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# *An optimal algorithm to generate tilings*

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january 2003

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## Abstract

A lot of progress has been made in tiling theory in the last ten years after Thurston ([Thu90]), building on previous work by Conway and Lagarias ([CL90]), introduced height functions as a tool to encode and study tilings.

This allowed the authors of this paper, in previous work ([Rém99], [Des01]), to prove that the set of lozenge (or domino) tilings of a hole-free, general-shape domain in the plane can be endowed with a distributive lattice structure.

In this paper, we see that this structure allows us in turn to construct an algorithm that is optimal with respect to both space and execution time to generate all the tilings of a domain  $D$ . We first recall some results about tilings and then we describe the algorithm.

**Keywords:** tiling, height function, flip, generation

## Résumé

De gros progrès ont été obtenus ces dix dernières années après que Thurston ([Thu90]), à partir du travail précédent de Conway et Lagarias ([CL90]) eut introduit les fonctions de hauteur comme outil pour coder et étudier les pavages.

Ces travaux ont permis, aux auteurs de ce papier, de prouver dans des articles précédents ([Rém99], [Des01]) que l'ensemble des pavages par losanges (ou dominos) d'un domaine quelconque sans trou du plan peut être muni d'une structure de treillis distributif.

Dans cet article, nous montrons que cette structure nous permet de construire un algorithme optimal en termes d'espace et de temps d'exécution qui engendre tous les pavages d'un domaine sans trou. Nous rappelons d'abord quelques résultats sur les pavages, puis nous décrivons l'algorithme. .

**Mots-clés:** pavage, fonction de hauteur, flip, génération

# An optimal algorithm to generate tilings

Sébastien Desreux <sup>\*</sup>      Eric Rémila <sup>†</sup>

## Abstract

A lot of progress has been made in tiling theory in the last ten years after Thurston ([Thu90]), building on previous work by Conway and Lagarias ([CL90]), introduced height functions as a tool to encode and study tilings.

This allowed the authors of this paper, in previous work ([Rém99], [Des01]), to prove that the set of lozenge (or domino) tilings of a hole-free, general-shape domain in the plane can be endowed with a distributive lattice structure.

In this paper, we see that this structure allows us in turn to construct an algorithm that is optimal with respect to both space and execution time to generate all the tilings of a domain  $D$ . We first recall some results about tilings and then we describe the algorithm.

## 1 Background

### Tiles

The plane is endowed with either the square or triangular regular lattice whose cells are colored black and white as on a chessboard. This induces a direction on the edges of the lattice: They are directed clockwise around black cells and (consequently) counter-clockwise around white cells. A *domain* or *region* is a finite and simply connected union of cells of the lattice. The boundary of a domain  $D$  will be denoted by  $\partial D$ .

A *lozenge* (resp. *domino*) is a union of two cells of the triangular (resp. square) grid sharing an edge, which is called the *central axis* of the lozenge (resp. domino). This yields two possible shapes for dominoes (vertical or horizontal) and three shapes for lozenges. A *domino tile* (resp. *lozenge tile*) is a domino (resp. lozenge) of either shape. A *tiling* of a domain is a set of tiles that cover the whole area with neither gap nor overlap.

### Height functions

The *height functions*, introduced by W. P. Thurston ([Thu90]) and independently in the statistical physics literature (see [BH97] for a review) and precisely studied and generalized by several authors ([Cha96], [Pro01], [Rém99], [Des01]) are a very powerful tool to study tilings. A lozenge tiling  $T$  of a domain  $D$  can be encoded by a height

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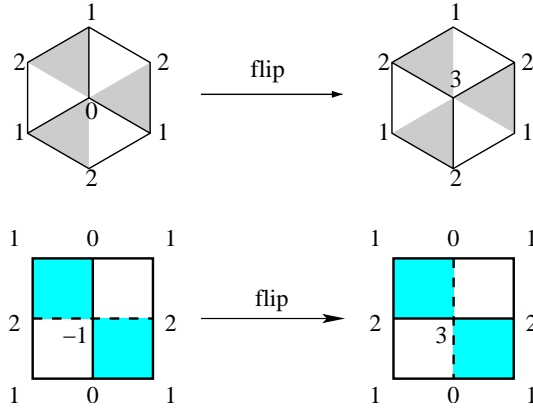


Figure 1: Flips in the triangular and square grids

function  $h_T$  defined as follows: Fix an origin vertex  $O$  on the boundary of  $D$  and set  $h_T(O) = 0$ ; if  $(v, v')$  is a directed edge such that  $[v; v']$  is the central axis of a lozenge of  $T$ , then  $h_T(v') = h_T(v) - 2$ ; otherwise,  $h_T(v') = h_T(v) + 1$ . This definition is coherent since it is coherent for each triangle and  $D$  is simply connected. Similarly, for dominoes,  $h_T(v') = h_T(v) - 3$  if  $[v; v']$  is the central axis of a domino,  $h_T(v') = h_T(v) + 1$  otherwise.

Height functions encode tilings: Only one such function is associated to a tiling and a tiling can be reconstructed from a height function by drawing only the edges whose endpoints have a height difference of 1.

### Lattice structure

Let  $(T, T')$  be a pair of tilings of  $D$ . We say that  $T \leq T'$  if  $h_T(v) \leq h_{T'}(v)$  for each vertex  $v$  of  $D$ . The functions  $h_{\inf(T, T')} = \min(h_T, h_{T'})$  and  $h_{\sup(T, T')} = \max(h_T, h_{T'})$  are themselves height functions that encode tilings ([Rém99], [Des01]), which implies that the set of the tilings of  $D$  has a structure of distributive lattice (see for instance [DP90] for an introduction to lattice theory).

### Flips

Let  $v$  be a vertex in the interior of  $D$  such that all the directed edges ending in  $v$  are central axes of lozenges (resp. dominoes) in a tiling  $T$ . A flip is the replacement of these three lozenges (resp. two dominoes) by three lozenges (resp. two dominoes) whose central axis are edges starting in  $v$ . A flip transforms a local minimum of the height function into a local maximum. See Figure 1.

A new tiling  $T_{\text{flip}}$  is thus obtained;  $T$  and  $T_{\text{flip}}$  are comparable for the order defined above. More generally,  $T \leq T'$  if and only if there exists an increasing sequence  $(T = T_0, T_1, \dots, T_p = T')$  of tilings such that  $\forall 0 \leq i < p$ ,  $T_{i+1}$  is deduced from  $T_i$  by a flip. As a corollary we have the *flip connectivity*: Given any pair  $(T, T')$  of tilings of  $D$ , one can pass from  $T$  to  $T'$  by a sequence of flips and, more precisely, the minimal number of flips to pass from  $T$  to  $T'$  is  $\sum_v |h_T(v) - h_{T'}(v)|/\lambda$ , where  $\lambda = 3$  for lozenges and  $\lambda = 4$  for dominoes.

### Thurston's algorithm

There exists a minimal tiling whose corresponding height function has no local maximum except on the boundary of  $D$ ; indeed, a downward flip could otherwise be performed on the local maximum, yielding a new minimal tiling. From this property one deduces a linear algorithm which constructs the minimal tiling if  $D$  can be tiled, or proves that  $D$  is not tileable ([Thu90]).

First, a vertex on  $\partial D$  must be selected and given an arbitrary height, usually 0. Then the heights of all the vertices on the boundary of  $D$  follow and they have the same height in all the tilings of  $D$ . Since the local maxima of the height function must lie on  $\partial D$ , let us select one such vertex and place tiles that cover it. There is only one way to proceed without introducing a local maximum in the interior of  $D$ . One can apply the same procedure to the remaining domain and a tiling is thus built if at all possible.

This tiling is the minimal element of the lattice of the tilings of  $D$ . A symmetric construction yields the maximal element, in which the height function has no local minimum except on the boundary of  $D$ .

## 2 Generalized Thurston algorithm

Thurston's algorithm allows one to construct a particular tiling of a domain  $D$ , using the fact that there exists a tiling whose height function has no local maximum except on  $\partial D$ . But ordinary tilings do have height functions which admit local maxima in the interior of  $D$ : Is there a way to generalize Thurston's algorithm so that it can construct any tiling of  $D$ ?

We will first reinterpret Thurston's algorithm with Birkhoff's representation theorem for finite distributive lattices, then exhibit a generalized version and finally give an example.

### Link with Birkhoff's representation theorem

Let  $T$  denote any tiling of a fixed domain  $D$ . Let  $V$  denote the set of the vertices in the interior of  $D$  and  $S \subset V$  the vertices on which the height function  $h_T$  reaches a local maximum. A downward flip can be applied on any element of  $S$  but on no element of  $V \setminus S$ . Therefore,  $T$  is a minimal tiling with respect to the heights on the vertices in  $S$ . In order to characterize  $T$ , it is enough to know  $h_T(v)$  for every  $v \in S$ . In lattice terms,  $T$  is the infimum of all the tilings of  $D$  which have fixed values on the elements of  $S$ .

What does the set  $S$  represent? First, let us suppose that  $S$  contains only one vertex  $v$ . Then  $T$  can be obtained by an upward flip from only one other tiling, namely the one obtained by applying a downward flip on  $v$  in  $T$ . This property defines a meet-irreducible element of the lattice of the tilings of  $D$ . If  $S$  contains more than one vertex, it can be viewed as a collection of meet-irreducible elements. Birkhoff's representation theorem (see for instance [DP90]) allows us to formalize this idea:

### ***Birkhoff's representation theorem***

*Any finite distributive lattice is isomorphic to the lattice of the ideals of the order of its meet-irreducible elements.*

In other words, it is legitimate to regard a tiling  $T$  as a collection (down set) of meet-irreducible elements. These admit a simple characterization in the case of tilings: A tiling

is a meet-irreducible element of the lattice if and only if its height function admits exactly one local maximum in the interior of  $D$ .

From this point of view, the minimal element of the lattice, which is precisely the tiling constructed by Thurston's algorithm, corresponds to the empty ideal.

### Generalized algorithm

We now undertake to construct any tiling of  $D$  using Thurston's idea to cover vertices so that no local maximum of the height function can appear.

A meet-irreducible element of the lattice of the tilings of  $D$  is characterized by a unique pair  $(v, h(v))$  where  $v$  is the only vertex in the interior of  $D$  on which the height function admits a local maximum and  $h(v)$  is the value of this local maximum.

Let  $S$  be a set of vertices in the interior of  $D$  and let  $\mathcal{S}$  be constituted of the pairs  $(v, h(v))$  where  $v \in S$  and  $h(v)$  is any number. According to Birkhoff's theorem, any tiling of  $D$  is characterized by a set  $\mathcal{S}$ . Our construction also allows sets  $\mathcal{S}$  which correspond to no tiling because  $h$  may vary too rapidly along an edge. We now suppose that  $\mathcal{S}$  is a fixed set such that there exists at least one tiling characterized by it.

If there exists at least one tiling whose height function coincides with  $h(v)$  for all  $v \in S$ , then there exists a smallest such tiling: It is the infimum of the tilings satisfying this property. In particular, the height function of this smallest tiling, let us note it  $T_{\mathcal{S}}$ , can't have a local maximum outside  $S$ , otherwise this would contradict the minimality of  $T_{\mathcal{S}}$ . We can therefore apply Thurston's idea in order to construct this tiling:

### Generalized Thurston algorithm

- **Input:** A domain  $D$ , a vertex  $v$  of  $\partial D$  whose height is 0, a subset  $S$  of the vertices in the interior of  $D$  and for each  $v \in S$ , an integer  $h(v)$ .
- **Initialization:** Compute the height function of  $\partial D$ . If a vertex receives two distinct heights, then  $D$  is not tileable.
- **Repeat:** Let  $v \in \partial D \cup S$  be a vertex on which the height function admits a global maximum. Place a tile whose frontier covers  $v$  in the only way that does not create a local maximum of the height function. Remove  $v$  from  $D$  and update  $\partial D$ .
- **Until:**  $D$  is tiled or one of its vertices was given two different heights, in which case  $D$  is not tileable.

As the original, this algorithm uses each cell of the domain only once so it is linear in the size of  $D$ . The space required is the one needed to store a height function, which is  $|D| \ln |D|$ , where  $|D|$  denotes the number of cells of  $D$ .

### Example

Let us illustrate the former algorithm with an example in the case of dominoes. Our domain  $D$  is a  $6 \times 4$  chessboard (see Figure 2); the vertex at the center of  $D$  has height 5. As in the case of Thurston's original algorithm, one can fix arbitrarily the height of one vertex on  $\partial D$  and then compute the heights of all the vertices on  $\partial D$ . Moreover, it is equivalent to proceed one tile at a time or to proceed in one step for all the tiles covering a vertex of maximal height.

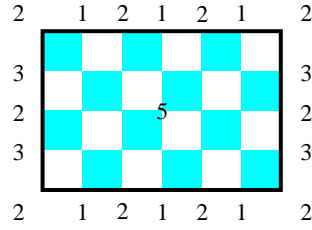


Figure 2: Generalized Thurston algorithm: step 1

Initially, the height function admits a global maximum at the center of the domain. This vertex must be covered by two dominoes, which can be either vertical or horizontal. The latter case would yield a vertex having a height greater than 5 so the former is the only possibility, otherwise we would construct a tiling satisfying the conditions but that is not minimal in this respect. We thus add two vertical dominoes and we update the height function on their boundaries.

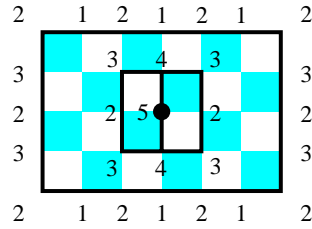


Figure 3: Generalized Thurston algorithm: step 2

The maximal value of the height function is now 4 (see Figure 3). The geometry of the domain compels us to add horizontal dominoes, but let us forget geometry and trust the algorithm. We can either add two vertical dominoes or one horizontal one. The former case would attempt to add a new local maximum, so only the latter is admissible. We thus add two horizontal dominoes, as in Figure 4.

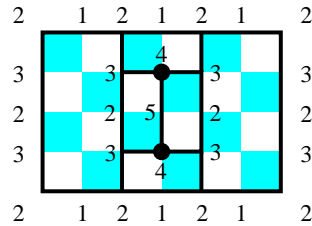


Figure 4: Generalized Thurston algorithm: step 3

In the last step, we proceed simultaneously all the vertices of height 3, with the same reasoning as above, and we obtain a complete tiling of  $D$  (see Figure 5).

We see that the height function of the tiling thus constructed has no local maximum except on the boundary and the center of  $D$ ; it is thus the smallest tiling satisfying the initial requirements.



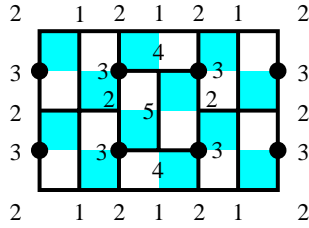


Figure 5: Generalized Thurston algorithm: step 4

If the conditions given initially had not corresponded to a tiling, a vertex would have been given two different heights.

### 3 Encoding of a tiling by a word

In order to generate all the tilings of a domain, a natural idea is to encode tilings by words and then to provide a way to find the successor of an element in the lexicographic order. In this section, we examine a way to encode tilings by words; the next section will provide a successor function.

Since height functions encode tilings, it suffices to fix an arbitrary order on the vertices of  $D$  (any order will do) to obtain an encoding of the tilings by words. This encoding follows closely the height functions and is therefore very natural.

We will use a slightly different construction, although the one mentioned above does work: Instead of the height function, we use a normalized height function, defined as follows:

$$H(v) = \frac{h(v) - h_{\min}(v)}{\lambda}$$

where  $h(v)$  is the value of the height function on the vertex  $v$ ,  $h_{\min}$  is the height function of the minimal tiling and  $\lambda$  is a normalization parameter, equal to 3 in the case of lozenges and to 4 in the case of dominoes. This simple homography has two advantages: First, it unifies the description of tilings by lozenges and by dominoes; second, the normalized height is closely connected to flips. Indeed,  $H(v)$  is the number of times a flip has been applied to  $v$  in any upward path going from the minimal tiling to the tiling considered. And, of course, there is a one-to-one correspondence between height functions and normalized height functions.

Moreover, it is convenient to represent a tiling by its phase:

**Definition 1 (Phase space)** Let  $D$  be a tileable domain,  $V$  the set of the vertices in the interior of  $D$  and  $\varphi$  a numerotation function from  $V$  to  $\{1, \dots, |V|\}$ .

The phase of a tiling  $T$  is the set of pairs  $(\varphi(v), H(\varphi(v)))$  where  $v \in V$ :

$$\Phi(T) = \{ (\varphi(v), H(\varphi(v))) \mid v \in V \}$$

The phase space associated with a domain  $D$  is the union of the  $\Phi(T)$  when  $T$  runs through the set  $\mathcal{T}(D)$  of the tilings of  $D$ :

$$\Psi(D) = \bigcup_{T(D)} \Phi(T)$$

We now give an example. Let us consider the domain of Figure 6, on which we have added an arbitrary order on the inner vertices.

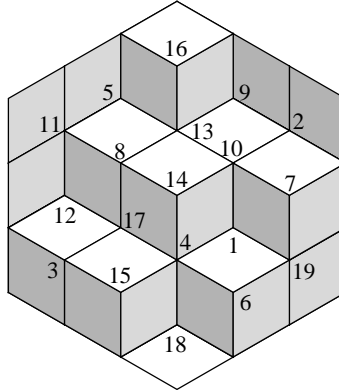


Figure 6: An arbitrary order on the inner vertices of a domain

The normalized heights can be easily computed, but in the case of lozenges they can also be read directly on the drawing. Indeed, since the lozenge group is isomorphic to  $\mathbb{Z}^3$  ([Thu90]), the result of a flip is, visually, to add a cube. Take for instance vertex 16: one cube has been added when starting from the minimal tiling of the domain, so its normalized height is 1.

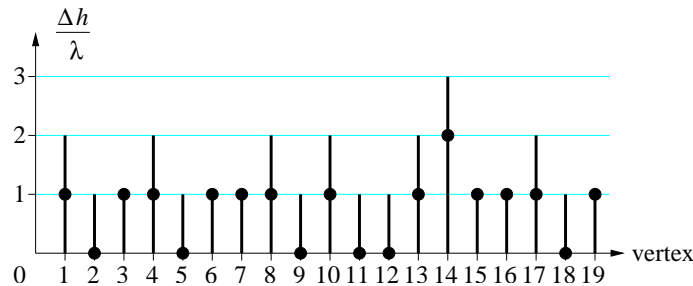


Figure 7: The phase of a tiling

In Figure 7, we have represented the normalized heights against the vertices' numbers. The dots correspond to the phase of the tiling of Figure 6. The vertical segments correspond to the values that the normalized heights of a vertex can take in a tiling. Its maximal value can be computed with Thurston's original algorithm in the maximal tiling version.

The encoding of the tiling follows easily from the phase diagram: it suffices to read each normalized height in the order of the numerotation function:

$$\mu(T) = H(1) \cdot H(2) \cdots H(|V|)$$

For the example of Figure 6, this yields the following word:

$$\begin{array}{cccccccccccccccccccc} \underline{1} & \underline{0} & \underline{1} & \underline{1} & \underline{0} & \underline{1} & \underline{1} & \underline{1} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{1} & \underline{2} & \underline{1} & \underline{1} & \underline{1} & \underline{0} & \underline{1} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \end{array}$$

Figure 8: Encoding of the tiling of Figure 6

## 4 Exhaustive generation

We have seen how to encode a tiling by a word. Since the lexicographic order is a linear extension of the lattice structure, generating all the tilings of a domain amounts to exhibiting a successor function.

### Successor of a word

Let  $w$  be an encoding of a tiling  $T$  of  $D$ . The successor of  $w$  in the lexicographic order need not encode a tiling itself. We call the *successor* of  $w$ , and we will denote it by  $s(w)$ , the smallest of the words greater than  $w$  and that encode a tiling. We will show how to construct it in two steps.

Since tilings are connected by flips, there exists a series of tilings going from  $w$  to  $s(w)$ . Since  $s(w)$  is minimal, the series must contain exactly one upward flip. The first step is thus to determine the local minima of the height function associated with  $w$  and to select the right-most position of the corresponding vertices in  $w$ . Applying one upward flip yields a word  $w'$  which differs from  $w$  on only one position, let us call it  $i_0$ :

$$\begin{cases} w'[i_0] = w[i_0] + 1 \\ w'[i] = w[i] & \text{for } i \neq i_0 \end{cases}$$

$w'$  is greater than  $w$  but it may be greater than  $s(w)$ . Consider for instance the example of Figure 6 again. An upward flip can be performed on vertex 18, which yields  $s(w)$ . At the next iteration, however, the right-most candidate is 12. If an upward flip is performed on this vertex, one obtains a tiling that is greater than  $s(s(w))$  since a downward flip could be performed on 18.

The second step in finding the successor lies in the use of the generalized Thurston algorithm. Suppose an upward flip has been performed (starting from  $w$ ) on the vertex  $i_0$ , yielding a word  $w'$ . The successor of  $w$  has the same values as  $w'$  on positions 1 to  $i_0$  included, but possibly smaller ones for positions  $> i_0$ . It is indeed the smallest of the word that coincide with  $w'$  on the positions 1 to  $i_0$ .

We use the generalized algorithm by feeding it the heights already computed for the vertices 1 to  $i_0$  and letting it compute all the remaining ones. At least one tiling exists under these conditions (the one associated with  $w'$ ) so the algorithm effectively yields a tiling, which has all the desired properties.

As an example, consider Figure 6 (encoded by  $w$ ) and suppose an upward flip has been performed on 18 and 12. We know the values of the normalized heights of  $s(s(w))$  for vertices 1 to 12, as shown in Figure 9.

In order to use the generalized algorithm, we start with values as shown in Figure 10.

The generalized algorithm yields the tiling of Figure 11. Once read, we obtain  $s(s(w))$  (see Figure 12).

$\underline{1}$   $\underline{0}$   $\underline{1}$   $\underline{1}$   $\underline{0}$   $\underline{1}$   $\underline{1}$   $\underline{1}$   $\underline{0}$   $\underline{1}$   $\underline{0}$   $\underline{1}$   $\underline{2}$   $\underline{2}$   $\underline{2}$   $\underline{2}$   $\underline{2}$   $\underline{2}$   $\underline{2}$   
 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Figure 9: The first coordinates of  $s(s(w))$

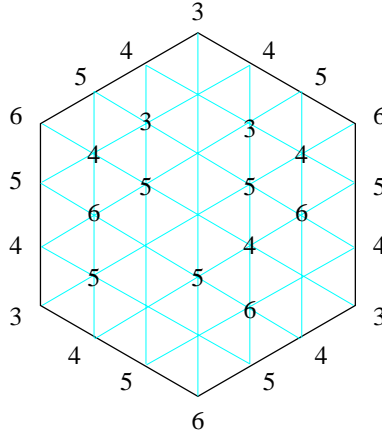


Figure 10: Utilization of the generalized Thurston algorithm

We can now summarize the process in an algorithm:

**Successor algorithm**

- **Input:** A tileable domain  $D$ , a numerotation function  $\varphi$  from the set  $V$  of the inner vertices of  $D$  to  $\llbracket 1; |V| \rrbracket$ , the minimal tiling of  $D$  and a word  $w$  coding a tiling  $T$  of  $D$ .
- **Step 1:** Compute the height function associated with  $T$  by using the normalized heights.
- **Step 2:** For  $i$  from  $|V|$  down to 1, examine whether vertex  $i$  is a local minimum of the height function; stop when such a vertex  $i_0$  has been found. If no vertex is found,  $w$  encodes the maximal tiling.
- **Step 3:** The first coordinates of  $s(w)$  are:

$$\begin{aligned}
 s(w)[i] &= w[i] && \text{for } 1 \leq i < i_0 \\
 s(w)[i_0] &= w[i_0] + 1
 \end{aligned}$$

- **Step 4:** Use the generalized Thurston algorithm in order to find the smallest tiling bearing the heights of  $s(w)$  for  $i = 1$  to  $i_0$ .
- **Step 5:** Encode the tiling obtained at step 4 by a word, which is  $s(w)$ .
- **Output:**  $s(w)$ .

Let us analyze this algorithm. We denote by  $|D|$  the number of vertices in  $D$ . Steps 1, 2, 3 and 5 require  $O(|D|)$  operations; step 4 uses the generalized Thurston algorithm,

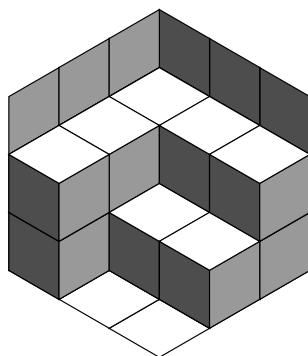


Figure 11: Result of the generalized Thurston algorithm

$$\begin{array}{cccccccccccccccccccc} \underline{1} & \underline{0} & \underline{1} & \underline{1} & \underline{0} & \underline{1} & \underline{1} & \underline{1} & \underline{0} & \underline{1} & \underline{0} & \underline{1} & \underline{1} & \underline{1} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{1} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \end{array}$$

Figure 12:  $s(s(w))$

which also runs in  $O(|D|)$  time, so the execution time is  $O(|D|)$  and the algorithm is linear. The space required is  $O(|D| \ln |D|)$ , which is the space needed to store a height function.

### Exhaustive generation

Since the successor function preserves the lexicographic order, generating all the tilings of  $D$  amounts to recursively calling it:

#### *Generation algorithm*

- **Input:** A domain  $D$ .
- **Initialization:** Compute the minimal tiling of  $D$  with Thurston's algorithm; stop if  $D$  is not tileable. Encode the minimal tiling by a word  $w$ . Numerote the vertices of  $D$ .
- **Current step:**  $w \leftarrow \text{Successor}(w)$  and decode  $w$  to obtain a tiling, until the Successor function does not produce a word.
- **Output:** the tilings of  $D$ .

Let us analyze the algorithm. The time and space complexity are controlled by the current step. Since the Successor function requires  $O(|D|)$  operations, the time complexity of the generation algorithm is  $O(|D|)$  times the number of tilings. In order to evaluate the space complexity, it is legitimate to suppose that each tiling is discarded as soon as its successor is generated since it is not used afterward. The space complexity is thus  $O(|D| \ln |D|)$ .

## 5 Related algorithms

The encoding of a tiling by a word, the successor function and the generalization of Thurston's algorithm can be used to generate more than the tilings of  $D$ : Indeed, all the characteristic elements of the lattice.

### Meet-irreducible elements

The meet-irreducible elements of the lattice are those whose height function admits exactly one local maximum in the interior of  $D$ . For each vertex in the interior of  $D$ , one can compute the height in the minimal and maximal tilings of  $D$  using Thurston's original algorithm. The possible values for  $h(v)$  vary 3 by 3 in the case of lozenges, 4 by 4 in the case of dominoes, so all the possibilities can easily be computed. For each pair defined by  $v$  and an admissible height, there exists a meet-irreducible element of the lattice, which can be computed using the generalized Thurston algorithm.

### Order of the meet-irreducible elements

The order on meet-irreducible elements is inherited from the lattice structure. In the case of lozenges (resp. dominoes), such an element can have at most 3 (resp. 2) successors in the order of the meet-irreducible elements. Generating the full order thus amounts to examining whether the putative successors are indeed meet-irreducible elements of the tiling.

### Lattice

We have already generated all the tilings of  $D$ . In order to generate all the lattice, it suffices to know which tilings can be obtained from a fixed tiling  $T$  by an upward flip.

This is done by examining the height function: for each vertex on which the function admits a local minimum, there exists a single tiling that can be deduced from  $T$  by a single flip. Moreover, since the set of the tilings of  $D$  is connected by flips, all the links between tilings are thus obtained.

Another way to generate the lattice uses the order of meet-irreducible elements: by Birkhoff's representation theorem, the former is isomorphic to the order of the ideals of the latter and there exist optimal generic algorithms that generate the order of the ideals of an arbitrary order (see [HMNS01] and [KMNF92]).

### Intervals

The lattice as a whole is a particular case of interval of itself. In order to generate the elements of an interval, the successor function must be modified: it is enough to compute the supremum of the result of the former successor function and the minimal element of the interval. The links between the tilings can, as above, be computed either by using the local minima of the height function or by using the order of the meet-irreducible elements since the intervals of a finite distributive lattice are themselves finite distributive lattices.

## 6 Conclusion

We have provided several algorithms that make a non-trivial use of the lattice structure of the tilings. Thurston's original algorithm has been reinterpreted through Birkhoff's

representation theorem and generalized in order to construct any tiling of the domain.

The normalized height functions provide a unified description of tilings by dominoes and lozenges; they easily translate into a natural encoding of tilings by words.

Our generation algorithm is linear in the number of tilings and requires a space equivalent to the size of a single tiling. It can be extended to the generation of all the characteristic elements of the lattice: meet-irreducible elements and their order, the lattice structure and its intervals.

Furthermore, it should be rather straightforward to generalize the concepts and methods to domains with holes.

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