Undecidability of the global fixed point attractor problem on circular cellular automata.

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Abstract

A great amount of work has been devoted to the understanding of the long-time behavior of cellular automata (CA). As for any other kind of dynamical system, the long-time behavior of a CA is described by its attractors. In this context, it has been proved that it is undecidable to know whether every circular configuration of a given CA evolves to some fixed point (not unique). In this paper we prove that it remains undecidable to know whether every circular configuration of a given CA evolves to the *same* fixed point. Our proof is based on properties concerning NW-deterministic periodic tilings of the plane. As a corollary it is concluded the (already proved) undecidability of the periodic tiling problem (nevertheless, our approach could also be used to prove this result in a direct and very simple way).

Keywords: cellular automata, periodic tilings of the plane.

Résumé

De nombreux travaux ont été consacrés à la compréhension de l'évolution à long terme des automates cellulaires (AC). Comme pour les autres types de systémes dynamiques, cette évolution à long terme est décrite par ses attracteurs. Dans ce contexte, il a été démontré indécidable de savoir si toute configuration périodique d'un AC donné èvolue vers un point fixe (peut-être non unique). Dans cet article, nous prouvons l'indécidabilité de savoir si toute configuration périodique evolue vers le *même* point fixe. Notre preuve s'appuie sur les propietés des pavages NW-déterministe et périodiques du plan. Comme corollaire, nous obtenons l'indécidabilité (déjà connue) de la pavabilité périodique (cependant notre approche permet d'arriver á ce résultat de façon simple et directe).

Mots-clés: automates cellulaires, pavages périodiques du plan.

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Abstract

A great amount of work has been devoted to the understanding of the long-time behavior of cellular automata (CA). As for any other kind of dynamical system, the long-time behavior of a CA is described by its attractors. In this context, it has been proved that it is undecidable to know whether every circular configuration of a given CA evolves to some fixed point (not unique). In this paper we prove that it remains undecidable to know whether every circular configuration of a given CA evolves to the *same* fixed point. Our proof is based on properties concerning NW-deterministic periodic tilings of the plane. As a corollary it is concluded the (already proved) undecidability of the periodic tiling problem (nevertheless, our approach could also be used to prove this result in a direct and very simple way).

Key words: cellular automata, periodic tilings of the plane.

1 Introduction

Cellular automata (CA) are discrete dynamical systems. They are defined by a *lattice of cells* and a *local rule* by which the *state* of a cell is determined as a function of the state of its neighborhood. A *configuration* of a CA is an assignment of states to the cells of the lattice. The *global transition function* is a map from the space of all configurations to itself obtained by applying the local rule simultaneously to all the cells. This global transition function corresponds to the CA *dynamics*.

Because of the dynamical system nature of CA, a great amount of work has been devoted to the understanding of its *long-time behavior* (consider, for instance, the well known Wolfram's classification of [Wol84]). The long-time behavior of any dynamical system is described by its attractors. In this context, for the two (and higher) dimensional CA, it was proved in [CPY89] the undecidability of the *nilpotency problem* (which, in practice, consists to decide whether every configuration of a given CA evolves to the same fixed point in a finite number of steps). Later J. Kari proved in [Kar92] the undecidability of the nilpotency problem for the one-dimensional case.

On the other hand, K. Sutner in [Sut90] restricted previous kind of study to *circular configurations* (those spatially periodic) because of their finitary description and therefore their possibility of being handled in the framework of ordinary computability theory. More precisely, by the use of non-standard simulations of Turing machines, it was proved that it is undecidable to know whether every circular configuration of a given one-dimensional CA evolves to some fixed point (not unique).

In this paper we prove that it remains undecidable to know whether every circular configuration of a given one-dimensional CA evolves to the *same* fixed point. Our result allows us to conclude the one of Sutner in a rather direct way.

The structure of our proof is inspired on the one developed by J. Kari in [Kar92]. In fact, our work is based on results concerning *tiling problems* and, in particular, on the useful *NW-deterministic* notion (roughly, a set of tiles is NW-deterministic if it is locally deterministic in one dimension). More precisely, here we prove that it is undecidable to know whether a given *NW-deterministic* set of tiles admits a *periodic* tiling of the plane. Despite the similarity with Kari's result, our objects are different in nature: the CA configurations considered here are *circular* and the tilings of the plane are *periodic*. In this particularity lies the difficulty of our proof.

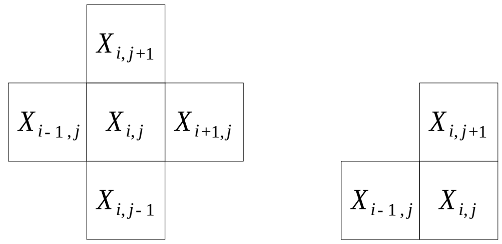
By the way, and as an obvious consequence, it can be concluded the undecidability of the *periodic tiling problem* (in which it is asked whether an *arbitrary* set of tiles admits a periodic tiling of the plane). This result was obtained by Gurevich-Koriakov in [GK72]. Nevertheless, we would like to remark that our approach could also be used to prove the Gurevich-Koriakov result in a *direct* way. In fact, when the NW-deterministic property is no more required, most of the technicities of the proof are no more needed and it becomes very simple.

2 Definitions

A one-dimensional cellular automaton with unitary radius neighborhood, or simply a CA, is a couple (Q, δ) where Q is a finite set of states and $\delta : Q^3 \to Q$ is a transition function. A configuration of a CA (Q, δ) is a bi-infinite sequence $\mathcal{C} \in Q^{\mathbb{Z}}$, and its global transition function $G_\delta : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ is such that $(G_\delta(\mathcal{C}))_i = \delta(\mathcal{C}_{i-1}, \mathcal{C}_i, \mathcal{C}_{i+1})$ for all $i \in \mathbb{Z}$. For $t \in \mathbb{N}^* = \mathbb{N} - \{0\}$ it is defined recursively $G^t_\delta(\mathcal{C}) = G_\delta(G^{(t-1)}_\delta(\mathcal{C}))$ with $G^0_\delta(\mathcal{C}) = \mathcal{C}$. A set of different configurations $\{\mathcal{C}^{(0)}, \cdots, \mathcal{C}^{(T-1)}\}$ is said to be a cycle of length T if $G^t_\delta(\mathcal{C}^{(0)}) = \mathcal{C}^{(t)}$ for $t \in \{0, \cdots, T-1\}$ and $G_\delta(\mathcal{C}^{(T-1)}) = \mathcal{C}^{(0)}$. A fixed point is a cycle of unitary length. We say that a configuration $\mathcal{C}$ is circular if there exists a $P \in \mathbb{N}^*$ for which $\mathcal{C}_i = \mathcal{C}_{i+P}$ for all $i \in \mathbb{Z}$.

In the *global fixed point attractor problem* it is asked whether every circular configuration of a given CA evolves to the *same* fixed point.

This work is mainly based on properties concerning *periodic tilings of the plane*. A tile is a labeled unit sized square. A tiling system is a pair $(\mathcal{T}, \varphi)$ where $\mathcal{T}$ is a finite set of tiles and $\varphi : \mathcal{T}^4 \to \mathcal{T}$ is a partial function called local matching. A tiling of the plane by $(\mathcal{T}, \varphi)$ is an assignment $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ satisfying for all $i, j \in \mathbb{Z}$: $\varphi(\mathcal{X}_{i-1,j}, \mathcal{X}_{i,j+1}, \mathcal{X}_{i+1,j}, \mathcal{X}_{i,j-1}) = \mathcal{X}_{i,j}$ (see figure 1-i). A tiling system $(\mathcal{T}, \varphi)$ is said to be NW-deterministic if for every pair $x, y \in \mathcal{T}$ there exists at most one tile $z \in \mathcal{T}$ accepting x has left neighbor and y as upper neighbor. In other words, for NW-deterministic tiling system $(\mathcal{T}, \varphi)$, the domain of the partial local matching function can be assumed to be $\mathcal{T}^2$. A tiling of the plane by a NW-deterministic set of tiles $(\mathcal{T},\varphi)$ is an assignment $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ satisfying for all $i,j \in \mathbb{Z}$: $\varphi(\mathcal{X}_{i-1,j},\mathcal{X}_{i,j+1}) = \mathcal{X}_{i,j}$ (see figure 1-ii).



(i) (ii)

Fig. 1 Local matching. (i) The general case. (ii) The NW-deterministic case.

A tiling $\mathcal{X}$ is said to be periodic if there exist horizontal and vertical translations for which $\mathcal{X}$ remains invariant. Formally, we say that $\mathcal{X} \in \mathcal{T}^{\mathbb{Z}^2}$ is periodic if there exists $P \in \mathbb{N}^*$ such that $\mathcal{X}_{i,j} = \mathcal{X}_{i+P,j} = \mathcal{X}_{i,j+P}$ for all $i,j \in \mathbb{Z}$.

In the *NW-deterministic periodic tiling problem* it is given a NW-deterministic tiling system and it is asked whether it admits a periodic tiling of the plane.

3 The Global Fixed Point Attractor Problem

It is direct to notice that every circular configuration of a CA evolves in a finite number of steps to a finite cycle. In [Sut90] it is proved that it is undecidable to know whether every circular configuration of a CA evolves to a fixed point. In this section we show that it remains undecidable to know whether every circular configuration of a CA evolves to the *same* fixed point. Our result will allow us to conclude the one of [Sut90] directly.

The reduction to the *global fixed point attractor problem* is done from the *NW-deterministic periodic tiling problem.*

Proposition 1 The *NW-deterministic periodic tiling problem* is undecidable.

Proof In section 4. □

Proposition 2 The *global fixed point attractor problem* is undecidable.

Proof Let $(\mathcal{T},\varphi)$ be a NW-deterministic tiling system. Let us consider now the CA (Q,δ) with $Q = \{\mathcal{T} \cup \{s\}\}$ such that $s \notin \mathcal{T}$, and with the transition function δ defined as follows:

$$\delta(x,y,z) = \begin{cases} \varphi(x,y) & \text{if } x,y,z \in \mathcal{T} \text{ and } \varphi(x,y) \text{ is well defined.} \\ s & \text{otherwise.} \end{cases}$$

It is not difficult to notice that $(\mathcal{T},\varphi)$ admits a periodic tiling of the plane if and only if there exists a circular configuration of (Q,δ) not evolving to the trivial fixed point $(\cdots sss \cdots)$. □

The *local fixed point attractor problem*, in which it is asked whether every circular configuration of a CA evolves to a (not necessarily unique) fixed point, was proved to be undecidable by K. Sutner in [Sut90]. Our result allows us to conclude Sutner's one in a direct way by considering the following lemma:

Lemma 1 Given a CA, it is decidable to know whether it admits a unique circular configuration as a fixed point.

Proof Given a CA (Q, δ), it suffices to consider the directed graph $G = (V, E)$ with $V \subseteq Q^3$ satisfying $(x, y, z) \in V$ if and only if $\delta(x, y, z) = y$ and $((x_1, y_1, z_1), (x_2, y_2, z_2)) \in E$ if and only if $y_1 = x_2$ and $z_1 = y_2$. There is a complete equivalence between the cycles of G and the circular fixed points of (Q, δ). $\square$

Corollary 1 The *local fixed point attractor problem* is undecidable.

Proof Let us denote as P the *global fixed point attractor problem* restricted to instances (CA) having a *unique* fixed point. By proposition 2 together with lemma 1 the undecidability of P can be concluded. On the other hand, the *global* and the *local* versions of the *fixed point attractor problem* when restricted to CA having a unique fixed point are equivalents. $\square$

4 The NW-Deterministic Periodic Tiling Problem

The goal of this section is to prove the undecidability of the *NW-deterministic periodic tiling problem*. As it was done in [Kar92], in order to make the proof more understandable, we are going to use an equivalent notion of NW-determinism. From now on we say that a tiling system $(\mathcal{T}, \varphi)$ is NW-deterministic if for every $a, b, c \in \mathcal{T}$ there exists at most one tile $d \in \mathcal{T}$ matching as in figure 2-i. In this case φ can be considered as a three arguments partial function and we note $\varphi(a, b, c) = d$.

Notice that if $(\mathcal{T}, \varphi)$ is a NW-deterministic tiling system in this new sense then there exists an equivalent tiling system $(\tilde{\mathcal{T}}, \tilde{\varphi})$ which is NW-deterministic in the original sense. In fact, let $\tilde{\mathcal{T}} = \mathcal{T}^2$ and let $\tilde{\varphi} : \tilde{\mathcal{T}}^2 \to \tilde{\mathcal{T}}$ be defined for all $x, a, b, c \in \mathcal{T}$ as follows (see figure 2-ii):

$$\tilde{\varphi}((x, a), (b, c)) = (a, \varphi(a, b, c))$$

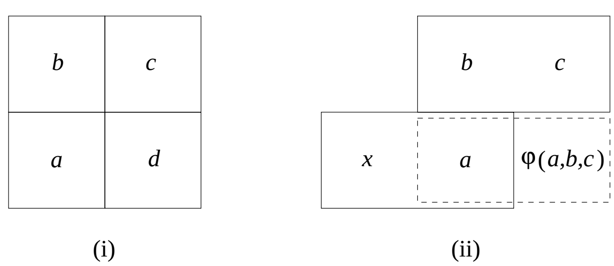


Fig. 2 (i) $\varphi(a, b, c) = d$. (ii) Equivalence between the two NW-deterministic notions.

It is direct to see that there exists a periodic tiling for $(\mathcal{T}, \varphi)$ if and only if there exists a periodic tiling for $(\tilde{\mathcal{T}}, \tilde{\varphi})$.

Let $(\mathcal{T}_1, \varphi_1)$ and $(\mathcal{T}_2, \varphi_2)$ be a pair of NW-deterministic tiling systems. We define a natural superposition operation $\otimes$ which preserves the NW-deterministic property in such a way that $(\mathcal{T}, \varphi) = (\mathcal{T}_1, \varphi_1) \otimes (\mathcal{T}_2, \varphi_2)$ with $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ and for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathcal{T}$:

$$\varphi((x_1, x_2), (y_1, y_2), (z_1, z_2)) = (\varphi_1(x_1, y_1, z_1), \varphi_2(x_2, y_2, z_2))$$

The undecidability of the *NW-deterministic periodic tiling problem* is going to be proved by a reduction from the *halting problem on Turing machines*. Before showing this reduction, we must construct a pair of NW-deterministic sets of tiles satisfying very particular conditions.

4.1 The NW-Deterministic Set of Tiles $\mathcal{A}$

Lemma 2 There exists a NW-deterministic set of tiles $\mathcal{A} = \mathcal{A}_1 \cup A_2$ such that:

- A_1 admits only nonperiodic tilings of the plane.
- For any $n > 1$ there exists a square of size 2^n tiled by $\mathcal{A}$ satisfying:
  - It has periodic boundary conditions. In other words, this square pattern can be repeated in order to tile the plane periodically.
  - The tiles of $\mathcal{A}_2$ appear only on the right and bottom borders of the square as it is schematically showed in figure 3.

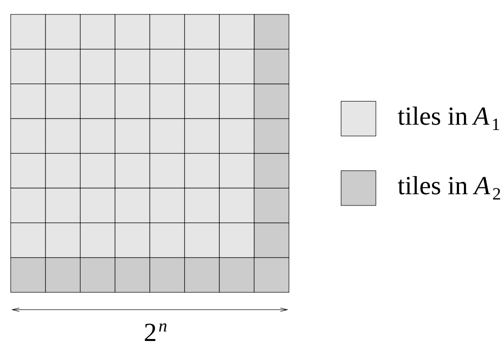


Fig. 3 A square tiled by $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ with periodic boundary conditions.

Proof The set $\mathcal{A}_1$ to be considered corresponds to the one introduced in [Kar92] which is almost identical to the well known Robinson's set ([Rob71]) denoted here as $\mathcal{A}_0$ and appearing in figure 4. Notice that $\mathcal{A}_0$ has cardinality 32 (8 crosses and and 24 arms) because all the rotations of each tile are admissible. Notice also that we refer to "set of tiles" instead of "tiling system" because the tiles themselves encode the local matching function (arrow heads must meet arrow tails). In [Rob71] it was proved that $\mathcal{A}_0$ admits only nonperiodic tilings of the plane.

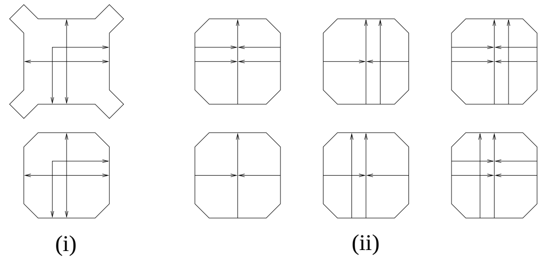


Fig. 4 Robinson's set $\mathcal{A}_0$. (i) Crosses. (ii) Arms.

By simply adding colors to the upper-left and bottom-right corners, in [Kar92] it is shown how to transform the set $\mathcal{A}_0$ into a NW-deterministic one $\mathcal{A}_1$ preserving the nonperiodicity property. More precisely, to the arms horizontally oriented (those with the principal arrow lying horizontally) it is added an H label on its upper-left corner and a V label on its bottom-right one. To the arms vertically oriented the V label is added on the upper-left corner while the H label is added on the bottom-right corner. Finally the crosses are duplicated by adding the same label (V and H) on both corners. In figure 5 appears the way the modification is done for three particular tiles belonging to each of previous cases. Notice that now, in order to tile correctly, adjacent corners must have the same color.

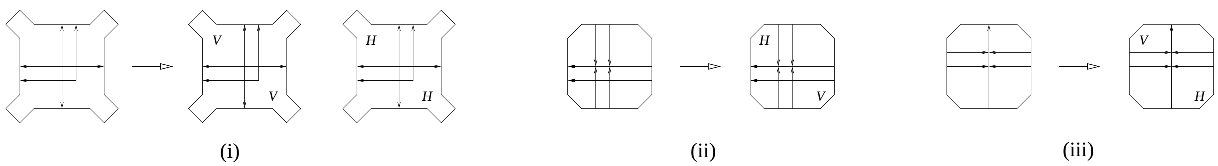


Fig. 5 Transforming $\mathcal{A}_0$ into $\mathcal{A}_1$. (i) A cross tile. (ii) An horizontally oriented arm. (iii) A vertically oriented arm.

Let us define the set of tiles $\mathcal{A}_2$ as the one of cardinality 5 that appears in figure 6.

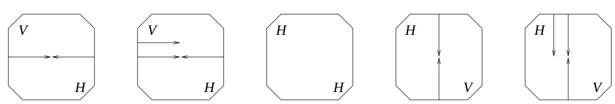


Fig. 6 The set $\mathcal{A}_2$.

The NW-determinism of $\mathcal{A} = \mathcal{A}_1 \cup A_2$ follows directly: it suffices to check. The periodic square of size 2^n with tiles of $\mathcal{A}_2$ just on the right and bottom borders appears in figure 7 for $n = 2$ and for $n = 3$. For an arbitrary n the proof has to be done by induction. Finally, in figure 8 it is shown that previous pattern effectively has periodic boundary conditions. $\square$

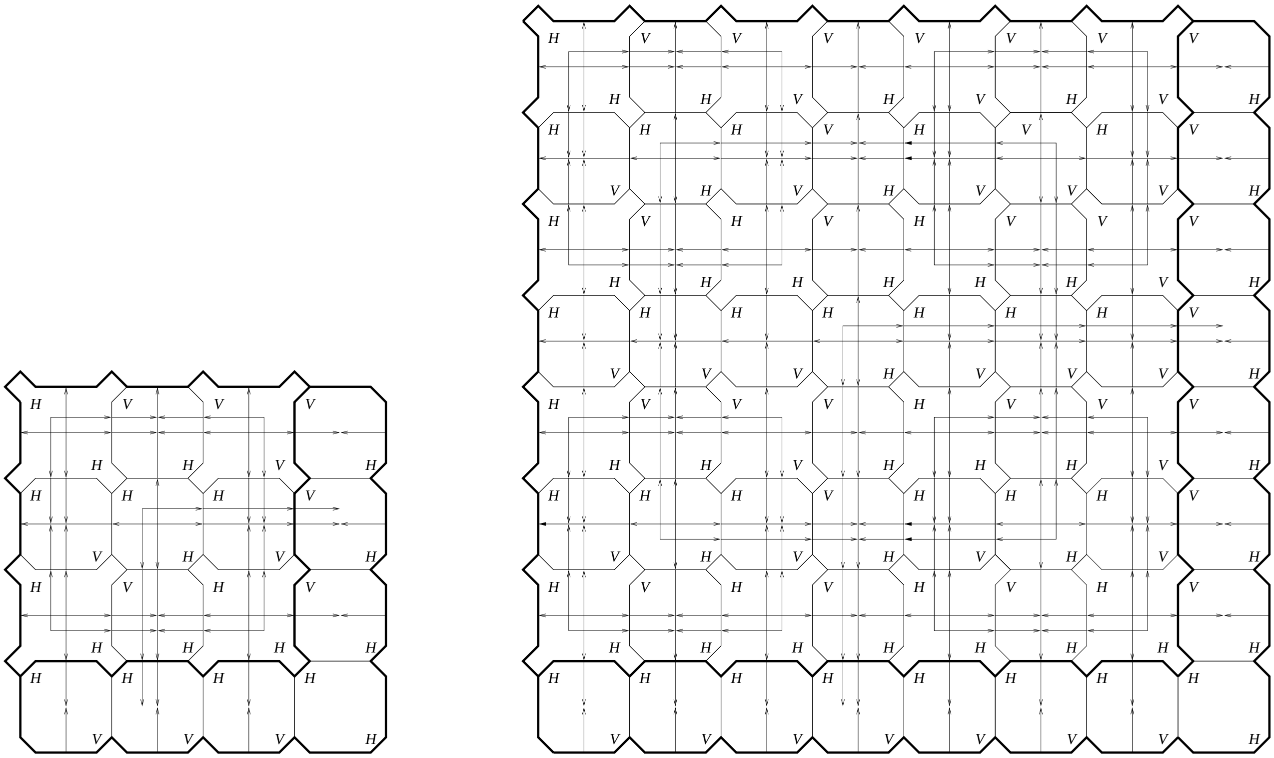


Fig. 7 The periodic pattern for $n = 2$ and $n = 3$.

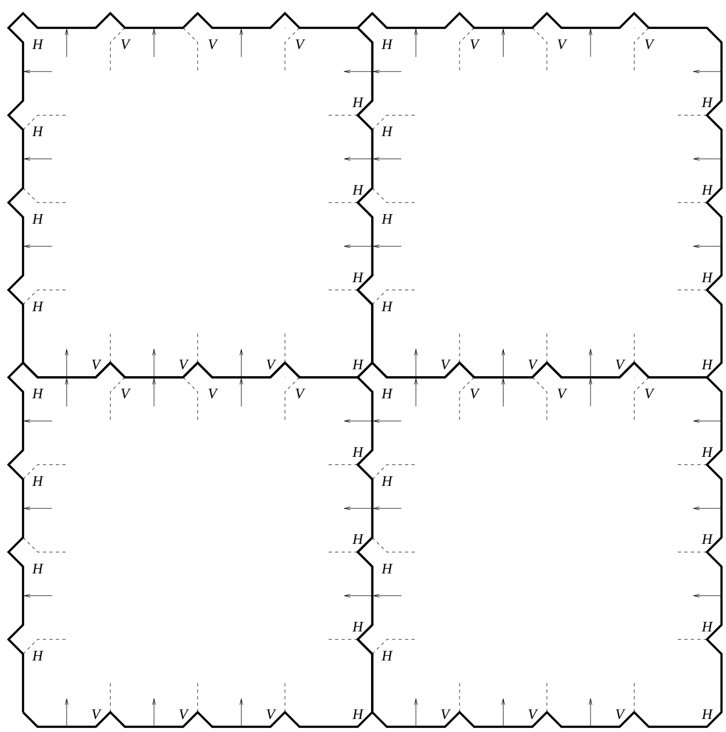


Fig. 8 The periodicity of the bounded conditions.

4.2 The NW-Deterministic Set of Tiles $\mathcal{AB}$

Here we are going to construct a NW-deterministic set of tiles $\mathcal{AB}$ admitting periodic tilings of the plane and satisfying that, in any of these possible periodic tilings, some particular patterns called "boards" always appear. Let us start by some definitions:

Definition 1 Let $\mathcal{A} = \mathcal{A}_1 \cup A_2$ be the NW-deterministic set of tiles of previous section. Let $\mathcal{B} = \mathcal{B}_{\text{int}} \cup \mathcal{B}_{\text{NW}} \cup \mathcal{B}_{\text{SE}}$ be the set of figure 9 made of internal tiles, NW-border tiles and SE-border tiles. We denote as $\mathcal{AB}$ the set obtained by the following superpositions:

$$\mathcal{AB} = \underbrace{\{\mathcal{A}_1 \otimes \mathcal{B}_{\text{int}}\}}_{\mathcal{AB}_{\text{int}}} \cup \underbrace{\{\mathcal{A}_1 \otimes \mathcal{B}_{\text{NW}}\} \cup \{\mathcal{A}_2 \otimes \mathcal{B}_{\text{SE}}\}}_{\mathcal{AB}_{\text{bord}}}$$

The tiles belonging to $\mathcal{AB}_{\text{int}}$ are called $\mathcal{AB}$-internal tiles while the tiles belonging to $\mathcal{AB}_{\text{bord}}$ are called $\mathcal{AB}$-border tiles.

Definition 2 An $\mathcal{AB}$-board is a square tiled by $\mathcal{AB}$ with $\mathcal{AB}$-border tiles appearing only at the border of the square as it is shown schematically in figure 10. Notice that from now on, as it is done in figure 10, for any tile in $\mathcal{AB}$ the presence of the $\mathcal{A}$-component will be represented by a unique shadowed background (no matter if the $\mathcal{A}$-component corresponds to a tile of $\mathcal{A}_1$ or $\mathcal{A}_2$).

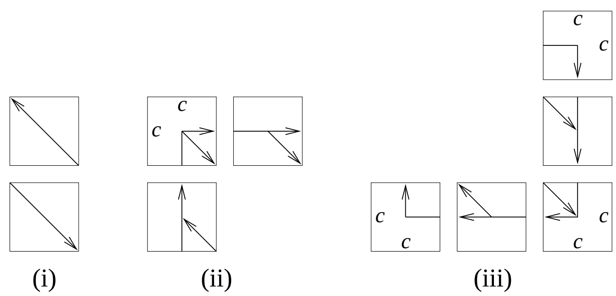


Fig. 9 The set of tiles $\mathcal{B}$. (i) Internal tiles (ii) NW-border tiles. (ii) SE-border tiles.

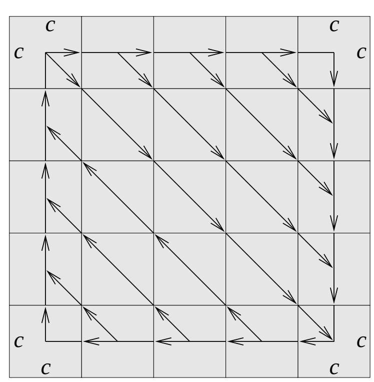


Fig. 10 An $\mathcal{AB}$-board.

In the two following lemmas we prove that the set $\mathcal{AB}$ satisfies our requirements:

Lemma 3 The set $\mathcal{AB}$ is NW-deterministic and for all $n > 1$ there exists an $\mathcal{AB}$-board of size 2^n with periodic boundary conditions.

Proof For the NW-determinism notice that $\mathcal{B}$ is NW-deterministic and $\mathcal{AB} \subseteq \mathcal{A} \otimes \mathcal{B}$. On the other hand, for any $n > 1$, in order to obtain an $\mathcal{AB}$-board of size 2^n with periodic boundary conditions it suffices to transform a square of size 2^n tiled by $\mathcal{A}$ with periodic boundary conditions (see figure 3) into an $\mathcal{AB}$-board by superposing in the suitable way the tiles of $\mathcal{B}$. $\Box$

Lemma 4 In any periodic tiling of the plane by $\mathcal{AB}$ an $\mathcal{AB}$-board must appear.

Proof Let $\mathcal{P}$ be a periodic tiling of the plane by $\mathcal{AB}$. First notice that at least one $\mathcal{AB}$-border tile t_0 must appear in $\mathcal{P}$. In fact, if this is not the case then the plane would be tiled periodically by $\mathcal{A}_1 \otimes \mathcal{B}_{\text{int}}$. But this is not possible because $\mathcal{A}_1$ does not admit periodic tilings of the plane. Notice also that t_0 can be assumed to be a *corner tile* (see figure 11-i). In fact, let us suppose that in $\mathcal{P}$ there are no corner tiles. If we define as *curve* any path in $\mathcal{P}$ determined by the (vertical and horizontal) arrows of the $\mathcal{B}$-components of the $\mathcal{AB}$-border tiles and if we denote as C_0 the curve that passes through t_0, then C_0 has to be an infinite line. By periodicity, there must exists a parallel line identical to C_0 and, because of the assumption that no corner tiles appear in $\mathcal{P}$, it follows one of the two contradictions of figure 11-ii. Let us consider the curve C_0 that passes through the corner tile t_0. It is now possible to prove (by the same kind of previous geometrical arguments) that C_0 has to be a square which, by definition, delimits an $\mathcal{AB}$-board. $\Box$

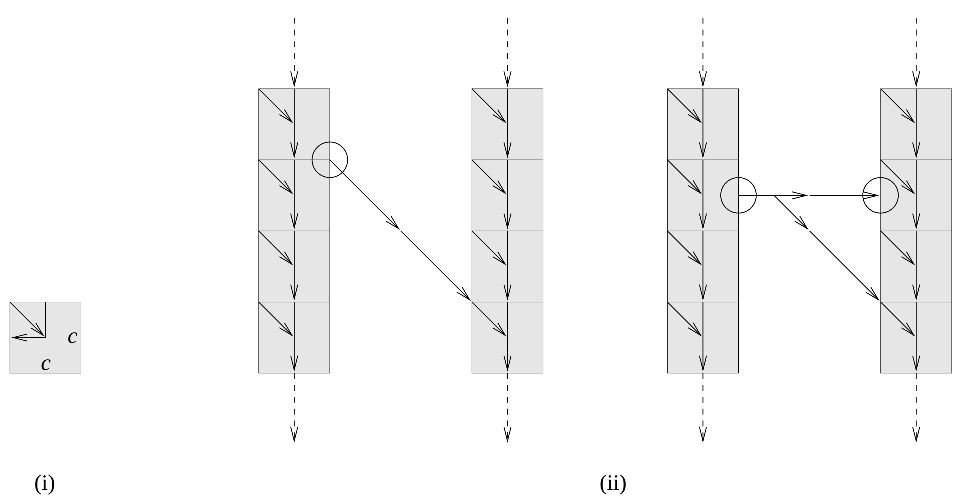


Fig. 11 (i) A corner tile. (ii) $\mathcal{P}$ does not admit infinite lines without corner tiles.

4.3 The Reduction

Now we are able to prove the undecidability of the *NW-deterministic periodic tiling problem.* We do it by a reduction from the known undecidable *halting problem on Turing machines* in which an arbitrary Turing machine $\mathcal{M} = (\Sigma, B, Q, q_0, q_h, \delta)$ is given and it is asked whether $\mathcal{M}$ reaches the halting state q_h when starting on a blank bi-infinite tape $(\cdots BB \cdots)$ and in the initial state q_0. Notice that $\delta : \Sigma \times Q \to \Sigma \times Q \times \{L, R, S\}$ represents the transition function of $\mathcal{M}$ with Σ being the alphabet, Q the set of states, and $\{L, R, S\}$ the possible movements (left, right, stay).

Proposition 3 The *NW-deterministic periodic tiling problem* is undecidable.

Proof Let $\mathcal{M}$ be an arbitrary Turing machine. Let $\mathcal{M}^* = (\Sigma, B, Q, q_0, q_h, q_f, \delta)$ be the same as $\mathcal{M}$ with the only difference that it never halts. More precisely, when it reaches the halting state q_h it erases the tape and it stays in a *final-quiescent configuration* (i.e: in a particular final state q_f and scanning the cell located at the origin of the blank tape). By a suitable composition of a set of tiles $\mathcal{T}^*$ (which codifies the Turing machine $\mathcal{M}^*$) and the set of tiles $\mathcal{AB}$ (introduced in previous section) we are going to obtain a NW-deterministic set of tiles $\mathcal{H}$ admitting a periodic tiling of the plane if and only if $\mathcal{M}^*$ reaches the final-quiescent configuration.

Let $\mathcal{T}^*$ be the set of tiles that codifies $\mathcal{M}^*$ and which appears in figure 12: *alphabet tiles* are generated for each $s \in \Sigma$; *merging tiles* for every pair $(s, q) \in \Sigma \times Q$; *right, left and stay tiles* are associated to the tuples (s_1, q_1, s_2, q_2, R), (s_1, q_1, s_2, q_2, L) and (s_1, q_1, s_2, q_2, S) satisfying respectively $\delta(s_1, q_1) = (s_2, q_2, R)$, $\delta(s_1, q_1) = (s_2, q_2, L)$, and $\delta(s_1, q_1) = (s_2, q_2, S)$.

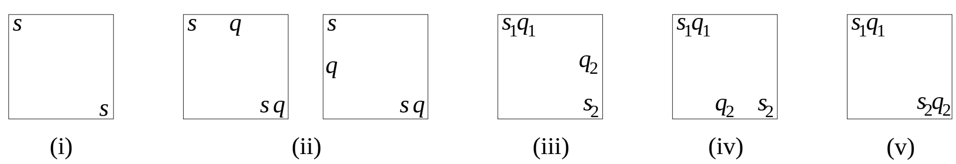


Fig. 12 The set of tiles $\mathcal{T}^*$. (i) Alphabet tiles. (ii) Merging tiles. (iii) Right tiles. (iv) Left tiles. (v) Stay tiles.

As it is showed in figure 13, the computation of $\mathcal{M}^*$ can be codified as a tiling of the bottom-right quadrant of the plane ($\mathbb{N}^2$). In fact, if a t-frame is a region of the form $\{(i, j) \in \mathbb{N}^2 : i = t \text{ or } j = t\}$ with $t \geq 0$, then instantaneous configurations of $\mathcal{M}^*$ appear codified in successive t-frames. In each t-frame the origin of the tape is represented in the cell (t, t). The left part of the tape is represented in the vertical part of the frame while the right part is represented in the horizontal part of the frame. All the tiles of a frame correspond to alphabet tiles excepting the scanning cell and, eventually, the neighbor with which it is interacting. Notice that these tilings can be seen as an *alternative* representation of the Turing machine dynamics.

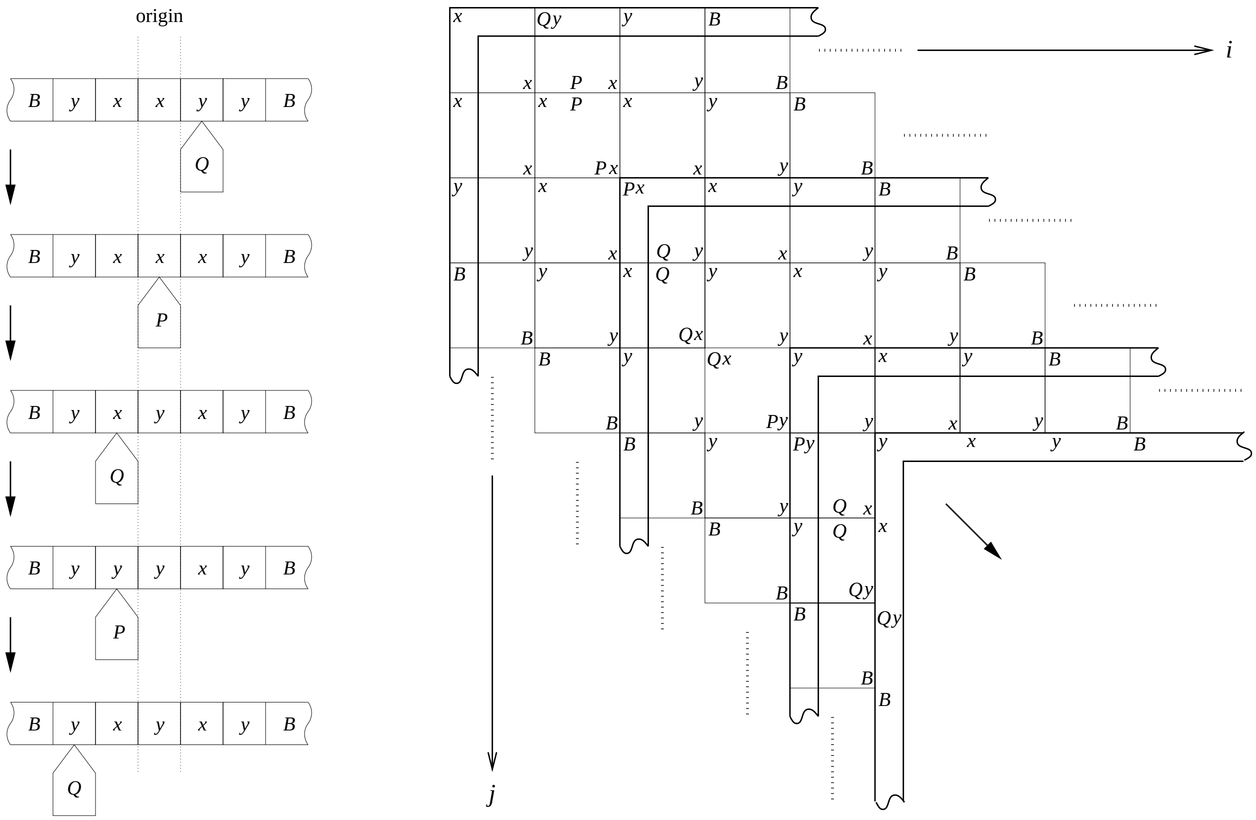


Fig. 13 Equivalence between a Turing machine computation and a tiling of the bottom-right quadrant of the plane.

Let the set of tiles $\mathcal{H} = \mathcal{H}_{\text{int}} \cup \mathcal{H}_{\text{bord}}$ be the one with $\mathcal{H}_{\text{int}} = \mathcal{AB}_{\text{int}} \otimes \mathcal{T}^*$ and with $\mathcal{H}_{\text{bord}}$ being obtained by superposing labels to some tiles of $\mathcal{AB}_{\text{bord}}$ as it appears explicitly in figure 14.

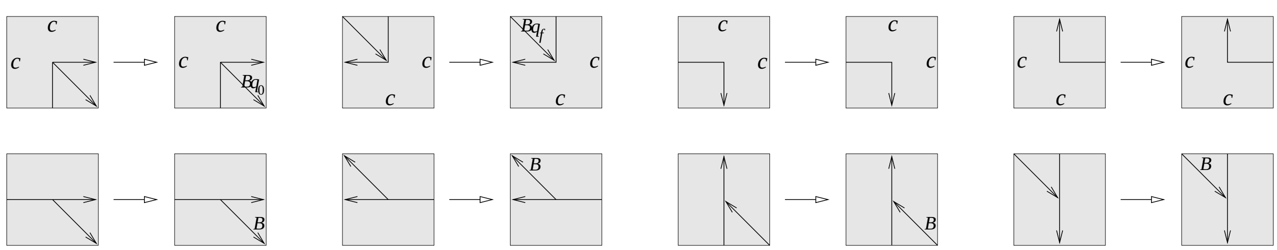


Fig. 14 Modification of $\mathcal{AB}_{\text{bord}}$ in order to obtain the $\mathcal{H}_{\text{bord}}$.

The tiles belonging to $\mathcal{H}_{\text{int}}$ are called $\mathcal{H}$-internal tiles, while the tiles belonging to $\mathcal{H}_{\text{bord}}$ are called $\mathcal{H}$-border tiles. As for the set $\mathcal{AB}$, we define an $\mathcal{H}$-board as a square tiled by $\mathcal{H}$ with the $\mathcal{H}$-border tiles appearing only at the border of the square as it is schematically shown in figure 15.

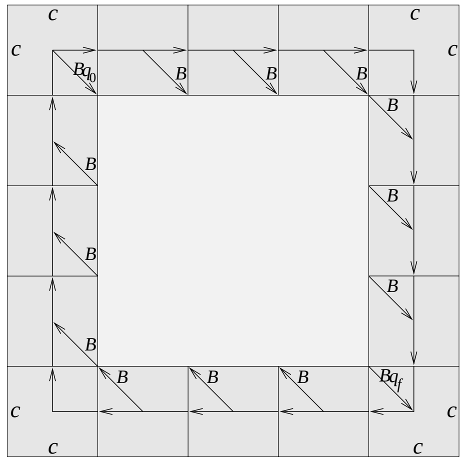


Fig. 15 An $\mathcal{H}$-board.

Notice that $\mathcal{H}$ is a NW-deterministic set of tiles. This fact can be easily checked by considering that $\mathcal{T}^*$ is NW-deterministic (because $\mathcal{M}^*$ is a deterministic machine) and that the same holds for the set $\mathcal{AB}$ (see lemma 3).

It remains to prove that $\mathcal{M}^*$ reaches the final-quiescent configuration when it starts from the blank tape if and only if $\mathcal{H}$ admits a periodic tiling of the plane. In fact, if $\mathcal{M}^*$ reaches the final-quiescent configuration then there exists a square $\mathcal{S}$ tiled by $\mathcal{T}^*$ with the boundary conditions that appears schematically in figure 16-i. Without loss of generality we can assume that the size of $\mathcal{S}$ is $(2^n - 2)$ for some $n > 1$. In fact, if the size of the original square in which the halting computation was represented is k then we can construct another one of size $(k+1)$ as it is explained in figure 16-ii. Now from $\mathcal{S}$ it is direct to obtain an $\mathcal{H}$-board of size 2^n (see figure 16-iii). Moreover, considering that there exists an $\mathcal{AB}$-board of size 2^n with periodic boundary conditions (see lemma 3) we can assume that the $\mathcal{H}$-board has periodic boundary conditions and it can be repeated in order to tile the plane periodically.

Let us now suppose that $\mathcal{H}$ admits a periodic tiling of the plane $\mathcal{P}$. It follows that an $\mathcal{H}$-board must appear in $\mathcal{P}$. In fact, if this is not the case we would contradict lemma 4. More precisely, if we suppose that in $\mathcal{P}$ no $\mathcal{H}$-board appears and we *extract* all the Turing machines symbols of $\mathcal{P}$ we would obtain a periodic tiling of the plane by $\mathcal{AB}$ having no $\mathcal{AB}$-boards. Finally, from an $\mathcal{H}$-board it is direct to obtain a square tiled by $\mathcal{T}^*$ encoding an halting computation of $\mathcal{M}^*$ (see figure 16). $\square$

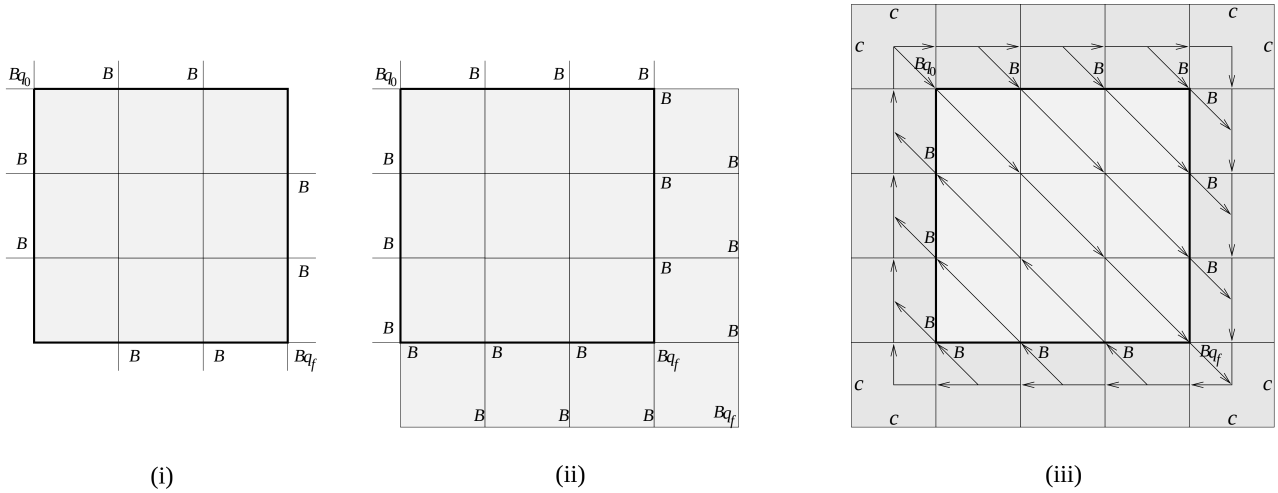


Fig. 16 (i) A square tiled by $\mathcal{T}^*$ representing an halting computation of $\mathcal{M}^*$. (ii) A bigger square. (iii) The associated $\mathcal{H}$-board.

Remark 1 Notice that the set $\mathcal{H}$ *always* admits a tiling of the plane. In fact, it suffices to use $\mathcal{H}_{\text{int}}$ in order to tile *nonperiodically* the plane by representing the evolution of $\mathcal{M}^*$ which, by construction, *never halts*.

Remark 2 As an obvious consequence of proposition 3 it can be concluded the undecidability of the *periodic tiling problem* (in which it is asked whether an *arbitrary* set of tiles admits a periodic tiling of the plane). This result was obtained in [GK72]. Nevertheless, we would like to remark that our approach could also be used to prove the Gurevich-Koriakov result in a *direct* way. In fact, it suffices to notice that when the NW-deterministic property is no more required, most of the technicities of the proof are no more needed and it becomes very simple (for instance, the set $\mathcal{A}$ has just to be nonperiodic and it does not need an explicit representation).

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