of a 3-connected planar graph.

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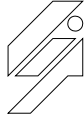
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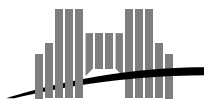
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Listing all the minimal separators of a 3-connected planar graph

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Février 2004

Abstract

I present an efficient algorithm which lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Keywords: 3-connected planar graphs, minimal separator enumeration*

Résumé

Je présente un algorithme d'énumération des séparateurs minimaux des graphes planaires 3-connexes dont la complexité est $O(n)$ par séparateur.

Mots-clés: graphes planaires 3 connexes, séparateurs minimaux, énumération

1 Introduction

In this paper, we address the problem of finding the minimal separators of a 3-connected planar graph G .

In the last ten years, minimal separators have been an increasingly used tool in graph theory with many algorithmic applications (for example [4], [7], [8], [10]).

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [4], Bodlaender and al. conjecture that for a class of graph which a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the notion of potential maximal clique (see [2]) and showed that if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can indeed be solved in polynomial time. They later showed in [3] that if a graph has a polynomial number of minimal separators, then it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Some research has been done to compute the set of the minimal separators of a graph ([1], [5], [6],[9]). In [1], Berry and al. proposed an algorithm of running time $O(n^3)$ per separator which uses the idea of generating a new minimal separator from an older one S by looking at the separator $S \cup N(x)$ for $x \in S$. This separator is not minimal but the neighbourhoods of the connected components it defines are. This simple process can generate all the minimal separators of a graph. The counterpart is that a minimal separator can be generated many times.

In this paper, I adapt this idea to 3-connected planar graphs but to avoid the problem of recalculation, I define the set $\mathcal{S}_a(S, O)$ of the a, b -minimal separators S' for some b that are such that the connected component of a in $G \setminus S'$ contains the connected component of a in $G \setminus S$ but avoids the set O . This way I put restrictions on the minimal separators I compute to ensure I do not compute the same minimal separator over and over.

2 Definitions

Throughout this paper, $G = (V, E)$ will be a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$.

A set $S \subseteq V$ is an a, b minimal separator if a and b are in two distinct connected components of $G \setminus S$ and no proper subset of S separates them. The connected component of a in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. A set S is a minimal separator if there exists a and b which make it an a, b -minimal separator or, which is equivalent, if it has at least two full connected components. An $a, *$ -minimal separator of a graph $G = (V, E)$ is a set of vertices S such that there exists $b \in V$ which makes it an a, b -minimal separator. The set of the $a, *$ -minimal separators is denoted by \mathcal{S}_a and the set of the minimal separators of G is denoted by $\mathcal{S}(G)$.

We can order the $a, *$ -minimal separators in the following way:

$$S_1 \preceq S_2 \quad \text{if} \quad C_a(S_1) \subseteq C_a(S_2).$$

For S an $a, *$ -minimal separators and $O \subseteq V$, the set $\mathcal{S}_a(S, O)$ is the set of the $a, *$ -minimal separator S' such that $S \preceq S'$ and $O \cap C_a(S') = \emptyset$. And if $x \in V$, the set $\mathcal{S}_a^x(S, O)$ is the set of $S' \in \mathcal{S}_a(S, O)$ such that $x \in C_a(S')$.

Remark 1 If $x \in S$, then $\mathcal{S}_a(S, O)$ is the disjoint union

$$\mathcal{S}_a(S, O \cup \{x\}) \sqcup \mathcal{S}_a^x(S, O).$$

And more precisely, if $(S_i)_{i \in I}$ are the minimal elements of $\mathcal{S}_a^x(S, O)$, we have

$$\mathcal{S}_a(S, O) = \mathcal{S}_a(S, O \cup \{x\}) \sqcup \left(\bigcup_{i \in I} \mathcal{S}_a(S_i, O) \right).$$

This gives us the skeleton of an algorithm to compute the set $\mathcal{S}_a(S, O)$.

Remark 2 *If S belongs to $\mathcal{S}_a^x(S, O)$, then $\mathcal{S}_a^x(S, O) = \mathcal{S}_a(S, O)$.*

The algorithm is based on remarks 1 and 2. To have \mathcal{S}_a , the algorithm computes the sets $\mathcal{S}_a(S, \emptyset)$ for every S minimal in \mathcal{S}_a . During this calculation, it will have to compute $\mathcal{S}_a(S, O)$ with $O \subseteq S$. To do so, it chooses $x \in S \setminus O$ and calculates $\mathcal{S}_a^x(S, O)$ and $\mathcal{S}_a(S, O \cup \{x\})$. The set $\mathcal{S}_a^x(S, O)$ is itself a union of $\mathcal{S}_a(S_i, O)$. But to obtain such a decomposition, one needs to find the minimal elements of $\mathcal{S}_a^x(S, O)$, which the following property does.

Property 1 *Let $G = (V, E)$ be a graph, S an $a, *$ -minimal separator, $O \subset S$ and $x \in S \setminus O$.*

Every minimal element of $\mathcal{S}_a^x(S, O)$ is the neighbourhood of a connected component of $G \setminus \{N(C) \cup C\}$ with $C = C_a(S) \cup \{x\}$.

Proof. Let $S_1 \in \mathcal{S}_a^x(S, O)$ be an a, b -minimal separator.

Let $C' = C_b(N(C))$ and let $S' = N(C')$. $S' \subseteq N(C)$. By construction, S' is an a, b -separator. Moreover, $C_a(S')$ and $C_b(S')$ are two full connected components which proves that S' is an a, b -minimal separator.

Let p be a path in $C_b(S_1)$ with b as one of its ends. The vertices of S_1 are at least at distance 1 of C so the vertices of p are at least at distance 2 of C . Because $S' \subseteq N(C)$, $p \cap S' = \emptyset$. Finally, since $b \in C'$, so does p and $C^b(S_1) \subseteq C^b(S')$. The a, b -minimal separators being a lattice for the relation \preceq , S_1 is greater than S' . Moreover, since $O \cap C_a(S_1) = \emptyset$, $O \cap C_a(S') = \emptyset$ and $S' \in \mathcal{S}_a^x(S, O)$.

If S_1 is minimal, then $S_1 = S'$ and S_1 is then the neighbourhood of a connected component of $G \setminus \{N(C) \cup C\}$ as required. \square

The property 1 gives us a good way to find the minimal elements of $\mathcal{S}_a^x(S, O)$, using the skeleton of remark 1, we can design an algorithm to compute the set $\mathcal{S}_a(S, O)$. It could look like:

ALGORITHM: _calc3_

```

begin
  if  $S \setminus O = \emptyset$  then
    return( $\{S\}$ )
  else
    let  $x \in S \setminus O$ 
     $\mathcal{S} \leftarrow \text{\_calc3\_}(G, a, S, O \cup \{x\})$ 

    for each  $S_i$  in find_min_elements( $G, a, x, S, O$ )
       $\mathcal{S} \leftarrow \mathcal{S} \cup \text{\_calc3\_}(G, a, S_i, O)$ 
    return( $\mathcal{S}$ )
end

```

But there are several problems to solve.

- i. First, we do not know whether the sets $\mathcal{S}_a(S_i, O)$ are disjoint or not. If not, we could compute a minimal separator many times which would lead to a bad complexity.
- ii. To implement the function `find_min_elements`, property 1 states that we can use a graph search of G .

But if $\mathcal{S}_a(S, O) = \{S\}$, the algorithm will try to find a minimal element in $\mathcal{S}_a^x(S, O)$ for every $x \in S \setminus O$. Each call to `find_min_elements` costs $O(m)$ and in the end, we would have spent $O(nm)$ to realise that $\mathcal{S}_a(S, O) = \{S\}$.

Property 3 ensures that for 3-connected planar graphs, point (i) is true and the section 3.3 shows how to determine that $\mathcal{S}_a^x(S, O)$ is empty in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let Σ be the plane. A *plane graph* $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of Σ between two vertices of V_Σ in such a way that the interiors of two distinct edges do not meet. We will denote by \tilde{G}_Σ the drawing of G_Σ . A *planar graph* is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of G_Σ is a connected component of $\Sigma \setminus \tilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Property 2 *In a 3-connected planar graph, minimal separators are minimal for inclusion.*

Proof. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let a, b, c and d be vertices such that S' is an a, b -minimal separator and S is a c, d -minimal separator. Since S is not an a, b -minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. $C_c(S) = C_c(S')$ and $N(C_c(S)) = S$.

But then G admits $K_{3,3}$ as a minor for if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices a', b' and c' , all these vertices have S in their neighbourhood and since G is 3-connected $|S| \geq 3$. This contradicts that fact that G is planar. \square

Property 3 *Let $G = (V, E)$ be a 3-connected planar graph, $a \in V$, S an $a, *$ -minimal separator, $O \subseteq S$ and $x \in S \setminus O$.*

If S_1 and S_2 are two minimal elements of $\mathcal{S}_a^x(S, O)$, then

$$\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O) = \emptyset.$$

Proof. Suppose that S_1 and S_2 are two distinct minimal elements of $\mathcal{S}_a^x(S, O)$.

By property 1, S_1 and S_2 are subsets of $S' = N(C_a(S) \cup \{x\})$.

Let b be a vertex such that S_1 is an a, b -minimal separator. Since S_1 and S_2 are not comparable, S_2 is not an a, b -separator. Indeed, since the set of all a, b -minimal separators is a lattice, $\min(S_1, S_2)$ would be in $\mathcal{S}_a^x(S, O)$ which would contradict the fact that S_1 and S_2 are minimal elements of $\mathcal{S}_a^x(S, O)$.

Suppose that $S_3 \in \mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O)$ is an a, c -minimal separator.

Since S_1 and S_2 are included in S' , $S'_3 = N(C_c(S'))$ is an a, c -minimal separator greater than S_1 and S_2 and smaller than S_3 so $S'_3 \in \mathcal{S}_a^x(S, O)$.

But S'_3 is included in S_1 and S_2 which is impossible in a 3-connected graph by property 2. \square

3.2 The intermediate graph

Definition 1 *Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a plane 3-connected graph. Let F be the set of its faces. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup F$. We place an edge between a vertex $v \in V$ and $f \in F$ if and only if the vertex v is incident to the face f .*

For G' a subgraph of G_I , the set $\tilde{G}' \cap V_\Sigma$ will be denoted by $V(G')$.

Property 4 *Let μ be a cycle of G_I such that the curve $\tilde{\mu}$ separates at least two vertices a and b of V_Σ .*

The set $V(\mu)$ is an a, b -separator of G_Σ .

Proof. Let p be a path in G_Σ from a to b . Since a and b are not in the same connected component of $\Sigma \setminus \tilde{\mu}$, \tilde{p} intersects $\tilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from a to b meets $V(\mu)$ and so $V(\mu)$ is an a, b -separator. \square

Property 5 *Let S be an a,b -minimal separator of G . There exists a simple cycle μ of G_I such that the Jordan curve it defines separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.*

Proof. Let C be the connected component of a in $G \setminus S$. Contract C into a supervertex v_C to build the graph $G_{/C}$. In $G_{/C}$, there is a cycle $\mu_{/C}$ of $(G_{/C})_I$ such that $V(\mu_{/C}) = N(v_C)$. Therefore, in G_I the neighbourhood of C has the structure of a cycle μ .

Suppose $\tilde{\mu}$ is not a Jordan curve, the border μ' of the connected component of b in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates a and b . But then property 4 shows that $V(\mu')$ which is a strict subset of S is an a,b -separator. This contradicts the fact that S is a a,b -minimal separator. \square

Property 5 shows that the minimal separators of a 3-connected planar graph are cycles of the intermediate graph which gives a criteria to say when a set is not a minimal separator. It gives nothing more for some cycles of G_I correspond to no minimal separator of G .

There are several ways to find an exact criteria for minimal separators. The following section gives one which is well suited for our purpose.

3.3 Ordered separators

Definition 2 *An ordered separator of G is a sequence of distinct vertices*

$$(v_0, \dots, v_{p-1}), (p > 2)$$

such that

- i. there exists a face to which v_i and $v_{i+1[p]}$ are incident;*
- ii. v_i and v_j are incident to a common face only if $i = j + 1[p]$ or $j = i + 1[p]$;*
- iii. there is no face incident to $v_i, v_{i+1[p]}$ and $v_{i+2[p]}$.*

The notation $i[p]$ means i modulo p .

We say that a set $S = \{v_0, \dots, v_{p-1}\}$ is an ordered separator if there exists a permutation σ such that $(v_{\sigma(0)}, \dots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = (v_0, \dots, v_{p-1})$ is an ordered separator of G , then S is naturally associated to the set $\{v_0, \dots, v_{p-1}\}$. We will either use an ordered separator as a sequence or as the corresponding set.

Remark 3 *If $p > 3$, the third condition is a corollary of the second for v_i et $v_{i+2[p]}$ would be too far apart.*

Lemma 1 *Every minimal separator S of G is ordered.*

Proof. Let S be an a,b -minimal separator of G .

The property 5 states that there exists a simple cycle of G_I

$$\mu = (v_0, f_0, \dots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \dots, v_{p-1})$ is an ordered separator corresponding to S .

- i. The construction of T ensures that v_i and v_{i+1} are incident to a common face (f_i).*
- ii. Suppose that v_i et v_j are incident to a common face f and that $i+1 \neq j[p]$ and $j+1 \neq i[p]$. $\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \dots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \dots, v_i, f)$ are laces of G_I . Moreover, since either μ_1 or μ_2 separates a and b , there exists an a,b -separator strictly included in S which is absurd.*
- iii. With the remark 3, we can suppose that $p = 3$.*

Suppose that v_0, v_1 et v_2 are all incident to a common face f . If we add a vertex f to G that we connect to the vertices v_0, v_1 and v_2 , the graph remains planar which is absurd for this graph has $K_{3,3}$ as a minor. Indeed, the connected component of a , the connected component of b and the vertex f are all incident to v_0, v_1 and v_2 which builds up a $K_{3,3}$.

The sequence T is an ordered separator corresponding to S .
Conversely,

□

Lemma 2 *Every ordered separator of G is a minimal separator of G .*

Proof. Let $S = (v_0, \dots, v_{p-1})$ be an ordered separator of G .

First, S is a separator. Otherwise, $G \setminus S$ would be connected or empty. In both cases all the vertices of S would be incident to a common face.

Let S' be a minimal separator included in S . By lemma 1, S' is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator S is a minimal separator. □

From lemma 1 and 2, we have the following property:

Property 6 *A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of G .*

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $\mathcal{S}_a^x(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \setminus O$).

Property 7 *Let $S = (v_0, \dots, v_{p-1})$ be an ordered $a, *$ -separator of a 3-connected planar graph $G = (V, E)$.*

Let $O = (v_0, \dots, v_i)$, ($i < p - 1$) be an initial sequence of S .

If there exists a face which is incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and v_j with $0 < j < i$, then $\mathcal{S}_a^{v_{i+1}}(S, O) = \emptyset$.

Proof. Suppose that S' is a minimal element of $\mathcal{S}_a^{v_{i+1}}(S, O)$ and f is incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and v_j with $0 < j < i$.

By property 1, $S' \subseteq N(C_a(S) \cup \{v_{i+1}\})$ and by lemma 1, S' is an ordered separator. So $S' = (v_0, \dots, v_i, y_1, \dots, y_l)$.

Since S is an ordered separator, no y_k can be incident to f .

But since there is a face to which y_k and y_{k+1} are incident and since there is a face to which v_i and y_1 are incident, in clockwise order, all the vertices y_k are between v_i and y . But there is no face to which y_l and v_0 are incident and S' is not an ordered separator. □

Conversely,

Property 8 *Let $S = (v_0, \dots, v_{p-1})$ be an ordered $a, *$ -separator of a 3-connected planar graph $G = (V, E)$.*

Let $O = (v_0, \dots, v_i)$, ($i < p - 1$) be an initial sequence of S .

If there is no face incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and v_j ($0 < j < i$), then there is an ordered separator in $S \cup N(v_{i+1}) \setminus C_a(S)$ which contains O .

Proof. The neighbours (y_1, \dots, y_l) of v_{i+1} taken in clockwise order are such that y_i and y_{i+1} are incident to the same face. Moreover, since v_{i+1} and v_i are both incident to a face f_1 and since v_{i+1} and v_{i+2} are both incident to a face f_2 , there is a sequence $P = (v_i, x_1, \dots, x_k, v_0)$ such that there exists a face incident to any two consecutive vertices of P and such that P uses only vertices of $N(v_{i+1}) \setminus C_a(S)$ and v_{i+2}, \dots, v_{p-1} . One such sequence is $(v_i, y_j, y_{j+1}, \dots, y_k, v_{i+2}, \dots, v_{p-1}, v_0)$.

Let P be such a sequence between v_i and v_0 of minimal length. Together with (v_1, \dots, v_{i-1}) , P forms an ordered separator of G as required. □

4 The algorithm

Now we have all we need to build up an algorithm to compute the set $\mathcal{S}_a(S, O)$ with $O \subseteq S$.

ALGORITHM: `_calc3_`

input:

G a 3-connected planar graph

a a vertex of G

$S = (v_0, \dots, v_{p-1})$ an ordered separator such that $a \notin S$

$O = (v_0, \dots, v_i)$ with $i \leq p - 1$ a subset of S

The vertices which have an incident face in common with v_i ($i \geq 1$) are tagged i unless they can be tagged j ($1 \leq j \leq i - 1$).

Theses vertices are the forbidden vertices.

The vertices of $C_a(S)$ are also tagged " $C_a(S)$ ".

output:

$\mathcal{S}_a(S, O)$

begin

if $i = p - 1$ **then**

return $\{S\}$

else

$x \leftarrow v_{i+1}$

tag if necessary the faces incident to x with $i + 1$

$\mathcal{S} \leftarrow \text{_calc3_}(G, a, S, (v_0, \dots, v_i, x))$

untag the faces incident to x

for each $y \in N(x)$ not tagged " $C_a(S)$ "

if y is tagged $j < i$ **then**

return \mathcal{S}

for each S' in `find_min_elements` (G, a, x, S, O)

$\mathcal{S} \leftarrow \mathcal{S} \cup \text{_calc3_}(G, a, S', (v_0, \dots, v_i))$

end

Property 9 *The algorithm `_calc3_` is correct. It computes the set $\mathcal{S}_a(S, O)$ of a 3-connected planar graph.*

Proof. The algorithm is just an application of remark 1. □

Property 10 *The algorithm can be implemented to compute the set $\mathcal{S}_a(S, O)$ in time $O(n|\mathcal{S}_a(S, O)|)$.*

Proof. For each minimal separator S , the algorithm does the following:

- i. the function `find_min_elements` produces S ;
- ii. for every $x \in S \setminus O$, there is a recursive call to `_calc3_` to extend the set O ;
- iii. S is returned.

The function `find_min_elements` does a graph search to compute the sets S_i , and to tag the vertices in $C_a(S_i)$. It orders S_i and tag the forbidden vertices. In a planar graph, the number m of edges satisfies $0 \leq m \leq 3n - 6$, so all this costs $O(n)$.

Each call to `_calc3_` costs $O(d(x))$ to tag and untag the faces incident to x , and $O(d(x))$ to check whether $\mathcal{S}_a^x(S, O)$ is empty or not. Since every time a different x is chosen, the recursive calls to `_calc3_` cost $O(n)$.

The overall complexity of function `_calc3_` is $O(n|\mathcal{S}_a(S, O)|)$. □

The following algorithm uses the function `_calc3_` to compute the set of all minimal separators of a planar graph G .

ALGORITHM: all_min_sep3**input:** G a 3-connected planar graph**output:**the set of the $a, *$ -minimal separators of G **begin** $S \leftarrow \emptyset$ **find** $a \in V$ **with** $d(a) < 6$ **for each** minimal separator $S \subseteq N(a)$ $S \leftarrow S \cup \text{calc3_}(G, a, S, \emptyset)$ **for each** $y \in N(a)$ **for each** minimal separator $S \subseteq N(y)$ $S \leftarrow S \cup \text{calc3_}(G, y, S, \emptyset)$ **return**(S)**end**

Theorem 1 all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time $O(n|\mathcal{S}(G)|)$

Proof. Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex a , $S \in \mathcal{S}(G)$ either belongs to \mathcal{S}_a or runs through a . In the second case, it is a $b, *$ -minimal separator for a neighbour b of a .

Moreover, there exists a vertex a of degree at most five in a planar graph. Let b_1, \dots, b_p be its neighbours.

By computing $\mathcal{S}_a \cup (\bigcup_{i \in [1..p]} \mathcal{S}_{b_i})$, a minimal separator can be calculated at most six times which gives the claimed complexity. \square

5 Conclusion

In the conclusion of [1], Berry and al. note that their algorithm may compute a minimal separator up to n times and that this could be improved. This paper confirms this feeling for this is exactly what I have gained for 3-connected planar graphs. However it would be more satisfying to compute the minimal separators of all planar graphs. I feel that a slightly modified version of my algorithm could compute them. I also feel, just like Berry and al., that there could be a better general algorithm to compute the minimal separators of a graph.

This paper gives another proof that planar graphs and their minimal separators in particular are peculiar. I feel that topological properties such as property 5 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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