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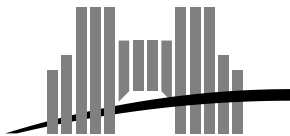
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***Cellular automata in the Cantor,  
Besicovitch and Weyl spaces***

François Blanchard, Enrico Formenti,      Juillet 1998  
Petr Kůrka

Research Report N° 98-31



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# Cellular automata in the Cantor, Besicovitch and Weyl spaces

François Blanchard, Enrico Formenti, Petr Kůrka

Juillet 1998

## Abstract

The Besicovitch and Weyl pseudometrics on the space  $A^{\mathbb{Z}}$  of biinfinite sequences measure the density of differences in either the central or arbitrary segments of given sequences. The Besicovitch and Weyl spaces are obtained from  $A^{\mathbb{Z}}$  by factoring through the equivalence of zero distance. We consider cellular automata as dynamical systems on the Besicovitch and Weyl spaces and compare their topological and dynamical properties with those in the Cantor space.

**Keywords:** Cellular automata, dynamical systems

## Résumé

Les pseudo-métriques de Besicovitch et Weyl sur l'espace  $A^{\mathbb{Z}}$  des suites biinfinies mesure la densité des différences dans la partie centrale ou dans une partie arbitraire d'une suite donnée. Les espaces de Besicovitch et Weyl sont obtenus en factorisant  $A^{\mathbb{Z}}$  par la relation d'équivalence "être à une distance nulle". Nous considérons les AC comme des systèmes dynamiques sur ces espaces et nous comparons leurs propriétés topologiques et dynamiques avec celles dans l'espace de Cantor.

**Mots-clés:** Automates cellulaires, systèmes dynamiques

# Cellular automata in the Cantor, Besicovitch and Weyl spaces

François Blanchard, Enrico Formenti, Petr Kůrka

9th July 1998

## Abstract

The Besicovitch and Weyl pseudometrics on the space  $A^{\mathbb{Z}}$  of biinfinite sequences measure the density of differences in either the central or arbitrary segments of given sequences. The Besicovitch and Weyl spaces are obtained from  $A^{\mathbb{Z}}$  by factoring through the equivalence of zero distance. We consider cellular automata as dynamical systems on the Besicovitch and Weyl spaces and compare their topological and dynamical properties with those in the Cantor space.

## 1 Introduction

A cellular automaton consists of a biinfinite array of cells containing letters from a finite alphabet, which are updated according to a local interaction rule. Cellular automata have been of considerable interest both as models of physical and biological phenomena and in symbolic dynamics as homomorphisms of the shift (Hedlund [8]). They display a large spectrum of dynamical behaviors ranging from stable to chaotic dynamics and they could also support universal computation. For survey, see Wolfram [13], Culik II, Hurd and Yu [5] or Blanchard, Maass and Kůrka [2].

When a cellular automaton is conceived as a dynamical system, the space of biinfinite sequences is equipped with product topology, which makes it homeomorphic to the Cantor space. In the Cantor space, the shift map has many chaoticity properties like sensitivity to initial conditions and topological transitivity. However, the shift may be regarded as a shift of the observation point, in which the configuration does not change at all. To distinguish the shift from chaotic cellular automata, which really change the structure of configurations, Cattaneo et al. [4], consider a shift invariant Besicovitch pseudometric, which has been also used in the study of almost periodic functions (see e.g. Besicovitch [1]). Besicovitch pseudometrics measures the density of differences in the central part of two given sequences. The Besicovitch space is obtained factoring the space of biinfinite sequences by the equivalence of zero distance. A variant of this approach is the Weyl

pseudometric, which measures the density of differences in arbitrary segments of two given sequences. Downarowicz and Iwanik [6] show that the Weyl space is pathwise connected and incomplete. The Besicovitch space is also pathwise connected but complete. Both spaces are infinite-dimensional and neither separable nor locally compact.

Cellular automata are continuous with respect to both Besicovitch and Weyl pseudometrics, so they yield dynamical systems in both Besicovitch and Weyl spaces. In the present paper we compare topological and dynamical properties of cellular automata in these spaces with those in the Cantor space. When passing from either Besicovitch or Weyl spaces to Cantor space, cellular automata preserve chaoticity properties like topological transitivity and sensitivity. Vice versa, when passing from the Cantor space to either Weyl or Besicovitch space, cellular automata preserve chain transitivity and stability properties like equicontinuity, existence of equicontinuity points and stability of periodic points. Finally, in neither the Weyl nor Besicovitch space holds the Hedlund theorem saying that cellular automata are exactly continuous maps commuting with the shift. We thus obtain in these spaces a class of shift commuting maps, which are not given by any local rule.

## 2 Dynamical systems

A dynamical system is a continuous map  $f : X \rightarrow X$  of a nonempty metric space  $X$  to itself. The  $n$ -th iteration  $f^n : X \rightarrow X$  of  $f$  is defined by  $f^0(x) = x$ ,  $f^{n+1}(x) = f(f^n(x))$ . A point  $x \in X$  is fixed, if  $f(x) = x$ . It is periodic, if  $f^n(x) = x$  for some  $n > 0$ . The least positive  $n$  with this property is called the period of  $x$ . The orbit of  $x$  is the set  $\mathfrak{o}(x) = \{f^n(x) : n \geq 0\}$ . A set  $Y \subseteq X$  is positively invariant, if  $f(Y) \subseteq Y$ . A point  $x \in X$  is equicontinuous ( $x \in \mathcal{E}(f)$ ) if the family of maps  $f^n$  is equicontinuous at  $X$ , i.e.  $x \in \mathcal{E}(f)$  iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in B_\delta(x))(\forall n > 0)(d(f^n(y), f^n(x)) < \varepsilon).$$

The map  $f$  is equicontinuous iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)(\forall y \in B_\delta(x))(\forall n > 0)(d(f^n(y), f^n(x)) < \varepsilon).$$

For an equicontinuous system  $\mathcal{E}(f) = X$ . Conversely if  $\mathcal{E}(f) = X$  and  $X$  is compact, then  $f$  is equicontinuous. A system  $(X, f)$  is sensitive (to initial conditions), iff

$$(\exists \varepsilon > 0)(\forall x \in X)(\forall \delta > 0)(\exists y \in B_\delta(x))(\exists n > 0)(d(f^n(y), f^n(x)) \geq \varepsilon).$$

A sensitive system has no equicontinuous point, there exist, however, systems with no equicontinuity points which are not sensitive. A system  $(X, f)$

is (positively) expansive iff

$$(\exists \varepsilon > 0)(\forall x \neq y \in X)(\exists n \geq 0)(d(f^n(x), f^n(y)) \geq \varepsilon)$$

A positively expansive system on a perfect space is sensitive.

A system  $(X, f)$  is (topologically) transitive, if for any nonempty open sets  $U, V \subseteq X$  there exists  $n \geq 0$  such that  $f^{-n}(U) \cap V \neq \emptyset$ . If  $X$  is perfect and if the system has a dense orbit, then it is transitive. Conversely, if  $(X, f)$  is topologically transitive and if  $X$  is compact, then  $(X, f)$  has a dense trajectory. Indeed, the set  $\{x \in X : \overline{\mathbf{o}(x)} = X\}$  is residual in this case. An  $\varepsilon$ -chain (from  $x_0$  to  $x_n$ ) is a sequence of points  $x_0, \dots, x_n \in X$  such that  $d(f(x_i), x_{i+1}) < \varepsilon$  for  $0 \leq i < n$ . A system  $(X, f)$  is chain transitive, if for any  $\varepsilon > 0$  and any  $x, y \in X$  there exists a  $\varepsilon$ -chain from  $x$  to  $y$ .

A fixed point  $x \in X$  is stable if it is equicontinuous and there exists its neighbourhood  $U \ni x$ , such that for every  $y \in U$ ,  $\lim_{n \rightarrow \infty} f^n(y) = x$ . A periodic point  $x$  with period  $n$  is stable if it is stable for  $f^n$ .

### 3 Cantor, Weyl and Besicovitch spaces

Let  $A$  be a finite alphabet with at least two letters. The binary alphabet is denoted by  $\mathbf{2} = \{0, 1\}$ . For  $n \in \mathbb{N}$ , denote by  $A^n$  the set of words over  $A$  of length  $n$ ,  $A^* = \cup_{n \geq 0} A^n$  the set of finite words over  $A$ . We consider also words  $u \in A^{[j,k]}$  indexed by an interval of integers  $[j, k]$ . Denote by  $A^{\mathbb{Z}}$  the set of biinfinite sequences of letters from  $A$ . The  $i$ -th letter of a point  $x \in A^{\mathbb{Z}}$  is denoted by  $x_i$ , and  $x_{[j,k]} = x_j \dots x_k \in A^{[j,k]}$  is the segment of  $x$  between indices  $j$  and  $k$ . For  $u \in A^{[j,k]}$ ,  $u^\infty$  is the infinite repetition of  $u$ , i.e.  $(u^\infty)_{m+n(k-j+1)} = u_m$  for  $n \in \mathbb{Z}$  and  $m \in [j, k]$ . The cylinder of  $u \in A^{[j,k]}$  is the set

$$[u] = \{x \in A^{\mathbb{Z}} : x_{[j,k]} = u\} .$$

The Cantor metric on  $A^{\mathbb{Z}}$  is defined by

$$d_C(x, y) = 2^{-k} \quad \text{where } k = \min\{|i| : x_i \neq y_i\}$$

so  $d_C(x, y) < 2^{-k}$  iff  $x_{[-k,k]} = y_{[-k,k]}$ . The cylinders are clopen sets for  $d_C$ . It is well known that all Cantor spaces (with different alphabets) are homeomorphic. The Cantor space is compact, totally disconnected and perfect.

The Weyl pseudometric on  $A^{\mathbb{Z}}$  is given by

$$d_W(x, y) = \limsup_{l \rightarrow \infty} \max_{k \in \mathbb{Z}} \frac{\#\{j \in [k+1, k+l] : x_j \neq y_j\}}{l} .$$

Here  $\#$  means the number of elements of a set, so  $d_W(x, y) < \varepsilon$  iff

$$(\exists l_0 \in \mathbb{N})(\forall l \geq l_0)(\forall k \in \mathbb{Z})(\#\{j \in [k+1, k+l] : x_j \neq y_j\} < l\varepsilon)$$

For  $x \in A^{\mathbb{Z}}$  denote by  $\tilde{x} = \{y \in A^{\mathbb{Z}} : d_W(y, x) = 0\}$  and  $X_W = \{\tilde{x} : x \in A^{\mathbb{Z}}\}$  the Weyl space over alphabet  $A$ . Clearly every two Weyl spaces (with different alphabets) are homeomorphic. The Weyl pseudometric could be considered also on the set  $A^{\mathbb{N}}$  of unilateral sequences. For  $x, y \in A^{\mathbb{N}}$  put

$$d_W(x, y) = \limsup_{l \rightarrow \infty} \max_{k \in \mathbb{N}} \frac{\#\{i \in [k+1, k+l] : x_i \neq y_i\}}{l}.$$

The map  $\varphi : A^{\mathbb{Z}} \rightarrow A^{\mathbb{N}}$  defined by  $\varphi(x) = x_0 x_{-1} x_1 x_{-2} x_2 \dots$  yields a homeomorphism between unilateral and bilateral Weyl spaces. In fact  $\varphi$  is uniformly continuous, so it preserves completeness.

The Besicovitch pseudometric on  $A^{\mathbb{Z}}$  is given by

$$d_B(x, y) = \limsup_{l \rightarrow \infty} \frac{\#\{j \in [-l, l] : x_j \neq y_j\}}{2l+1}$$

so  $d_B(x, y) < \varepsilon$  iff

$$(\exists l_0)(\forall l \geq l_0)(\#\{j \in [-l, l] : x_j \neq y_j\} < (2l+1)\varepsilon).$$

For  $x \in A^{\mathbb{Z}}$  put again  $\tilde{x} = \{y \in A^{\mathbb{Z}} : d_W(y, x) = 0\}$  and  $X_B = \{\tilde{x} : x \in A^{\mathbb{Z}}\}$  the Besicovitch space over alphabet  $A$ . Clearly every two Besicovitch spaces (with different alphabets) are homeomorphic and they are also homeomorphic to the unilateral Besicovitch space obtained by the following pseudometric

$$d_B(x, y) = \limsup_{l \rightarrow \infty} \frac{\#\{i \in [0, l-1] : x_i \neq y_i\}}{l}, \quad x, y \in A^{\mathbb{N}}.$$

Since  $d_B(x, y) \leq d_W(x, y)$ , the identity yields a continuous map  $I : X_W \rightarrow X_B$ .

Both Weyl and Besicovitch spaces are homogeneous. For any  $u \in \mathbf{2}^{\mathbb{Z}}$ ,  $f : \mathbf{2}^{\mathbb{Z}} \rightarrow \mathbf{2}^{\mathbb{Z}}$  defined by  $f(x)_i = x_i + u_i \pmod{2}$  is a homeomorphism, which sends  $0^\infty$  to  $u$ . Using Toeplitz sequences, Downarowicz and Iwanik [6] show that the Weyl space is pathwise connected. Using the same technique we prove the same result for the Besicovitch space and we show that both spaces are infinite dimensional.

A sequence  $x \in A^{\mathbb{N}}$  is Toeplitz, if every its subword occurs periodically, i.e, if

$$(\forall n \in \mathbb{N})(\exists p > 0)(\forall j \in \mathbb{N})(x_{n+jp} = x_n)$$

Toeplitz sequences are constructed by filling in successively periodic parts. For an alphabet  $A$  denote by  $\tilde{A} = A \cup \{*\}$ . For  $x, y \in \tilde{A}^{\mathbb{N}}$ ,  $T(x, y) \in \tilde{A}^{\mathbb{Z}}$  is the point obtained by replacing the stars in  $x$  by  $y$ . Let  $t_i$  be the increasing sequence of all integers for which  $x_{t_i} = *$ . Then put

$$T(x, y)_{t_i} = \begin{cases} x_i & \text{if } x_i \neq * \\ y_i & \text{otherwise.} \end{cases}$$

Consider a map  $f : \{0, 1\}^* \rightarrow \tilde{A}^{\mathbb{Z}}$  defined by induction  $f(\lambda) = *^\infty$ ,

$$\begin{aligned} f(x_0 \dots x_{n+1}) &= T(f(x_0 \dots x_n), (0*)^\infty) \quad \text{if } x_n = 0 \\ f(x_0 \dots x_{n+1}) &= T(f(x_0 \dots x_n), (*1)^\infty) \quad \text{if } x_n = 1 \end{aligned}$$

Thus

$$\begin{aligned} f(0) &= 0 * 0 * 0 * 0 * 0 * 0 * \dots \\ f(1) &= *1 * 1 * 1 * 1 * 1 * 1 * \dots \\ f(00) &= 000 * 000 * 000 * \dots \\ f(01) &= 0 * 010 * 010 * 01 \dots \\ f(10) &= 01 * 101 * 101 * 1 \dots \\ f(11) &= *111 * 111 * 111 \dots \end{aligned}$$

For a real number  $x \in [0, 1]$  with binary expansion  $x = \sum_{i=1}^{\infty} x_i 2^{-i}$  put  $f(x) = \lim_{n \rightarrow \infty} f(x_1 \dots x_n)$ . If  $2^n x$  is not integer for any  $n$ , then  $x$  has a unique expansion and  $f(x) \in \{0, 1\}^{\mathbb{Z}}$ . On the other hand if  $2^n x$  is an integer for some  $n$ , then  $x$  has two binary expansions, but  $f(x)$  is the same for both expansions. It contains exactly one hole, which could be filled in so that  $f(x)$  is periodic. If  $|x - y| < 2^{-m}$ , then  $x_{[1, m]} = y_{[1, m]}$ , so  $d_W(x, y) < 2^{-m+1}$ , so  $f : [0, 1] \rightarrow X_W$  is continuous.

**Proposition 1** *The Weyl and Besicovitch spaces are pathwise connected and infinite dimensional.*

Proof: Consider the continuous map  $f : [0, 1] \rightarrow X_W$  constructed above. For a given  $u \in \mathbf{2}^{\mathbb{Z}}$  we construct a continuous map  $g : [0, 1] \rightarrow \mathbf{2}^{\mathbb{Z}}$  by  $g(x)_i = u_i f(x)_i$ , so  $X_W$  and therefore also  $X_B$  is pathwise connected. To show that  $X_W$  is infinite dimensional, construct for any  $n$  an embedding  $g : [0, 1]^n \rightarrow X_W$  of an  $n$ -dimensional cube by

$$g(x_1, \dots, x_n) = f(x_1)_0 \dots f(x_n)_0 f(x_1)_1 \dots f(x_n)_1 \dots \diamond$$

The following proof is adapted from Marcinkiewicz [12].

**Proposition 2** *The Besicovitch space is complete.*

Proof: We use the unilateral Besicovitch space. Let  $x^{(n)} \in A^{\mathbb{N}}$  be a Cauchy sequence. There exists a subsequence  $x^{(n_j)}$  such that  $d_B(x^{(n_{j+1})}, x^{(n_j)}) < 2^{-j-1}$ . Choose a sequence  $l_j$  of positive integers such that  $l_{j+1} \geq 2l_j$  and for every  $l > l_j$

$$\#\{i \in [0, l) : x_i^{(n_{j+1})} \neq x_i^{(n_j)}\} < l \cdot 2^{-j-1} .$$



It follows that for  $k > j$  and  $l \geq l_k$

$$\#\{i \in [0, l) : x_i^{(n_k)} \neq x_i^{(n_j)}\} < l \cdot 2^{-j} .$$

Define  $x \in A^{\mathbb{N}}$  by  $x_t = x_t^{(n_j)}$  if  $l_j \leq t < l_{j+1}$  and arbitrarily if  $t < l_0$ . If  $k > j$  and  $l_k \leq l < l_{k+1}$ , then

$$\begin{aligned} \#\{i \in [0, l) : x_i \neq x^{(n_j)}\} &\leq l_j + l_{j+2} \cdot \#\{i \in [0, l_{j+2}) : x_i^{(n_{j+1})} \neq x_i^{(n_j)}\} + \cdots + \\ &\quad l_k \#\{i \in [0, l_k) : x_i^{(n_{k-1})} \neq x_i^{(n_j)}\} + \\ &\quad l \#\{i \in [0, l) : x_i^{(n_k)} \neq x_i^{(n_j)}\} \\ &\leq l_j + (l_{j+2} + \cdots + l_k + l)2^{-j} \leq l_j + 3l \cdot 2^{-j} \end{aligned}$$

It follows  $d_B(x, x^{(n_j)}) \leq 3 \cdot 2^{-j}$ , so  $x^{(n_j)}$  converges to  $x$  and since  $x^{(n)}$  is a Cauchy sequence, it converges to  $x$  as well.  $\diamond$

To show further properties of the Weyl and Besicovitch spaces, we use Sturmian sequences (see e.g. de Luca [11] or Blanchard and K urka [3]). For an irrational  $x \in (0, 1)$  define  $S(x) \in \mathbf{2}^{\mathbb{N}}$  by

$$S(x)_n = \begin{cases} 0 & \text{if } 0 < nx - k < 1 - x \text{ for some } k \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

$S(x)$  is called Sturmian sequence with density  $x$ .

**Lemma 1** *If  $x, y \in (0, 1)$  and  $x/y$  are all irrational, then*

$$D_W(S(x), S(y)) = D_B(S(x), S(y)) = x(1 - y) + (1 - x)y .$$

Proof: Consider dynamical system (rotation of the torus)

$$T(a, b) = (a + x \bmod 1, b + y \bmod 1)$$

defined on the torus  $\mathbb{R}^2/\mathbb{Z}^2$ . Then  $T$  is uniquely ergodic with Lebesgue measure as the invariant measure. We have  $S(x)_n \neq S(y)_n$  iff

$$T^n(0, 0) \in [0, 1 - x] \times [1 - y, 1] \cup [1 - x, 1] \times [0, 1 - y] .$$

This set has Lebesgue measure  $x(1 - y) + y(1 - x)$ .  $\diamond$

**Proposition 3** *The Weyl and Besicovitch spaces are neither separable nor locally compact.*

Proof: For any  $0 < a < b < 1$  there exists an uncountable set  $E_{ab} \subseteq (a, b)$  such that for all  $x, y \in E_{ab}$ ,  $x, y$  and  $x/y$  are all irrationals. From Lemma 1 we have that for every  $x, y \in E_{a,b}$  it holds

$$a(1 - b) < d_W(S(x), S(y)) = d_B(S(x), S(y)) < b(1 - a).$$

It follows that neither  $X_W$  nor  $X_B$  is separable (i.e. they do not have a countable base). Since  $b(1-a)$  can be arbitrarily small, and since both  $X_W$  and  $X_B$  are homogeneous, neither  $X_W$  nor  $X_B$  is locally compact.  $\diamond$

Let  $f : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  be a map which is  $W$ -continuous (or  $B$ -continuous). Then  $f(\tilde{x}) \subseteq \widetilde{f(x)}$ , so  $\tilde{f} : X_W \rightarrow X_W$  defined by  $\tilde{f}(\tilde{x}) = \widetilde{f(x)}$  is continuous and  $(X_W, \tilde{f})$  (or  $(X_B, \tilde{f})$ ) is a dynamical system. We refer to these systems when speaking about dynamical  $W$ -properties (or  $B$ -properties) of a map  $f$ . In virtue of the relation  $d_B(x, y) \leq d_W(x, y)$  we have

**Proposition 4** *Let  $f : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  be a map which is both  $W$ -continuous and  $B$ -continuous. Then*

1. *If  $f$  is  $W$ -equicontinuous, then it is  $B$ -equicontinuous.*
2. *If  $f$  has a  $W$ -equicontinuity point, then it has a  $B$ -equicontinuity point.*
3. *If  $f$  is  $W$ -transitive, then it is  $B$ -transitive.*
4. *If  $f$  is  $W$ -chain transitive, then it is  $B$ -chain transitive.*
5. *If  $f$  is  $B$ -expansive, then it is  $W$ -expansive.*
6. *If  $f$  is  $B$ -sensitive, then it is  $W$ -sensitive.*

In both Weyl and Besicovitch spaces we have

**Proposition 5** *Let  $(X, f)$  be a dynamical system on a non-separable space  $X$ . If  $(X, f)$  is transitive, then it is sensitive.*

Proof: There exists  $\varepsilon > 0$  and an uncountable set  $E \subseteq X$  such that for every  $x, y \in E$ ,  $x \neq y$  we have  $d(x, y) > 4\varepsilon$ . We show that  $\varepsilon$  is a sensitivity constant for  $(X, f)$ . Let  $x \in X$ . For every  $n \geq 0$  there is at most one  $z \in E$  whose distance from  $f^n(x)$  is less than  $2\varepsilon$ . Since  $E$  is uncountable, there exists  $z \in E$  such that  $d(f^n(x), z) > 2\varepsilon$  for all  $n \geq 0$ . By transitivity, in every neighborhood  $U$  of  $x$  there exists  $y \in U$  such that  $d(f^n(y), z) < \varepsilon$  for some  $n$ . It follows

$$d(f^n(x), f^n(y)) \geq d(f^n(x), z) - d(z, f^n(y)) \geq 2\varepsilon - \varepsilon = \varepsilon \quad \diamond$$

## 4 Cellular automata

A cellular automaton is a  $C$ -continuous map  $f : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  which commutes with the shift  $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  defined by  $\sigma(x)_i = x_{i+1}$ . Every cellular automaton is defined by some local rule  $F : A^{2r+1} \rightarrow A$  with radius  $r \geq 0$  by

$$f(x)_i = F(x_{i-r} \dots x_{i+r})$$

It follows that any cellular automaton is continuous for both Weyl and Besicovitch pseudometrics. We compare now topological and dynamical properties of cellular automata in the Cantor, Weyl and Besicovitch spaces. We refer to these properties using subscripts  $C$ ,  $W$  or  $B$ .

**Proposition 6** *A cellular automaton  $f : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  is surjective iff it is  $W$ -surjective iff it is  $B$ -surjective (i.e. iff  $\tilde{f} : X_W \rightarrow X_W$  or  $\tilde{f} : X_B \rightarrow X_B$  is surjective).*

Proof: Clearly, if  $f$  is surjective, then so is  $\tilde{f}$ . Suppose that  $\tilde{f} : X_W \rightarrow X_W$  is surjective. By a theorem of Hedlund [8],  $f$  is surjective iff every block  $u \in A^*$  has a preimage. Consider the periodic point  $x = u^\infty$ . By the assumption there exists  $y \in A^{\mathbb{Z}}$  such that  $d_W(f(y), x) = 0$ . It follows that in  $y$  one can find blocks which are mapped to  $u$ .  $\diamond$

**Proposition 7** *If a cellular automaton  $f$  is  $C$ -equicontinuous, then it is  $W$ -equicontinuous and therefore also  $B$ -equicontinuous.*

Proof: By the assumption for  $\varepsilon = 1$  there exists  $\delta = 2^{-m}$  such that for every  $x, y \in A^{\mathbb{Z}}$  if  $x_{[-m, m]} = y_{[-m, m]}$ , then  $f^n(x)_0 = f^n(y)_0$  for every  $n \geq 0$ . Therefore, if  $x_{[j-m, k+m]} = y_{[j-m, k+m]}$ , then  $f^n(x)_{[j, k]} = f^n(y)_{[j, k]}$  for every  $n \geq 0$ . For a given  $\varepsilon > 0$  put  $\delta = \frac{\varepsilon}{2m+2}$  and suppose that  $d_W(x, y) < \delta$ , so there exists  $l_0$  such that for all  $l \geq l_0$  and all  $k \in \mathbb{Z}$

$$\#\{i \in [k+1, k+l] : x_i \neq y_i\} < l\delta$$

Thus in the interval  $[k+1, k+l]$ ,  $f^n(x)$  may differ from  $f^n(y)$  only in one of the end intervals  $[k+1, k+m]$ ,  $[k+l-m+1, k+l]$  or in an interval  $[i-m, i+m]$  for some  $i$  with  $x_i \neq y_i$ . It follows that

$$\text{card}\{i \in [k+1, k+l] : f^n(x)_i \neq f^n(y)_i\} < l\delta(2m+1) + 2m$$

If  $l\delta > 2m$ , then

$$\frac{\#\{i \in [k+1, k+l] : f^n(x)_i \neq f^n(y)_i\}}{l} < \delta(2m+2) = \varepsilon$$

so  $d_W(f^n(x), f^n(y)) < \varepsilon$ . Thus  $f$  is  $W$ -equicontinuous.  $\diamond$

**Proposition 8** *If a cellular automaton  $f$  has a  $C$ -equicontinuity point, then it has a  $W$ -equicontinuity point and therefore also a  $B$ -equicontinuity point.*

Proof: Let  $r$  be the radius of  $f$  and  $z \in A^{\mathbb{Z}}$  be a  $C$ -equicontinuity point of  $f$ . For  $\varepsilon = 2^{-r}$  there exists  $\delta = 2^{-m}$  such that whenever  $y_{[-m, m]} = z_{[-m, m]} = u \in A^{2m+1}$ , then  $f^n(y)_{[-r, r]} = f^n(z)_{[-r, r]}$  for all  $n \geq 0$ . We show that  $x = u^\infty$  is a  $W$ -equicontinuity point. For given  $\varepsilon > 0$  put  $\delta = \frac{\varepsilon}{4m-2r+1}$ . If  $d_W(x, y) < \delta$ , then there exists  $l_0$  such that for all  $l \geq l_0$  and all  $k \in \mathbb{Z}$

$$\#\{i \in [k+1, k+l] : x_i \neq y_i\} < l\delta.$$

Every change in one of the blocks  $x_{[k+1, k+2m+1]} = u$  with  $k = j(2m+1)$  may change only this block or  $m-r$  positions in any of its two neighboring blocks, i.e. at most  $4m-2r+1$  positions. Thus

$$\frac{\#\{i \in [k+1, k+l] : f^n(x)_i \neq f^n(y)_i\}}{l} < \delta(4m-2r+1) = \varepsilon$$

and  $d_W(f^n(x), f^n(y)) < \varepsilon$ .  $\diamond$

The following result is implicit in Hurley [9]

**Lemma 2** *If  $x \in A^{\mathbb{Z}}$  is a  $C$ -stable periodic point of a cellular automaton  $f$ , then  $\sigma(x) = x$ .*

Proof: We can assume that  $x$  is a fixed point, since  $f^n$  is a cellular automaton too. If  $x$  is a stable fixed point with an attracting neighborhood  $x \in [u] \subseteq U$ , then  $\sigma(x)$  is a stable fixed point with attracting neighborhood  $\sigma([u])$ . For  $k$  large enough  $[u] \cap \sigma^k[u]$  and  $[u] \cap \sigma^{k+1}[u]$  are both nonempty. For  $y \in [u] \cap \sigma^k[u]$  and  $z \in [u] \cap \sigma^{k+1}[u]$  we get

$$\sigma^k(x) = \lim_{n \rightarrow \infty} f^n(y) = x = \lim_{n \rightarrow \infty} f^n(z) = \sigma^{k+1}(x)$$

so  $\sigma(x) = x$ .  $\diamond$

**Proposition 9** *If  $x \in A^{\mathbb{Z}}$  is a  $C$ -stable periodic point of a cellular automaton  $f$ , then  $\tilde{x}$  is both  $W$ -stable and  $B$ -stable.*

Proof: By Lemma 2,  $x = a^\infty$  for some  $a \in A$ . By the proof of Proposition 8,  $\tilde{x}$  is both  $W$ -equicontinuous and  $B$ -equicontinuous. Since  $x$  is  $C$ -stable, there exists  $m > 0$  such that for  $a^{2m+1} \in A^{[-m, m]}$ ,  $\lim_{n \rightarrow \infty} f^n(y) = x$  for every  $y \in [a^{2m+1}]$ . It follows that there exists  $s$  such that  $f^s[a^{2m+1}] \subseteq [a^{2m+3}]$  with  $a^{2m+3} \in A^{[-m-1, m+1]}$ , so  $a$ 's spread in  $s$  steps at least one field in both directions. For the Weyl pseudometric consider a neighborhood

$$U = \{y \in A^{\mathbb{Z}} : d_W(y, x) < \frac{1}{2m+1}\}$$

For  $y \in U$  there exists  $l$  such that for every  $k$

$$\#\{i \in [k+1, k+l(2m+1)] : y_i \neq a\} < l$$

so every subword of  $y$  of length  $l(2m+1)$  contains  $1^{2m+1}$  as a subword. It follows that for  $t > s(l-1)(2m+1)$ ,  $f^t(y) = x$ , so  $x$  is  $W$ -stable. For the Besicovitch pseudometric use neighbourhood  $U = \{y \in A^{\mathbb{Z}} : d_B(y, x) < \frac{1}{2m+1}\}$ .  $\diamond$

**Proposition 10** *If a cellular automaton  $f$  is  $W$ -sensitive or ( $B$ -sensitive), then it is  $C$ -sensitive.*

Proof: If  $f$  is  $W$  sensitive, it has no  $W$ -equicontinuity point, so by Proposition 8 it has no  $C$ -equicontinuity point and by Theorem 3 in K urka [10] it is  $C$ -sensitive.  $\diamond$

**Proposition 11** *If a cellular automaton  $f$  is  $W$ -transitive or  $B$ -transitive, then it is  $C$ -transitive.*

Proof: Let  $f$  be  $B$ -transitive and  $u, v \in A^{[-m, m]}$ . We show that  $[u] \cap f^{-n}[v] \neq \emptyset$  for some  $n > 0$ . Consider spatially periodic points  $u^\infty, v^\infty$ . By the assumption for  $\varepsilon = \frac{1}{3(2m+1)}$  there exists  $x \in A^{\mathbb{Z}}$  and  $n > 0$  with  $d_B(x, u^\infty) < \varepsilon$  and  $d_B(y, v^\infty) < \varepsilon$ , where  $y = f^n(x)$ . It follows that there exists  $l > 0$  such that in the interval  $[-m - (2m+1)l, m + (2m+1)l]$  there is at most  $(2m+1)(2l+1)\varepsilon = \frac{2l+1}{3}$  differences, i.e.

$$\#\{i \in [-m - (2m+1)l, m + (2m+1)l] : x_i \neq (u^\infty)_i\} < \frac{2l+1}{3}$$

and

$$\#\{i \in [-m - (2m+1)l, m + (2m+1)l] : y_i \neq (v^\infty)_i\} < \frac{2l+1}{3}$$

Thus there exists at least one unperturbed block, i.e. there exists  $|l_1| \leq l$  such that for  $j = (2m+1)l_1$  we have

$$x_{[j-m, j+m]} = u, \quad f^n(x)_{[j-m, j+m]} = v$$

and  $\sigma^j(x) \in [u] \cap f^{-n}([v])$ .  $\diamond$

**Proposition 12** *If a cellular automaton  $f$  is  $C$ -chain transitive, then it is  $W$ -chain transitive and  $B$ -chain transitive.*

Proof: Let  $F : A^{[-r, r]} \rightarrow A$  be the local rule for  $f$ . A sequence  $x^{(i)} \in A^{\mathbb{Z}}$  is a  $2^{-m}$ -chain for  $d_C$  if  $x_j^{(n+1)} = F(x_{j-r}^{(n)}, \dots, x_{j+r}^{(n)})$  for  $|j| \leq m$ . Since only the sites  $|j| \leq m+r$  are involved, we identify  $2^{-m}$  chains with sequences  $x_{[-m-r, m+r]}^{(i)} \in A^{[-m-r, m+r]}$ . There exists a letter  $a \in A$  such that  $a^\infty$  is periodic. Denote by  $p$  its period. For a given  $\varepsilon > 0$  let  $m \in \mathbb{N}$  be such that  $\frac{2r}{2r+2m+1} < \varepsilon$ . By the assumption for every  $u \in A^{[-m-r, m+r]}$  there exists a  $2^{-m}$  chain  $u^{(1)}, \dots, u^{(n)} \in A^{[-m-r, m+r]}$  such that  $u^{(1)} = a^{2m+2r+1}$  and  $u^{(n)} = u$ . We can assume that  $n > p$ . Let  $w \in A^{[-b, b]}$  be a word which contains as subwords all the words  $u^{(n-p+1)}, \dots, u^{(n)}$ . By the assumption there exists a  $2^{-b+r}$ -chain from  $a^{2b+1}$  to  $w$ . Denote by  $q$  the length of this chain. If we restrict this chain to positions where  $u^{(j)}$  is located, we obtain a  $2^{-m}$ -chain of length  $l$  from  $a^{2r+2m+1}$  to  $u^{(j)}$ . It follows that there exists a  $2^{-m}$ -chain from  $a^{2m+2r+1}$  to  $u$  of all lengths  $q, q+1, \dots, q+p-1$  and since  $a^\infty$  has period  $p$  there exists chains from  $a^{2m+2r+1}$  to  $u$  of all lengths greater than  $l$ . If we consider also chains from  $v$  to  $a^\infty$ , we get that there exists  $q$  such that for every pair  $u, v \in A^{[-m-r, m+r]}$  there exists a  $2^{-m}$ -chain from  $u$  to  $v$ , whose length is exactly  $q$ . Given  $x, y \in A^{\mathbb{Z}}$  we construct now  $\varepsilon$ -chain  $x^{(1)}, \dots, x^{(q)}$  leading from  $x$  to  $y$  for the Weyl pseudometric. In every interval

$$[b_j, c_j] = [-m - r + j(2m + 2r + 1), m + r + j(2m + 2r + 1)]$$

where  $j \in \mathbb{Z}$ , we construct a  $2^{-m}$  chain  $x_{[b_j, c_j]}^{(n)}$  from  $x_{[b_j, c_j]}$  to  $y_{[b_j, c_j]}$ , so  $x^{(1)} = x$  and  $x^{(q)} = y$ . Moreover  $f(x^{(n)})_k = x_k^{(n+1)}$  for every  $k \in [b_j + m, c_j - m]$ , so  $x^{(n)}$  is a  $\varepsilon$ -chain for  $d_W$ .  $\diamond$

**Proposition 13** *No cellular automaton is  $B$ -positively expansive.*

Proof: Let  $f : \mathbf{2}^{\mathbb{Z}} \rightarrow \mathbf{2}^{\mathbb{Z}}$  be a  $B$ -positively expansive cellular automaton with expansivity constant  $\varepsilon$ . Choose an integer  $q$  with  $\frac{1}{q+1} < \varepsilon$  and consider points  $x, y \in \mathbf{2}^{\mathbb{Z}}$ , which are symmetric (i.e.  $x_{-i} = x_i$  and  $y_{-i} = y_i$ ) and their nonnegative parts are

$$\begin{aligned} x_{[0, \infty)} &= 0^{q^1} 1^{q^0} 0^{q^3} 1^{q^2} 0^{q^5} \dots \\ y_{[0, \infty)} &= 1^{q^1} 0^{q^0} 1^{q^3} 0^{q^2} 1^{q^5} \dots \end{aligned}$$

Then  $d_B(0^\infty, x) = d_B(1^\infty, y) = \frac{1}{1+q}$ . Let  $F : A^{2r+1} \rightarrow A$  be local rule of  $f$ . We have four cases:

1.  $F(0 \dots 0) = 0$  and  $F(1 \dots 1) = 0$ ; in this case  $f(x) = 0^\infty$  and hence for any  $t \in \mathbb{N}$ ,  $d_B(f^t(0^\infty), f^t(x)) = 0 < \varepsilon$ .
2.  $F(0 \dots 0) = 0$  and  $F(1 \dots 1) = 1$ ; in this case  $f(x) = x$  and hence for any  $t \in \mathbb{N}$ ,  $d_B(f^t(0^\infty), f^t(x)) = \frac{1}{1+q} < \varepsilon$ .
3.  $F(0 \dots 0) = 1$  and  $F(1 \dots 1) = 1$ ; in this case  $f(y) = 1^\infty$  and hence for any  $t \in \mathbb{N}$ ,  $d_B(f^t(1^\infty), f^t(y)) = 0 < \varepsilon$ .
4.  $F(0 \dots 0) = 1$  and  $F(1 \dots 1) = 0$ ; in this case  $f(0^\infty) = 1^\infty$ ,  $f(1^\infty) = 0^\infty$ ,  $f(x) = y$ ,  $f(y) = x$ , hence  $\forall t \in \mathbb{N}$ ,  $d(f^t(0^\infty), f^t(x)) = \frac{1}{1+q} < \varepsilon$ .  
 $\diamond$

We give some examples showing that the preceding propositions cannot be converted.

**Example 1** *The identity map  $f(x) = x$ .*

The identity is  $W$ -chain transitive (since the Weyl space is connected), but not  $C$ -chain transitive (since the Cantor space is totally disconnected). Thus the converse of Proposition 12 is false.

**Example 2** *The shift map  $\sigma(x)_i = x_{i+1}$*

The shift map is a  $W$ -isometry, so it is  $W$ -equicontinuous and it is neither  $W$ -transitive nor  $W$ -sensitive. On the other hand it is  $C$ -transitive and  $C$ -sensitive. Thus the Propositions 7, 8 and 10 cannot be converted. Observe that  $\tilde{\sigma} : X_W \rightarrow X_W$  has an infinite number of fixed points. Any sequence  $k_n$  of positive integers which grows fast enough, yields a fixed point

$$x = \dots 1^{k_3} 0^{k_2} 1^{k_1} 0^{k_0} 1^{k_1} 0^{k_2} 1^{k_3} \dots$$

**Example 3** *The permutive cellular automaton  $f(x)_i = x_{i-1} + x_i + x_{i+1}$*

is  $B$ -sensitive (see Cattaneo et al [4]). We do not know whether it is  $B$ -transitive.

**Example 4** *The multiplication cellular automaton*  $f(x)_i = x_{i-1}x_ix_{i+1}$

The system has a  $C$ -stable fixed point  $0^\infty$ , and  $W$ -stable and  $B$ -stable fixed point  $\widetilde{0}^\infty$ . In  $X_A$   $\widetilde{f}$  has many other fixed points like  $0^\infty 1^\infty$ ,  $1^\infty 0^\infty$  and for fast enough increasing sequence  $k_n$  points

$$x = \dots 1^{k_3} 0^{k_2} 1^{k_1} 0^{k_0} 1^{k_1} 0^{k_2} 1^{k_3} \dots$$

**Example 5** *Gilman cellular automaton*  $f(x)_i = x_{i+1}x_{i+2}$

Here the fixed point  $0^\infty$  is  $W$ -stable but not  $C$ -stable. Thus Proposition 9 cannot be converted.

In the Cantor topology it is well known that any continuous shift-commuting map on  $A^\mathbb{Z}$  is a cellular automaton. This is no longer true for the Weyl pseudometric

**Example 6** *Let the application*  $f : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$ , *where*  $A = \{0, 1, s\}$ , *be defined as follows*

$$\begin{array}{ll} f(x)_i = a + b + c & \text{if } x_{[i-j-1, i+k+1]} = as^jbs^k c \\ f(x)_i = a + b & \text{if } x_{[i-j-1, \infty)} = as^jbs^\infty \\ f(x)_i = b + c & \text{if } x_{(-\infty, i+k+1]} = s^\infty bs^k c \\ f(x)_i = b & \text{if } x_{(-\infty, \infty)} = s^\infty bs^\infty, x_i = b \\ f(x)_i = s & \text{if } x_i = s \end{array}$$

where  $a, b, c \in \mathbf{2}$ .

This map can be considered as the embedding of the addition of the two nearest neighbors on  $\{0, 1\}$  into  $A^\mathbb{Z}$ , where the letter  $s$  plays a neutral role: it stays unmodified by  $f$  but lets the information pass on between occurrences of 0 and 1. By definition  $f$  commutes with the shift; a coordinate of  $f(x)$  does not depend on any bounded set of neighbors, so  $f$  is not a CA. We claim it is both  $W$ -continuous and  $B$ -continuous. First let  $x \in A^\mathbb{Z}$  and suppose  $x'_i = x_i$  except for  $i = 0$ ; then  $f(x')_i \neq f(x)_i$  for at most three values of  $i$ : 0, the first occurrence of a 0 or 1 to the left and the first one to the right. Now consider  $y \in A^\mathbb{Z}$  and an integer  $n > 0$ ; for each interval of coordinates  $[k, k+n-1]$ ,  $k \in \mathbb{Z}$  one has

$$\#\{j \in [k+1, k+l] : f(x)_j \neq f(y)_j\} \leq 3 \cdot \#\{j \in [k+1, k+l] : x_j \neq y_j\} + 2$$

The first term of the right-hand sum is a very rough majoration of the differences between  $f(x)$  and  $f(y)$  arising in this interval from differences between  $x$  and  $y$  in the same interval; the term 2 majorates the number

of differences arising in the interval because of differences between  $x$  and  $y$  outside this interval. Dividing by  $n$  and taking the lim sup one obtains  $d_W(f(x), f(y)) \leq 3d_W(x, y)$  and  $d_B(f(x), f(y)) \leq 3d_B(x, y)$ , so  $f$  is both  $W$ -continuous and  $B$ -continuous. This example has an interesting dynamical property: there is a unique  $W$ -equicontinuous point for  $f$ . One easily shows that the fixed point  $\widetilde{s^\infty}$  has this property; all other points in the Weyl space have not, because they inherit the sensitivity property of their coordinates on  $A = \{0, 1\}$ .

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