Explicit Pure Type Systems for the Lambda-Cube

Romain Kervarc, Pierre Lescanne

Laboratoire de l'Informatique du Parallélisme

École Normale Supérieure de Lyon
Unité Mixte de Recherche CNRS-INRIA-ENS LYON-UCBL n° 5668

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Romain KERVARC
Pierre LESCANNE

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École Normale Supérieure de Lyon
46 Allée d'Italie, 69364 Lyon Cedex 07, France
Téléphone : +33(0)4.72.72.80.37
Télécopieur : +33(0)4.72.72.80.80
Adresse électronique : `lip@ens-lyon.fr`

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Abstract

Pure type systems are a general formalism allowing to represent many type systems – in particular, Barendregt's λ-cube, including Girard's system $\mathcal{F}$, dependent types, and the calculus of constructions. We built a variant of pure type systems by adding a cut rule associated to an explicit substitution in the syntax, according to the Curry-Howard-de Bruijn correspondence. The addition of the cut requires the addition of a new rule for substitutions, with which we are able to show type correctness and subject reduction for all explicit systems. Moreover, we proved that the explicit λ-cube obtained this way is strongly normalizing.

Keywords: λ-calculus, explicit substitutions, pure type systems, higher order types, calculus of constructions, λ-cube, strong normalization.

Résumé

Les sytèmes de types purs sont un formalisme général permettant de représenter de nombreux systèmes – en particulier, le λ-cube de Barendregt, incluant le système $\mathcal{F}$ de Girard, les types dépendants et le calcul des constructions. Nous avons construit une vatiante des systèmes de types purs en ajoutant une règle de coupure associée à une substitution explicite dans la syntaxe, selon la correspondance de Curry-Howard-de Bruijn. L'adjonction de la règle de coupure impose d'ajouter une nouvelle règle pour les substitutions, avec laquelle nous pouvons montrer la correction des types et la réduction du sujet pour tous les systèmes explicites. En outre, nous avons montré que le λ-cube explicite obtenu de cette manière est fortement normalisant.

Mots-clés: λ-calcul, substitutions explicites, systèmes de types purs, types d'ordre supérieur, calcul des constructions, λ-cube, forte normalisation.

Contents

Part I
Introduction

The calculus of constructions was designed by Coquand and Huet [1] as an extension of Girard's system $\mathcal{F}$ [2] in order to provide a very general typed language for proof assistants based on λ-calculus. Its main feature is that it admits several kinds of dependency between types and terms, namely types depending on types, terms depending on types and types depending on terms. Thanks to the Curry-Howard isomorphism, this yields a very general constructive logic with several kinds of quantifications. Whereas the calculus of constructions was designed as a monolithic framework, Barendregt [3] proposed a hierarchical presentation: the so-called *cube* starting from the simply typed λ-calculus adding axioms between sorts and culminating at the calculus of construction. In Barendregt's cube each vertex corresponds to a kind of dependency given by axioms telling constraints over sorts. The systems that occur in building the cube are called pure type system (PTS in short).

As we said the cube is based on the λ-calculus. As it is, this approach misses two key points. On the computational side, the λ-calculus does not give a complete account of the process of substitution since substitutions are not part of the calculus: instead substitutions are described in the meta-theory. Making substitution first-class citizen yields calculi known under the generic name of *calculi of explicit substitution* which themselves fork into two main families, namely with de Bruijn indices [4, 5] and with explicit names [6, 7]. On the logical side, the traditional cube misses the important *cut rule*, which comes naturally as the typing rule for explicit substitution. Unlike Verstergaard and Wells [8] who advocate for de Bruijn indices when dealing with cut, we have chosen to consider the calculus of explicit substitution with explicit names $\lambda\mathsf{x}$ due to Bloo and Rose [6].

Following Bloo [9], we are going to describe a variant of pure type systems that he and we call *explicit pure type systems* (EPTS in short) despite our system contains one more rule than Bloo's. First EPTS replace implicit substitutions in rules by explicit ones, this is specifically the case for the rule (Π-E). Second, EPTS contain the *cut* rule, basically the same as Bloo's rule *substitution* but different. In addition, EPTS contain a new rule called *xpand*, the introduction of which was made necessary by the need to insure subject reduction. It is an avatar of a rule previously introduced in $\lambda\mathsf{x}$ under the name *drop* by Lengrand *et al.* [10]; *xpand* allows proving a type judgment, the term part of which is a closure, i.e. a term with an explicit substitution on the top. So with all the rules taken together, we are able to prove correctness and subject reduction for *explicit pure type systems*. With axioms on sorts we are able to build a cube similar to Barendregt's cube for lambda calculus. Thus the main result of this paper is a proof of the strong normalization of the calculi that correspond to the vertices of that cube.

Related works Other approaches considering cuts are Di Cosmo and Kesner [11] and Di Cosmo, Kesner and Polonovski [12], Verstergaard and Wells [8] and Herbelin [13]. In [14], Muñoz studies dependent types and explicit substitutions. Those approaches do not consider the whole cube and they are all but [13] and part of [12] in the framework of de Bruijn indices. Note also that Lengrand *et al.* [10], speak about cut. Anyway the closest work related to ours is this of Bloo [15, 9], but he considers only explicit substitution in terms, not in types.

Part II
The λ-cube in pure type systems

We shall here introduce the notion of pure type system. Those type systems present the interest of being a general systematic and elegant presentation of various systems, among which very interesting systems, such as Barendregt λ-cube (cf. figure 1 below).

In order to get more information about type systems for the λ-calculus with usual implicit substitution, the reader may have a look at Barendregt's paper [3].

1 Pure type systems

Pure type systems are defined the following way:

Definition 2.1: *Pure type system*
A *pure type system* is a triple $\mathfrak{T} = (\mathcal{S}, \mathcal{A}, \mathcal{R})$, where the elements of $\mathcal{S}$ ar called *sorts*, those of $\mathcal{A} \subseteq \mathcal{S}^2$, *axioms* and those of $\mathcal{R} \subseteq \mathcal{S}^3$, *rules*.

Definition 2.2: $\lambda\mathfrak{T}$*-calculus*
Let $\mathfrak{T} = (\mathcal{S}, \mathcal{A}, \mathcal{R})$ be a pure type system. For all sort $\sigma \in \mathcal{S}$, let $\mathcal{U}_\sigma$ be an infinite countable set of variables of *nature* σ. The set $\mathcal{E}(\mathfrak{T})$ of $\mathfrak{T}$*-expressions* is the set defined by the following grammar:

$$E ::= {}^\sigma x \mid \sigma \mid E\,E \mid \lambda^\sigma x{:}E.E \mid \Pi^\sigma x{:}E.E$$
$${}^\sigma x \in \mathcal{U}_\sigma$$
$$\sigma \in \mathcal{S}$$

Bound variables, free variables and α-conversion are defined as usual.

In all the following, we will forget the sort annotations on variables and consider the set $\mathcal{U}$ of all variables. Moreover we shall apply Barendregt's convention that a variable does not appear both free and bound in the same term, and that two distinct bound variables do not have the same name.

Definition 2.3: β*-reduction in implicit PTS*
β-reduction is induced by the following rule: $(\lambda x{:}A.B)\,C \ \stackrel{\beta}{\to}\ B[x{:=}C]$.

Definition 2.4: *Typing in implicit PTS*
A *type assertion* is a couple of expressions, denoted so: $M : N$, the first and second elements of which are called its *subject* and its *predicate*.
A *typing context* is a finite sequence – and not a set – of type assertions, the subjects of which are distinct variables. Its *domain*, denoted $\mathrm{dom}(\Gamma)$ is the set of the subjects of the assertions in which it consists; its *support* is the sorted sequence of them. The empty context is denoted by $()$. Context concatenation is denoted by a comma and one assumes that two concatenated contexts have distinct domains.
A *type judgement* is an expression of the form $\Gamma \vdash M : N$ obtained by derivation from the inference rules of table 1. M and N are respectively called *subject* and *predicate* of the judgment.

Definition 2.5: *Strong normalization*
Let $\mathfrak{T}$ be a pure type system. $\mathfrak{T}$ is said to be *strongly normalizing* if it satisfies the following property:

$$\frac{(\sigma, \tau) \in \mathcal{A}}{\vdash \sigma : \tau} \text{(axiom-}\mathcal{A}\text{)}$$

$$\frac{\Gamma \vdash A : \rho \quad \Gamma, x : A \vdash B : \sigma \quad (\rho, \sigma, \tau) \in \mathcal{R}}{\Gamma \vdash \Pi x{:}A.B : \tau} \text{(}\Pi\text{-}\mathcal{R}\text{)}$$

$$\frac{\Gamma \vdash A : \sigma}{\Gamma, x : A \vdash x : A} \text{(hypothesis)}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : \sigma \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : C \vdash A : B} \text{(weakening)}$$

$$\frac{\Gamma \vdash (\Pi x{:}A.B) : \sigma \qquad \Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B)} \text{(}\Pi\text{ - I)}$$

$$\frac{\Gamma \vdash M : (\Pi x{:}A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]} \text{(}\Pi\text{ - E)}$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : \overline{\sigma} \quad A \overset{\beta}{\equiv} B}{\Gamma \vdash M : B} \text{(conversion)}$$

Table 1: Typing rules for (implicit) PTS

Let Γ be a typing context, and A, B two expressions such that $\Gamma \vdash_{\mathfrak{T}} A : B$. Then A and B do not admit any infinite reduction with respect to the β-reduction.

2 Construction of the λ-cube

The following definitions allow to construct the λ-cube in a systematic and elegant way:

Definition 2.6: *Non-empty, elementary, full PTS*
An *elementary rule* is a rule of the form (σ, τ, τ), denoted shortly by $[\sigma, \tau]$. Given a set $\mathcal{S}$ of sorts, the set of the elementary rules involving sorts of $\mathcal{S}$ is denoted $\mathcal{R}^e_{\mathcal{S}}$.
A pure type system is said to be non-empty if it contains at least one rule. A pure type system $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ is said *elementary* if $\mathcal{R} \subseteq \mathcal{R}^e_{\mathcal{S}}$ and *full* if $\mathcal{R} = \mathcal{R}^e_{\mathcal{S}}$.

All λ-cube pure type systems have the same sets of sorts and axioms: $\mathcal{S}_c = \{*, \square\}$ and $\mathcal{A}_c = \{* : \square\}$. Their rules are element of $\mathcal{R}^e_{\mathcal{S}_c} = \{[*, *], [*, \square], [\square, *], [\square, \square]\}$.

$\lambda\rightarrow$ contains the only rule $[*, *]$. Then, following an upward arrow adds the rule $[\square, *]$; a rightward arrow, the rule $[*, \square]$ and a backward arrow, the rule $[\square, \square]$. So λF, $\lambda\underline{\omega}$ and λP contain two elementary rules; λP2, λP$\underline{\omega}$ and λFω, three; and λC is the full system.

3 Properties of the λ-cube systems

Barendregt established in [3] that pure type systems of the λ-cube satisfy the following properties:

Theorem 2.1: *Type correctness*
Let $k \in \mathbb{N}$, Γ be a context and A, B be two terms such that $\Gamma \vdash A : B$. Then $\exists \sigma \in \mathcal{S}$, $\Gamma \vdash B : \sigma$ or $B = \sigma$.

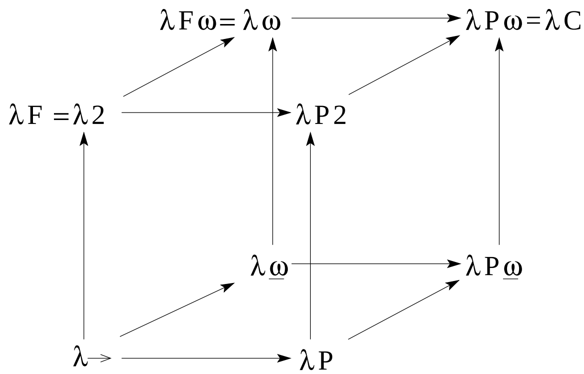

Figure 1: Barendregt's **λ**-cube

Theorem 2.2: *Subject reduction*
Let Γ be a context and A, A', B three terms such that $\Gamma \vdash A : B$ and $A \xrightarrow{\beta} A'$. Then $\Gamma \vdash A' : B$.

In fact the two theorems above hold for any pure type system. On the contrary, the following one is not always satisfied, but this is the case in the λ-cube.

Theorem 2.3: *Strong normalization*
The λ-cube pure type systems are strongly normalizing.

Part III
Explicit pure type systems

In this section, a notion of pure type system with explicit substitution will be introduced, and the properties of these objects studied.

1 Syntax and reduction

The basic definition of pure type systems remains unchanged: a pure type system is a triple $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ of sorts, axioms and rules.

Definition 3.1: $\lambda\mathfrak{T}\mathtt{x}$-*calculus*
Let $\mathfrak{T}$ be a pure type system and $\mathcal{U}$ an infinite countable set of *variables*.
The set $\mathcal{E}\mathtt{x}(\mathfrak{T})$ of *expressions with explicit substitutions* – or more simply *expressions* – of $\mathfrak{T}$ is defined by the following algebraic grammar:

$$E ::= {}^{\sigma}x \mid \sigma \mid E\,E \mid \lambda^{\sigma}x{:}E.E \mid \Pi^{\sigma}x{:}E.E \mid E\langle{}^{\sigma}x{:=}E\rangle$$
$$^{\sigma}x \in \mathcal{U}_{\sigma}$$
$$\sigma \in \mathcal{S}$$

One defines upon $\mathcal{E}\mathtt{x}(\mathfrak{T})$ the relation $\overset{\alpha\mathtt{x}}{\equiv}$ of $\alpha\mathfrak{T}\mathtt{x}$-*equivalence* inductively as follows:

- if $M = x \in \mathcal{U}$, $M \overset{\alpha\mathtt{x}}{\equiv} N$ if $N = x$;
- if $M = \sigma \in \mathcal{S}$, $M \overset{\alpha\mathtt{x}}{\equiv} N$ if $N = \sigma$;
- if $M = \lambda x{:}A.P$, $M \overset{\alpha\mathtt{x}}{\equiv} N$ if $N = \lambda y{:}B.R$ with $A \overset{\alpha\mathtt{x}}{\equiv} B$ and $P[x{:=}z] \overset{\alpha\mathtt{x}}{\equiv} R[y{:=}z]$ for all z except a finite number;
- if $M = \Pi x{:}A.P$, $M \overset{\alpha\mathtt{x}}{\equiv} N$ if $N = \lambda y{:}B.R$ with $A \overset{\alpha\mathtt{x}}{\equiv} B$ and $P[x{:=}z] \overset{\alpha\mathtt{x}}{\equiv} R[y{:=}z]$ for all z except a finite number;
- if $M = PQ$, $M \overset{\alpha\mathtt{x}}{\equiv} N$ if $N = RS$ with $P \overset{\alpha\mathtt{x}}{\equiv} R$ and $Q \overset{\alpha\mathtt{x}}{\equiv} S$;
- if $M = P\langle x{:=}Q\rangle$, $M \overset{\alpha\mathtt{x}}{\equiv} N$ if $N = R\langle y{:=}S\rangle$, with $Q \overset{\alpha\mathtt{x}}{\equiv} S$ and $P[x{:=}z] \overset{\alpha\mathtt{x}}{\equiv} R[y{:=}z]$ for all z except a finite number.

This relation is an equivalence relation, and one can therefore define the quotient set $\Lambda\mathfrak{T}\mathtt{x} = \mathcal{E}\mathtt{x}(\mathfrak{T})/\overset{\alpha\mathtt{x}}{\equiv}$, the elements of which are the terms of the $\lambda\mathfrak{T}\mathtt{x}$-calcul. All the operations of $\mathcal{E}\mathtt{x}(\mathfrak{T})$ can be canonically extended to $\Lambda\mathfrak{T}\mathtt{x}$.
In the rest of this section, we will forget about sort decoration for variables.

Definition 3.2: *Bound, free, available variables*
The set $bv(M)$ of the *bound* variables of a term M is inductively defined as follows:

- $bv(x) = bv(\sigma) = \emptyset$;
- $bv(\lambda x{:}L.M) {=} bv(\Pi x{:}L.M) {=} bv(L) {\cup} bv(M) {\cup} \{x\}$;
- $bv(M\,N) = bv(M) \cup bv(N)$;
- $bv(M\langle x{:=}N\rangle) = bv(M) \cup bv(N) \cup \{x\}$.

The set $fv(M)$ of the *free* variables of a term M is inductively defined as follows:

- $fv(x) = \{x\}; \quad fv(\sigma) = \emptyset;$
- $fv(\lambda x{:}L.M){=}fv(\Pi x{:}L.M){=}(fv(M)\backslash\{x\}) \cup fv(L);$
- $fv(M\,N) = fv(M) \cup fv(N);$
- $fv(M\langle x{:=}N\rangle) = (fv(M) \setminus \{x\}) \cup fv(N).$

The set $av(M)$ of the *available* variables of a term M is inductively defined as follows:

- $av(x) = \{x\}; \quad av(\sigma) = \emptyset;$
- $av(\lambda x{:}L.M){=}av(\Pi x{:}L.M){=}(av(M)\backslash\{x\}) \cup av(L);$
- $av(M\,N) = av(M) \cup av(N);$
- $av(M\langle x{:=}N\rangle){=}\begin{cases}(av(M)\backslash\{x\}){\cup}av(N) & \text{if} x{\in}av(M)\\ av(M) & \text{if} x{\notin}av(M).\end{cases}$

Available variables were introduced in [10]; they are more relevant than free variables for terms with substitution.

In what follows we shall again consider terms up to α-conversion and use Barendregt's convention.

The notion of reduction can now be defined:

Definition 3.3: *βx-reduction in explicit PTS*
One considers the following reduction rules:

(B)	$(\lambda x{:}A.B)\,C$	$\xrightarrow{B}$	$B\langle x{:=}C\rangle$
(quant)	$(\Pi y{:}A.B)\langle x{:=}C\rangle$	$\xrightarrow{X}$	$\Pi y{:}A\langle x{:=}C\rangle.B\langle x{:=}C\rangle$
(abs)	$(\lambda y{:}A.B)\langle x{:=}C\rangle$	$\xrightarrow{X}$	$\lambda y{:}A\langle x{:=}C\rangle.B\langle x{:=}C\rangle$
(app)	$(A\,B)\langle x{:=}C\rangle$	$\xrightarrow{X}$	$A\langle x{:=}C\rangle\,B\langle x{:=}C\rangle$
(subst)	$x\langle x{:=}N\rangle$	$\xrightarrow{X}$	N
(var)	$y\langle x{:=}N\rangle$	$\xrightarrow{X}$	$y \quad$ if $x \neq y$
(gc)	$M\langle x{:=}N\rangle$	$\xrightarrow{X}$	$M \quad$ if $x \notin av(M)$

The x-*reduction*, denoted $\xrightarrow{\mathtt{x}}$, is defined as the relation induced by $\xrightarrow{X}$. The $\beta\mathfrak{T}$x-*reduction*, $\xrightarrow{\beta\mathtt{x}}$, is defined as the relation induced by $\xrightarrow{X}$ and $\xrightarrow{B}$.

Here we make use of the (gc) rule instead of the (var) rule on its own. In fact the two systems – with and without (gc) – are equivalent.

2 Typing

We introduce now a few slight modifications to the notions of typing in pure type systems, and discuss the pertinence of these modifications.

Definition 3.4: *Typing in explicit PTS*
A *type assertion* is a couple denoted $M : N$ where M, the *subject*, belongs to $\Lambda\mathfrak{T}\mathtt{x}$ and N, the *predicate*, belongs to $\overline{\Lambda\mathfrak{T}\mathtt{x}} = \Lambda\mathfrak{T}\mathtt{x} \uplus \{\int\}$. The additionnary $\int$ element, named *pseudo-sort*, is a special type introduced in the frame of explicit substitutions for reasons explained afterwards.

A *typing context* is a finite sequence of type assertions, the subjects of which are distinct variable and the predicates of which are types of $\Lambda\mathfrak{T}\mathtt{x}$. Its *domain*, denoted $\mathrm{dom}(\Gamma)$ is the set of the subjects of the assertions in which it consists; its *support* is the sorted sequence of them, i.e. the image of the context through the first canonical surjection. The empty context is denoted by $()$. Context concatenation is denoted by a comma and one implicitely assumes that two concatenated contexts have distinct domains.

Type judgement are of the form $\Gamma \vdash_{\mathfrak{T}} M : N$. They are obtained by derivation from the inference rules enounced in table 2 below. Index $\mathfrak{T}$ will be omitted in non-ambiguous cases.

Remark:

Like in implicit pure type systems, typing contexts are ordered sequences, and not sets.

Notation:

If $\Gamma = (x_i : A_i)_{1 \leqslant i \leqslant n}$, then:

i. $\Gamma,\, x_{n+1} : A_{n+1}$ denotes $(x_i : A_i)_{1 \leqslant i \leqslant n+1}$;

ii. $\Gamma \setminus y$ denotes $(x_i : A_i)_{i \in I}$, where $I = \{i \in [\![1,\, n]\!] / x_i \neq y\}$.

Notation:

$\overline{\mathcal{S}}$ is the distinct union $\mathcal{S} \uplus \{\int\}$ and $\overline{\mathcal{A}}$ is the distinct union $\mathcal{A} \cup \{(\sigma,\, \int) \,/\, \sigma \in \mathcal{S}\}$.

$$\frac{(\overline{\sigma},\, \overline{\tau}) \in \overline{\mathcal{A}}}{\vdash \overline{\sigma} : \overline{\tau}}\ \text{(axiom)}$$

$$\frac{\Gamma \vdash A{:}\rho \quad \Gamma, x{:}A \vdash B{:}\sigma \quad (\rho, \sigma, \tau) \in \mathcal{R}; x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \Pi x{:}A.B : \tau}\ \text{(rule)}$$

$$\frac{\Gamma \vdash A : \sigma \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma,\, x{:}A \vdash x{:}A}\ \text{(hypothesis)}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : \sigma \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma,\, x : C \vdash A : B}\ \text{(weakening)}$$

$$\frac{\Gamma \vdash (\Pi x{:}A.B){:}\sigma \quad \Gamma,\, x{:}A \vdash M{:}B \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B)}\ (\Pi\text{ - I})$$

$$\frac{\Gamma \vdash M : (\Pi x{:}A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B\langle x{:=}N\rangle}\ (\Pi\text{ - E})$$

$$\frac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle}\ \text{(cut)}$$

$$\frac{\Gamma \vdash M{:}B \quad \Delta \vdash N{:}A \quad P\langle x{:=}N\rangle \xrightarrow{\mathtt{x}} M}{\Gamma \vdash P\langle x{:=}N\rangle : B}\ (\mathtt{x}\text{pand})$$

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : \overline{\sigma} \quad A \overset{\beta\mathtt{x}}{\equiv} B}{\Gamma \vdash M : B}\ \text{(conversion)}$$

Table 2: Typing rules for EPTS

Explicit PTS contain two new rules with respect to PTS.
The (cut) rule corresponds to Bloo's (substitution) rule in [15] with substitution in both subject (i.e. term expression) and predicate (i.e. type expression). It is the version with explicit names of Muñoz's (Clos_{Π}) rule in [14].
The (xpand) rule is a not so straightforward generalization of the (drop) rule introduced in [10].

In all the following we shall try and respect the following writing convention: small latine letters denote variables; capital latine letters denote terms; small greek letters denote sorts, and, if overlined, sorts or $\int$; capital greek letters denote typing contexts.

In order to be able to make inductive reasoning upon derivation trees, it will be useful to introduce a notion of height or depth. This notion will moreover have to take into account the fact that some rules do not change the structure of the subject of the judgment, whereas others do. Therefore we introduce the following *complexity* notion:

Definition 3.5: *Complexity*

Let $\overset{\vdots}{\Gamma \vdash A{:}B}$ be a type derivation. This derivation's *complexity* is the integer computed using the rules shown in table 3.
The *complexity* of a type judgement $\Gamma \vdash A : B$, denoted $\varkappa(\Gamma \vdash A : B)$, is defined as the least integer k such that $\Gamma \vdash A : B$ admits a derivation of complexity k.

Notation:
One shall label a tree node by $[k]$ to indicate that its complexity is equal to k and by $(\!(k)\!)$ to indicate that it is at most k. One also adopts the same notation for type judgments.
This complexity notion is introduced for technical reasons.

Definition 3.6: *Well-formed context*
Let $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ a typing context.
Γ is said to be *well-formed*, which is denoted $\Gamma \vdash$, if for all $i \in [\![1, n]\!]$, there exists $\overline{\sigma}_i \in \overline{\mathcal{S}}$ such that $(x_1 : A_1, \ldots, x_{i-1} : A_{i-1}) \vdash A_i : \overline{\sigma}_i$.

Notation:
Let $\Gamma = (x_i : A_i)_{1 \leqslant i \leqslant n}$ and $\Delta = (x_i : B_i)_{1 \leqslant i \leqslant n}$ be two contexts of same support.

(*i*) Let $\xrightarrow{R}$ be an reduction relation upon $\Lambda\mathfrak{T}\mathrm{x}$. One shall write $\Gamma \xrightarrow{R} \Delta$ if there exist $i \in [\![1, n]\!]$ such that $A_i \xrightarrow{R} B_i$ and for all $j \in [\![1, n]\!] \setminus \{i\}$, $A_j = B_j$.

(*ii*) Let $\overset{R}{\equiv}$ be an equivalence relation upon $\Lambda\mathfrak{T}\mathrm{x}$. One shall write $\Gamma \overset{R}{\equiv} \Delta$ if for all $i \in [\![1, n]\!]$, $A_i \overset{R}{\equiv} B_i$.

Notice that the notions of reduction and equivalence defined above are preserving the well-formedness of contexts.

Definition 3.7: *Relationships between contexts*
Let $\Gamma = (e_i)_{1 \leqslant i \leqslant m}$ and $\Delta = (f_j)_{1 \leqslant j \leqslant n}$ be two contexts.

- Γ is said to be a *sub-context* of Δ, and Δ a *super-context* of Γ, denoted $\Gamma \sqsubseteq \Delta$, if Γ is a subsequence of Δ, i.e. there exists a strictly monotonic application $\varphi : [\![1, m]\!] \to [\![1, n]\!]$ such that for all $i \in [\![1, m]\!]$, $e_i = f_{\varphi(i)}$;
- Γ is said to be a *restriction* of Δ, and Δ an *extension* of Γ, denoted $\Gamma \subseteq \Delta$, if any assertion of Γ also belongs to Δ, i.e. there exists an injection $\varphi : [\![1, m]\!] \to [\![1, n]\!]$ such that for all $i \in [\![1, m]\!]$, $e_i = f_{\varphi(i)}$;

$$\frac{(\overline{\sigma},\,\overline{\tau})\;\in\;\overline{\mathcal{A}}}{\vdash\overline{\sigma}:\overline{\tau}\;_{[0]}}\;\text{(axiom)}$$

$$\frac{\Gamma\vdash A{:}\rho\,_{[i]}\quad \Gamma,x{:}A\vdash B{:}\sigma\,_{[j]}\quad (\rho,\sigma,\tau){\in}\mathcal{R};\,x{\notin}\mathrm{dom}(\Gamma)}{\Gamma\vdash\Pi x{:}A.B:\tau\;_{[1+\max\{i,j\}]}}\;\text{(rule)}$$

$$\frac{\Gamma\vdash A:\sigma\;_{[k]}\qquad x\notin\mathrm{dom}(\Gamma)}{\Gamma,\,x:A\vdash x:A\;_{[k+1]}}\;\text{(hypothesis)}$$

$$\frac{\Gamma\vdash A:B\;_{[i]}\quad \Gamma\vdash C:\sigma\;_{[j]}\quad x\notin\mathrm{dom}(\Gamma)}{\Gamma,\,x:C\vdash A:B\;_{[i]}}\;\text{(weakening)}$$

$$\frac{\Gamma\vdash(\Pi x{:}A.B){:}\sigma\;_{[i]}\quad \Gamma,x{:}A\vdash M{:}B\;_{[j]}\quad x{\notin}\mathrm{dom}(\Gamma)}{\Gamma\vdash\lambda x{:}A.M:(\Pi x{:}A.B)\;_{[1+\max\{i,j\}]}}\;(\Pi\text{ - I})$$

$$\frac{\Gamma\vdash M:(\Pi x{:}A.B)\;_{[i]}\qquad \Gamma\vdash N:A\;_{[j]}}{\Gamma\vdash MN:B\langle x{:=}N\rangle\;_{[1+\max\{i,j\}]}}\;(\Pi\text{ - E})$$

$$\frac{\Gamma,\,x:A\vdash M:B\;_{[i]}\qquad \Gamma\vdash N:A\;_{[j]}}{\Gamma\vdash M\langle x{:=}N\rangle:B\langle x{:=}N\rangle\;_{[1+\max\{i,j+1\}]}}\;\text{(cut)}$$

$$\frac{\Gamma\vdash M:B\;_{[i]}\quad \Delta\vdash N:A\;_{[j]}\quad P\langle x{:=}N\rangle\xrightarrow{\mathrm{x}}M}{\Gamma\vdash P\langle x{:=}N\rangle:B\;_{[\max\{i,j+1\}]}}\;(\times\text{pand})$$

$$\frac{\Gamma\vdash M:A\;_{[i]}\quad \Gamma\vdash B:\overline{\sigma}\;_{[j]}\quad A\overset{\beta\mathrm{x}}{\equiv}B}{\Gamma\vdash M:B\;_{[\max\{i,j+1\}]}}\;\text{(conversion)}$$

Table 3: Complexity of derivations

- Γ is said to be an *prefix context* of Δ, and Δ a *prolongation* of Γ, denoted $\Gamma\prec\Delta$, if $m\leqslant n$ and for all $i\in[\![1,\,m]\!]$, $e_i{=}f_i$.

Remark:
If Γ is a context such that there exist A and B with $\Gamma\vdash A:B$, then Γ is well-formed.

Definition 3.8: *Complexity of terms*
The *complexity* of a term A in the well-formed context Γ typing A, denoted $\varkappa_\Gamma(A)$, is inductively defined as follows:

- if $A=\sigma\in\mathcal{S}$, $\varkappa_\Gamma(A)=0$;
- if $A=x$, necessarily $\Gamma=\Gamma_1,\,x:B,\,\Gamma_2$ and one sets $\varkappa_\Gamma(A)=\varkappa_{\Gamma_1}(B)$, which is defined because Γ is well-formed;
- if $A=\lambda x{:}C.D$, $\varkappa_\Gamma(A)=1+\varkappa_\Gamma(C)+\varkappa_{\Gamma,\,x:C}(D)$
- if $A=\Pi x{:}C.D$, $\varkappa_\Gamma(A)=1+\varkappa_\Gamma(C)+\varkappa_{\Gamma,\,x:C}(D)$
- if $A=CD$, $\varkappa_\Gamma(A)=1+\varkappa_\Gamma(C)+\varkappa_\Gamma(D)$
- if $A=C\langle x{:=}D\rangle$, $\varkappa_\Gamma(A)=1+\varkappa_{\Gamma,\,x:B}(C)+\varkappa_\Gamma(D)$

Remark:
The complexity of a term only depends on the complexity of its free variables. In particular, if Δ be a well-formed extension of Γ, $\varkappa_\Delta(A) = \varkappa_\Gamma(A)$.

3 The "pseudo-sort" type

The need for the extra pseudo-type $\int$ comes from the fact that in $\lambda\mathfrak{T}\mathsf{x}$, we apply substitutions not only to term expressions, but also to type expressions – whereas for instance Bloo, in [15], only applies explicit substitution to assertion subjects (i.e. term expressions), and not to assertion predicates (i.e. type expressions). This choice implies the possible apparition of types of the form e.g. $\sigma\langle x_1{:=}N_1\rangle \ldots \langle x_k{:=}N_k\rangle$, and we therefore wished to extend the PTS (conversion) rule:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \sigma \qquad A \overset{\beta\mathsf{x}}{\equiv} B}{\Gamma \vdash M : B}\text{(conversion)}$$

in order to be able to swap types A and B in case they both represent (i.e. are βx-equivalent to) a same sort σ.

The premise $\Gamma \vdash B : \sigma$ of the (conversion) rule ensures that only correct types are used. The problem is how to extend it enough to allow to swap A and B and nevertheless preserve the correctness.

We considered two proposals to extend $\Gamma \vdash B : \sigma$. The first one was $(\Gamma \vdash B : \sigma) \vee (B \in \mathcal{S})$. The problem with this rule is that it is not flexible enough, as it says that if B is not typable by a sort, then it has to be syntactically equal (not convertible) to an element of $\mathcal{S}$ (e.g. $*$, $\Box$ in the cube). This is too strong.
The second one was to say that if B is not typable by a sort, it has to be convertible to an element of $\mathcal{S}$. But the corresponding premise $(\Gamma \vdash B : \sigma) \vee (B \overset{\beta\mathsf{x}}{\equiv} \tau)$ is too loose, since from $\Gamma \vdash M : \sigma$, it would be possible to derive $\Gamma \vdash M : \sigma\langle x{:=}\omega\omega\rangle$, which is not strongly normalizing.

Therefore we went to replace $B \overset{\beta\mathsf{x}}{\equiv} \tau$ by something similar but with a strict control on the convertibility of B to τ, accepting any B of the form $\tau\langle x_1{:=}N_1\rangle \ldots \langle x_k{:=}N_k\rangle$ provided that all N_i should be typable.

As a solution we introduce the pseudo-sort $\int$, which intuitively means "disguised sort". But this pseudo-sort $\int$ should not be included in the set of sorts, because it is well-known that type systems with the sort of all the sorts are not consistent. We will manipulate $\int$ like a sort using (axiom) and (xpand), but we insist that **$\int$ is no sort**, since it appears nowhere in the the set of rules and never as a predicate of a context type assertion. Moreover, it allows type correction and satisfies the following statement, which is all that we wish.

Proposition 3.1: *Restricted use of* $\int$
For all context Γ and term M such that $\Gamma \vdash M : \int$, there exist $\sigma \in \mathcal{S}$, $x_1, \ldots, x_k \in \mathcal{U}$, $N_1, \ldots, N_k \in \Lambda\mathfrak{T}\mathsf{x}$ such that $M = \sigma\langle x_1{:=}N_1\rangle \cdots \langle x_k{:=}N_k\rangle$.

4 The (xpand) rule

The introduction of this rule answers a specific identification problem linked with explicit substitutions.

On the one hand, this rule generalizes the (drop) rule (cf. [10]), which enables to type some terms like $yz\langle x{:=}zy\rangle$.

On the other hand, and this is the main reason of introduction of this rule, it solves the following problem.

It is sometimes needed to invert the order in which two hypotheses are discarded. E.g., having typed in context $\Gamma, y : D, x : A$ the term $(\lambda x{:}A.B)\langle y{:=}C\rangle$, and willing to type the term $\lambda x{:}A\langle y{:=}C\rangle.B\langle y{:=}C\rangle$ – to ensure subject reduction – one needs to use first hypothesis $y : D$ then hypothesis $x : A$. But y may well have free occurences in A. In the case of implicit substitutions, a *subsitution lemma* solves the problem by establishing that if $\Gamma, y : D, \Delta \vdash M : N$ and $\Gamma \vdash C : D$, then $\Gamma, \Delta[C/y] \vdash M[C/y] : N[C/y]$. So it is possible to invert the order of two hypotheses and afterwards discard the hypotheses of Δ. But this lemma does not hold for explicit substitutions, because if M is e.g. an abstraction $\lambda x{:}P.Q$, the terms $(\lambda x{:}P.Q)\langle y{:=}C\rangle$ and $\lambda x{:}P\langle y{:=}C\rangle.Q\langle y{:=}C\rangle$ are not syntactically equal – whereas they would be with implicit substitutions. It is therefore necessary to lift the substitutions, and the (xpand) rule does it.

In particular, (xpand) allows us to prove the subject reduction for general pure type systems with explicit substitution, unlike Bloo [15, 9], who has a counter-example:

$$(\lambda x{:}a.(\lambda z{:}a.z)x)\langle a{:=}b\rangle \xrightarrow{\beta\mathfrak{T}\mathrm{x}} \lambda x{:}b.((\lambda z{:}a.z)x)\langle a{:=}b\rangle.$$

One could object that the following rule is not satisfactory from the point of view of type inference, because it performs a kind of subject expansion. But in fact this is not the case, as for type inference rules must be read upward: this rule simply allows to "push" the explicit substitution inward enough to be able to type the term – which solves Bloo's problem. Moreover, this corresponds to the intuitive perception of explicit systems as lazy systems, where substitutions are not performed when not needed.

In the conclusion of [14], Muñoz writes about problems related to a rule that he calls (Clos_{Π}), which is in the framework of $\lambda\sigma$-like calculi of explicit substitution our cut rule. The problems he mentions are solved bu (xpand).

Part IV

Properties of explicit PTS

In this section we will study the properties of our explicit pure type systems and show especially that they verify type correctness and subject reduction.

1 Type derivation lemmas

Lemma 4.1: *Free variables*
Let $\Gamma = x_1 : X_1, \ldots, x_n : X_n$ be such a context and A, B be two terms such that $\Gamma \vdash A : B$. Then:

i. $fv(A) \cup fv(B) \subseteq \mathrm{dom}(\Gamma)$;

ii. $\forall\, i \in [\![1, n]\!],\, fv(X_i) \subseteq \{x_j \,/\, 1 \leqslant j < i\}$.

Lemma 4.2: *Compose*
Let A, B, C be three terms and x, y be two variables such that $x, \notin av(B) \cup av(B)$.
Then $A\langle x{:=}B\rangle\langle y{:=}C\rangle \overset{\mathtt{x}}{\equiv} A\langle y{:=}C\rangle\langle x{:=}B\langle y{:=}C\rangle\rangle$
(note that the former condition is ensured by Barendregt's convention).

Lemma 4.3: *Initialization*
Let Γ be a well-formed context. Then:

i. if $(\overline{\sigma} : \overline{\tau}) \in \overline{\mathcal{A}}$, then $\Gamma \vdash \sigma : \tau$ $[0]$;

ii. if $(x : A) \in \Gamma$, then $\Gamma \vdash x : A$ $[1{+}\varkappa_\Gamma(A)]$.

Lemma 4.4: *Weakening*
Let Γ and Δ be two contexts such that $\Gamma \subseteq \Delta$. Let A, B be two such terms of $\overline{\Lambda\mathfrak{T}\mathtt{x}}$ that $\Gamma \vdash A : B$ $[k]$. Then $\Delta \vdash A : B$ $((k))$.

Lemma 4.5: *Generation*
Let Γ be a context, A and B be two terms of $\overline{\Lambda\mathfrak{T}\mathtt{x}}$ such that $\Gamma \vdash M : T$. Then:

$$
\begin{array}{rl}
\textit{(i)} & M = \sigma{\in}\overline{\mathcal{S}} \\
\Rightarrow & \exists\,\tau{\in}\overline{\mathcal{S}}, T \overset{\beta\mathtt{x}}{\equiv} \tau \wedge (\sigma : \tau){\in}\mathcal{A} \\
\textit{(ii)} & M = x{\in}\mathcal{U} \\
\Rightarrow & \exists\,\tau{\in}\mathcal{S}, \exists\,U{\in}\Lambda\mathfrak{T}\mathtt{x}, \Gamma \vdash U : \tau \ ((k{-}1)) \\
 & \wedge (x : U){\in}\Gamma \wedge T \overset{\beta\mathtt{x}}{\equiv} U \\
\textit{(iii)} & M = \Pi x{:}A.B \\
\Rightarrow & \exists\,(\rho,\, \sigma,\, \tau){\in}\mathcal{R}, \Gamma \vdash A : \rho \ ((k{-}1)) \\
 & \wedge \Gamma,\, x : A \vdash B : \sigma \ ((k{-}1)) \wedge T \overset{\beta\mathtt{x}}{\equiv} \tau \\
\textit{(iv)} & M = \lambda x{:}A.B \\
\Rightarrow & \exists\,\sigma{\in}\mathcal{S}, \exists\,C{\in}\Lambda\mathfrak{T}\mathtt{x}, \Gamma \vdash (\Pi x{:}A.C) : \sigma \ ((k{-}1)) \\
 & \wedge \Gamma,\, x : A \vdash B : C \ ((k{-}1)) \wedge T \overset{\beta\mathtt{x}}{\equiv} \Pi x{:}A.C \\
\textit{(v)} & M = AB \\
\Rightarrow & \exists\,C,\, D{\in}\Lambda\mathfrak{T}\mathtt{x}, \Gamma \vdash A : (\Pi x{:}C.D) \ ((k{-}1)) \\
 & \wedge \Gamma \vdash B : C \ ((k{-}1)) \wedge T \overset{\beta\mathtt{x}}{\equiv} D\langle x{:=}B\rangle \\
\textit{(vi)} & M = A\langle x{:=}B\rangle \\
\Rightarrow & \exists\,C{\in}\Lambda\mathfrak{T}\mathtt{x},\, D{\in}\overline{\Lambda\mathfrak{T}\mathtt{x}}, \Gamma,\, x : C \vdash A : D \ ((k{-}1)) \\
 & \wedge \Gamma \vdash B : C \ ((k{-}1)) \wedge T \overset{\beta\mathtt{x}}{\equiv} D\langle x{:=}B\rangle \\
\vee & \exists\,\Delta, \exists\,C,\, D,\, E{\in}\Lambda\mathfrak{T}\mathtt{x}, \Gamma \vdash E : D \ ((k{-}1)) \\
 & \wedge \Delta \vdash B : C \ ((k{-}1)) \wedge A\langle x{:=}B\rangle \xrightarrow{\mathtt{x}} E \wedge T \overset{\beta\mathtt{x}}{\equiv} D
\end{array}
$$

Lemma 4.6: *Substitution*
Let Θ and Ξ be three contexts, v be a variable and P, Q, R, S be four terms such that $\Theta,\, v : S,\, \Xi \vdash P : Q$ and $\Theta \vdash R : S$. Then $\Theta, \Xi\langle v{:=}R\rangle \vdash P\langle v{:=}R\rangle : Q\langle v{:=}R\rangle$.

2 Type correctness and subject reduction

In this section, we are going to state two fundamental results needed for a type system to be "appropriate". Type correctness theorem ensures that only terms having a meaning as a type – i.e. sorts and terms typable by a sort – may be used as type in dervations.

Theorem 4.7: *Type correctness*
Let $k \in \mathbb{N}$ be an integer, Γ be a context, A, B be two terms of $\overline{\Lambda\mathfrak{T}\mathtt{x}}$ such that $\Gamma \vdash A : B\ [k]$. Then $\exists\,\overline{\sigma} \in \overline{\mathcal{S}}$, $\Gamma \vdash B : \overline{\sigma}\ ((k))$.

Theorem 4.8: *Subject reduction*
Let Γ, Γ' be contexts and A, A', B be three terms of $\Lambda\mathfrak{T}\mathtt{x}$ such that $\Gamma \vdash A : B$, $\Gamma \xrightarrow{\beta\mathtt{x}}\!\!\!\rightarrow \Gamma'$ and $A \xrightarrow{\beta\mathtt{x}}\!\!\!\rightarrow A'$. Then $\Gamma' \vdash A' : B$.

Remark:
The dual property, subject expansion, is not satisfied.

Part V

Strong normalization for the explicit λ-cube

Here we show that the explicit pure type systems of the λ-cube are strongly normalizing. For this, it is enough to show that λCx is, as:

Lemma 5.1:
Let $\mathfrak{T}$ be an explicit pure type system of the λ-cube.
Let Γ, A, B be such that $\Gamma \vdash_{\mathfrak{T}} A : B$. Then $\Gamma \vdash_{\lambda\mathsf{Cx}} A : B$.

The proof of strong normalization in λCx will be achieved through a two-step reduction:

(*i*) show that λCx is strongly normalizing if $\lambda\omega$x is;

(*ii*) show that $\lambda\omega$x is strongly normalizing if $\mathcal{F}\mathsf{x}$ is.

$\mathcal{F}\mathsf{x}$ is an adaptation to $\lambda\mathsf{x}$ of Girard's system $\mathcal{F}$ (cf. [16]), which we defined in a former work [17] and which we proved to be strongly normalizing.
And hence the theorem will hold.

In all the following, we will only consider pure type systems from the λ-cube, that is, with the following sorts and axioms: $\mathcal{S} = \{*, \Box\}$ and $\mathcal{A} = \{* : \Box\}$ – and, of course, $\overline{\mathcal{S}} = \{*, \Box, \int\}$ and $\overline{\mathcal{A}} = \{* : \Box, * : \int, \Box : \int\}$ – and rules included in $\mathcal{R}^e_{\mathcal{S}} = \{[*, *], [*, \Box], [\Box, *], [\Box, \Box]\}$.

1 Preliminaries

In this proof, we are going to introduce for technical reasons many fresh auxiliary variable, *e.g.* Ø, ø, þ, ς, ß, æ on specific purpose.

Now we are going to define some notions which shall be useful in both steps of the proof.

One introduces a useful classification on expressions by the means of the notion of *degree*:

Definition 5.1: *Degree of an expression*
The *degree* $\delta(E)$ of a λ-cube expression E is inductively defined as follows:

- $\delta(\int) = 4$; $\delta(\Box) = 3$; $\delta(*) = 2$;
- $\delta({}^\sigma x) = \delta(\sigma) - 2$;
- $\delta(\lambda^\sigma x{:}A.B) = \delta(\Pi^\sigma x{:}A.B) = \delta(B\langle{}^\sigma x{:=}A\rangle) = \delta(B\ A) = \delta(B)$.

Notation:
For $I \subseteq \mathbb{N}$, let Δ_I denote the set $\{M \in \Lambda\mathfrak{T}\mathsf{x} \,/\, \delta(M) \in I\}$.

Definition 5.2: *(Hereditary) compatibility of a statement*
A statement $A : B$ is said to be *compatible* if $\delta(A) + 1 = \delta(B)$.
It is said to be *hereditarily compatible* if it is compatible and all its substatements occuring after a binder – λ or Π – are compatible.

Lemma 5.2: *Properties of the degree*
The following properties hold:

(i) $\Gamma \vdash_{\lambda\mathsf{Cx}} M : U \overset{\beta\mathsf{x}}{\equiv} \Box$ implies $\delta(M) = 2$;

(ii) if M is typable in λCx, then $M \xrightarrow{\beta\mathsf{x}} M'$ implies $\delta(M) = \delta(M')$;

(iii) if $\Gamma \vdash_{\lambda \mathsf{Cx}} A : B$ ($B \neq \int$), then $A : B$ and the statements of Γ are hereditarily compatible;

(iv) if $(\lambda x{:}A.B)$ is typable in λCx, then $\delta(x) = \delta(A)$.

(v) if $B\langle x{:=}A\rangle$ is typable in λCx and $x \in av(B)$, then $\delta(x) = \delta(A)$.

2 First step

In this proof, we are going to introduce three maps: $\flat\cdot$, $\sharp\cdot$ and $[\![\cdot]\!]$. Our purpose here is to reduce the strong normalization in λCx into $\lambda\omega$x by showing that if $\Gamma \vdash_{\lambda \mathsf{Cx}} A : B$ then $\sharp(\Gamma) \vdash_{\lambda\omega \mathsf{x}} [\![A]\!] : \sharp B$ and $A \xrightarrow{\beta \mathsf{x}} A'$ implies $[\![A]\!] \xrightarrow{\beta \mathsf{x}}{}^{+} [\![A']\!]$. The map $\flat\cdot$ will ensure type correctness.

We define a map $\flat : \Delta_{\{2,3\}} \longrightarrow \Lambda\mathfrak{T}$x as follows:

- $\flat * = \flat\Box = \flat\int = *$;
- $\flat(\Pi x{:}A.B) = \flat A \rightarrow \flat B$ if $\delta(A) = 2$
  $= \flat B$ else;
- $\flat(\lambda x{:}A.B) = \flat(B\ A) = \flat(B\langle x{:=}A\rangle) = \flat B$.

Intuitively this map "flattens" terms.

It is clear that if $\delta(M) = 2, 3$ then $\flat M$ is defined and, moreover, $fv(\flat M) = \emptyset$

Lemma 5.3:

The following propositions hold:

i. $\Gamma \vdash_{\lambda \mathsf{Cx}} A : U$ and $U \overset{\beta \mathsf{x}}{\equiv} \Box$ imply $\vdash_{\lambda\omega \mathsf{x}} \flat A : \Box$;

ii. let A be a term of $\Delta_{2,3}$ and $A \xrightarrow{\beta \mathsf{x}} B$; then $\flat A = \flat B$.

In the following, we introduce special variables, terms and context:

- a term $\varnothing = \Pi^{\Box} x : *.x$;
- a variable ${}^{\Box}ø$ with $ø : *$;
- a variable ${}^{*}þ$ with $þ : \varnothing$;
- a context $\Theta = [ø : *, þ : \varnothing]$;
- indexed terms $Þ_M$ defined as follows:
  - if $\Gamma \vdash_{\lambda\omega \mathsf{x}} M{:}*$, then $Þ_M{=}þM$;
  - if $\Gamma \vdash_{\lambda\omega \mathsf{x}} M{:}\Box$ and $M{=}*$, then $Þ_M{=}ø$;
  - if $\Gamma \vdash_{\lambda\omega \mathsf{x}} M{:}\Box$ and $M{=}A{\rightarrow}B$, then $Þ_M{=}\lambda x{:}A.Þ_B$.

Lemma 5.4:

If $\Gamma \vdash_{\lambda\omega \mathsf{x}} M : \sigma$, then $\Theta, \Gamma \vdash_{\lambda\omega \mathsf{x}} Þ_M : M$.

We can now define the next map $\sharp : \Delta_{\{1,2,3\}} \rightarrow \Lambda\mathfrak{T}$x as follows:

- $\sharp * = \sharp\Box = \sharp\int = ø$;
- $\sharp\,{}^{\Box}x = {}^{\Box}x$;

- $\sharp\,(\Pi x{:}A.B) = \Pi x{:}\flat A.\sharp\,A \rightarrow \sharp\,B$ if $\delta(A) = 2$
  $\Pi x{:}\sharp\,A.\sharp\,B$ if $\delta(A) = 1$
  $\sharp\,B$ else;

- $\sharp\,(\lambda x{:}A.B) = \lambda x{:}\flat A.\sharp\,B$ if $\delta(A) = 2$
  $\sharp\,B$ else;

- $\sharp\,(B\,A) = \sharp\,B$ if $\delta(A) = 0$
  $\sharp\,B\,\sharp\,A$ else;

- $\sharp\,(B\langle^{*}x{:=}A\rangle) = \sharp\,B$;
  $\sharp\,(B\langle^{\Box}x{:=}A\rangle) = \sharp\,B\,\langle x{:=}\sharp\,A\rangle$.

The map $\sharp$ can be extended to contexts as follows: define
$\sharp\,' : ({}^{*}x : A) \mapsto {}^{*}x : \sharp\,A$
$({}^{\Box}x : A) \mapsto {}^{\Box}x : \flat A,\ {}^{*}x : \sharp\,A$, then $\sharp\,(\Gamma) = \Theta,\ \sharp\,'(\Gamma)$.

(By induction upon the structure of $A \in \Delta_{\{1,2,3\}}$, it follows that $\sharp\,A$ is defined and ${}^{*}x \notin fv(\sharp\,A)$.)

Lemma 5.5:
The following properties hold:

i. if $A \xrightarrow{\beta\mathsf{x}} B$, then $\sharp\,A \xrightarrow{\beta\mathsf{x}}\!\!\!\rightarrow \sharp\,B$;

ii. let $\Gamma \vdash_{\lambda\mathsf{Cx}} B : U \overset{\beta\mathsf{x}}{\equiv} \Box$ or $B \overset{\beta\mathsf{x}}{\equiv} \Box$;
then $\Gamma \vdash_{\lambda\mathsf{Cx}} A : B$ implies $\sharp\,(\Gamma) \vdash_{\lambda\omega\mathsf{x}} \sharp\,A : \flat B$.

Now we can define a last map $[\![\cdot]\!] : \Delta_{\{0,1,2\}} \longrightarrow \Lambda\mathfrak{T}\mathsf{x}$ as follows:

- $[\![*]\!] = Ꝥ_{\emptyset}$;

- $[\![{}^{\Box}x]\!] = [\![{}^{*}x]\!] = {}^{*}x$;

- $[\![\Pi x{:}A.B]\!] =$
  $Ꝥ_{\emptyset\rightarrow\emptyset\rightarrow\emptyset}[\![A]\!]([\![B]\!]\langle^{\Box}x{:=}Ꝥ_{\flat A}\rangle\langle^{*}x{:=}Ꝥ_{\sharp\,A}\rangle)$ if $\sharp\,A = 2$
  $Ꝥ_{\emptyset\rightarrow\emptyset\rightarrow\emptyset}[\![A]\!]([\![B]\!]\langle^{*}x{:=}Ꝥ_{\sharp\,A}\rangle)$ else;

- $[\![\lambda x{:}A.B]\!] =$
  $(\lambda^{*}ð : \emptyset.\lambda^{\Box}x{:}\flat A.\lambda^{*}x{:}\sharp\,A.[\![B]\!])[\![A]\!]$ if $\sharp\,A{=}2$
  $(\lambda^{*}ð : \emptyset.\lambda^{*}x{:}\sharp\,A.[\![B]\!])[\![A]\!]$ else;

- $[\![BA]\!] = [\![B]\!]\sharp\,A[\![A]\!]$ if $\sharp\,A \neq 0$
  $[\![B]\!][\![A]\!]$ else;

- $[\![B\langle^{*}x{:=}A\rangle]\!] = [\![B]\!]\langle^{*}x{:=}[\![A]\!]\rangle$;
  $[\![B\langle^{\Box}x{:=}A\rangle]\!] = [\![B]\!]\langle^{\Box}x{:=}\sharp\,A\rangle\langle^{*}x{:=}[\![A]\!]\rangle$

where ${}^{*}ð$ above is a fresh variable.

Lemma 5.6:
The following properties hold:

(*i*) if $A \xrightarrow{\beta\mathsf{x}} B$, then $[\![A]\!] \xrightarrow{\beta\mathsf{x}}{}^{+} [\![B]\!]$;

(*ii*) $\Gamma \vdash_{\lambda\mathsf{Cx}} A : B$ implies $\sharp\,(\Gamma) \vdash_{\lambda\omega\mathsf{x}} [\![A]\!] : \sharp\,B$.

Proposition 5.7:
λCx is strongly normalizing if $\lambda\omega$x is.

3 Second step

Let us first recall briefly system $\mathcal{F}\mathsf{x}$.

We denote here by $\Lambda\mathrm{x}$ the set of the λ-terms with explicit substitution, the elements M of which are defined by the following algebraic grammar, where $\mathcal{V}$ is a set of term variables (denoted by small latin letters):

$$M ::= x \mid \lambda x.M \mid M\ M \mid M\,\langle x{:=}M\rangle \qquad (x \in \mathcal{V}).$$

$\mathcal{F}\mathsf{x}$ is a type system for $\Lambda\mathrm{x}$. The type set Φ is defined by the following algebric grammar, Υ been a set of type variables (denoted as types by small greek letters):

$$\tau ::= \alpha \mid \forall\alpha.\tau \mid \tau{\rightarrow}\tau \qquad (\alpha \in \Upsilon).$$

The inference rules of $\mathcal{F}\mathsf{x}$ are presented in table 4 below.

$$\frac{}{\Gamma,\, x:\sigma \vdash x:\sigma}\ \text{(hypothèse)}$$

$$\frac{\Gamma,\, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma{\rightarrow}\tau}\ (\rightarrow\text{ - I}) \qquad \frac{\Gamma \vdash M:\sigma{\rightarrow}\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}\ (\rightarrow\text{ - E})$$

$$\frac{\Gamma \vdash M:\tau \quad \alpha \in \Upsilon \text{ non libre dans } \Gamma}{\Gamma \vdash M:\forall\alpha.\tau}\ (\forall\text{ - I}) \qquad \frac{\Gamma \vdash M:\forall\alpha.\tau \quad \sigma \in \Phi}{\Gamma \vdash M:\tau[\alpha{:=}\sigma]}\ (\forall\text{ - E})$$

$$\frac{\Gamma,\, x:\sigma \vdash M:\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash M\langle x{:=}N\rangle:\tau}\ \text{(coupure)} \qquad \frac{\Gamma \vdash M:\tau \quad \Delta \vdash N:\sigma \quad x \notin av(M)}{\Gamma \vdash M\langle x{:=}N\rangle:\tau}\ \text{(largage)}$$

Table 4: Typing rules for $\mathcal{F}\mathsf{x}$

We recall the following result of [17]:

Theorem 5.8:

$\mathcal{F}\mathsf{x}$ is strongly normalizing.

Let us define two maps : $|\cdot| : \Delta_{\{0,1\}} \longrightarrow \Lambda\mathrm{x}$ and $\natural : \Delta_{\{1,2\}} \longrightarrow \Phi$ such that, on the first hand, if $\Gamma \vdash_{\lambda\omega\mathrm{x}} A : B$ then $\natural\Gamma \vdash_{\mathcal{F}} |A| : \natural B$, and, on the second hand, if $A \xrightarrow{\beta\mathrm{x}} A'$ then $|A| \xrightarrow{\beta\mathrm{x}} |A|'$. These maps is to reduce strong normalization in $\lambda\omega\mathrm{x}$ into this in $\mathcal{F}\mathsf{x}$.

We set aside in Υ a special type variable of Υ which should never be used except in the special cases that we are going to specify. Let us call it ς. Moreover, we also set aside two special term variables, namely *ß* and *æ*.

We define $|\cdot|$ as follows:

- $|\sigma| = ß$
- $|{}^{\sigma}x| = x_\sigma$ (note that x_σ is a term variable for any σ);
- $|\Pi^\sigma x{:}A.B| = (\lambda ß.(\lambda x_\sigma.|B|)æ)|A|$;
- $|\lambda^\sigma x{:}A.B| = (\lambda ß.\lambda x_\sigma.|B|)|A|$ if $\delta({}^{\sigma}x) = \delta(B)$
  $(\lambda ß.|B|)|A|$ else;
- $|B\ A| = |B|\ |A|$ if $\delta(B) = \delta(A)$
  $(\lambda ß.|B|)|A|$ else;

- $|B\langle^\sigma x{:=}A\rangle| = |B|\,\langle x_\sigma{:=}|A|\rangle$.

Lemma 5.9:
If $M \xrightarrow{\beta\mathsf{x}} M'$ then $|M| \xrightarrow{\beta\mathsf{x}}{}^{+} |M'|$.

As far as predicate terms are concerned, one can easily show by induction that a term typable by a sort is the closure of either a sort or a variable or a Π-abstraction or an application like $(\lambda^\Box x{:}A.B)C$. So with the type correctness theorem $\natural$ need not be defined but on terms of the former aspect; we therefore define it the following way:

- $\natural\sigma = \varsigma$
- $\natural^\Box x = \xi$ (one assumes that $\mathcal{V}$ can be injected in Υ, and this injection is denoted by replacing a latin letter by the corresponding greek letter)
- $\natural\Pi^\sigma x{:}A.B = \begin{array}{ll} \natural A {\to} \natural B & \text{if } \delta(B) = \delta(A) \\ \forall\xi.\natural B & \text{else;} \end{array}$
- $\natural((\lambda^* x{:}A.B)C) = \natural B$;
- $\natural((\lambda^\Box x{:}A.B)C) = \natural B[\natural C/\xi]$;
- $\natural(B\langle^* x{:=}C\rangle) = \natural B$;
- $\natural(B\langle^\Box x{:=}C\rangle) = \natural B[\natural C/\xi]$.

Lemma 5.10:
If A, B are predicate terms (i.e. terms typable by a sort) such that $A \overset{\beta\mathsf{x}}{\equiv} B$, then $\natural A = \natural B$.

We introduce now special type judgements in order to be able to translate contexts:

- $\bot$ will be an abbreviation for the type $\forall\xi.\xi$ and one will have $\text{æ} : \bot$;
- the translations of sorts will correspound: $^*\eth : \varsigma$;
- for this purpose, a particular typing context is defined: $\Xi = \{\text{æ} : \bot, {}^*\eth : \varsigma\}$.

Given the map $\natural'$ defined so: $\natural'(^\sigma x : T) = x_\sigma : \natural T$, we can now extend $\natural$ to contexts the following way: $\natural\Gamma = \Xi, \natural'(\Gamma)$.

Lemma 5.11:
For all type φ, $\Xi \vdash \text{æ} : \varphi$.

We can now state that the translation works correctly:

Lemma 5.12:
If $\Gamma \vdash_{\lambda\omega\mathsf{x}} A : B$ then $\natural\Gamma \vdash_{\mathcal{F}} |A| : \natural B$.

Proposition 5.13:
$\lambda\omega\mathsf{x}$ is strongly normalizing.

4 Strong normalization theorem

We have shown that all pure type systems of the λ-cube are strongly normalizing if and only if $\lambda C\mathsf{x}$ is, that $\lambda C\mathsf{x}$ is strongly normalizing if $\lambda\omega\mathsf{x}$ is and that $\lambda\omega\mathsf{x}$ is strongly normalizing if $\mathcal{F}\mathsf{x}$ is, which is the case. Therefore:

Theorem 5.14: *Strong normalization for the $\lambda\mathsf{x}$-cube*
All pure type systems of the explicit λ-cube are strongly normalizing.

Part VI

Conclusion

We have studied explicit pure type systems (EPTS) which are an extension of pure type systems (PTS) where λ-calculus is replaced by the calculus of explicit substitution $\lambda\mathsf{x}$.

We have defined in these EPTS an equivalent of Barendregt's λ-cube and we have proven that the explicit pure type systems of this cube are strongly normalizing.

In particular, we think that we have answered Muñoz's request in [14] when he writes: "work is necessary to understand the interaction with dependant types and meta-variables" in the context of explicit substitution.

Part VII

Appendix A: Proofs

I - Introduction

II - The λ-cube in pure type systems

2.1, 2.2, 2.3 These are proved by Barendregt in [3].

III - Explicit pure type systems

3.1 Easy by induction upon judgement derivation.

IV - Properties of explicit PTS

4.1 By induction upon the derivation of $\Gamma \vdash A : B$.

4.2 By structural induction upon A.

4.3 By inductive application of the (weakening) rule.

4.4 By induction upon the derivation of $\Gamma \vdash A : B$.

- (axiome): $\dfrac{(\overline{\sigma}, \overline{\tau}) \in \overline{\mathcal{A}}}{\vdash \overline{\sigma} : \overline{\tau}}$
  Let Δ be a well-formed extension of $()$, i.e. a well-formed context. By the initialization lemma (4.3), $\Delta \vdash \overline{\sigma} : \overline{\tau}\ [0]$.

- (rule): $\dfrac{\Gamma \vdash A{:}\rho\ \Gamma, x{:}A \vdash B{:}\sigma\ (\rho,\sigma,\tau) \in \mathcal{R}; x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \Pi x{:}A.B : \tau}$
  Let Δ be a well-formed extension of Γ. By induction hypothesis, $\Delta \vdash A : \rho\ ((k-1))$. So $\Delta, x : A$ is well-formed and $\Delta, x : A \vdash B : \sigma\ ((k-1))$ by induction hypothesis. So by (rule) $\Delta \vdash \Pi x{:}A.B : \tau\ ((k))$.

- (hypothesis): $\dfrac{\Gamma \vdash A : \sigma \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \vdash x : A}$
  Let Δ be a well-formed extension of $\Gamma, x : A$. Δ is a well-formed context containing $(x : A)$, so, by the initialization lemma (4.3) $\Delta \vdash x : A\ [1+\varkappa_\Delta(A)]$. Now $\varkappa_\Delta(A) = \varkappa_\Gamma(A) = k - 1$.

- (weakening): $\dfrac{\Gamma \vdash A : B \quad \Gamma \vdash C : \sigma \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : C \vdash A : B}$
  A well-formed extension Δ of $\Gamma, x : C$ is also a well-formed extension of Γ so, by induction hypothesis, $\Delta \vdash A : B\ ((k))$.

- (Π - I): $\dfrac{\Gamma \vdash (\Pi x{:}A.B){:}\sigma \quad \Gamma, x{:}A \vdash M{:}B \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B)}$
  Obvious by induction hypothesis – same as for (rule).

- (Π - E): $\dfrac{\Gamma \vdash M : (\Pi x{:}A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B\langle x{:=}N\rangle}$
  Obvious by induction hypothesis.

- (cut): $\dfrac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle}$
  Obvious by induction hypothesis.

- (xpand): $\dfrac{\Gamma \vdash M : B \quad \Delta \vdash N : A \quad P\langle x{:=}N\rangle \xrightarrow{\mathrm{x}} M}{\Gamma \vdash P\langle x{:=}N\rangle : B}$
  Obvious by induction hypothesis.

- (conversion): $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \overline{\sigma} \qquad A \overset{\beta\mathrm{x}}{\equiv} B}{\Gamma \vdash M : B}$
  Obvious by induction hypothesis.

4.5 The last rule applied in the derivation can only be (weakening), (conversion), or the expected rule – respectively, from (i) to (vii): (axiom), (hypothesis), (rule), (Π-I), (Π-E), (cut) or (xpand). It is easy to show by induction that the results hold with a restriction of Γ, and one can afterwards conclude with the weakening lemma (4.4).

4.6 This lemma is not used in the proof of **4.7**; it will be proved together with **4.8**.

4.7 By induction upon the derivation of $\Gamma \vdash A : B$.

- (axiom): $\dfrac{(\overline{\sigma}, \overline{\tau}) \in \mathcal{A}}{\vdash \overline{\sigma} : \overline{\tau} \ [0]}$
  Obvious with (axiom), as $(\overline{\tau}, \int) \in \overline{\mathcal{A}}$.

- (rule): $\dfrac{\Gamma \vdash A{:}\rho \quad \Gamma, x{:}A \vdash B{:}\sigma \quad (\rho,\sigma,\tau) \in \mathcal{R}; x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \Pi x{:}A.B : \tau \ [k]}$
  Obvious with (axiom), as $(\overline{\tau}, \int) \in \overline{\mathcal{A}}$.

- (hypothesis): $\dfrac{\Gamma \vdash A : \sigma \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \vdash x : A \ [k+1]}$
  Obvious.

- (weakening): $\dfrac{\Gamma \vdash A : B \quad \Gamma \vdash C : \sigma \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : C \vdash A : B \ [k]}$
  Obvious by induction hypothesis and (weakening).

- (Π - I): $\dfrac{\Gamma \vdash (\Pi x{:}A.B){:}\sigma \quad \Gamma, x{:}A \vdash M{:}B \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B) \ [k+1]}$
  Obvious.

- (Π - E): $\dfrac{\Gamma \vdash M : (\Pi x{:}A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B\langle x{:=}N\rangle \ {\scriptstyle [k+1]}}$
  By induction hypothesis, there exists $\overline{\tau}$ such that $\Gamma\vdash(\Pi x{:}A.B){:}\overline{\tau}\,{\scriptstyle ((k))}$. By the generation lemma (4.5), there exists a sort σ such that $\Gamma,\, x : A \vdash B : \sigma \ {\scriptstyle ((k-1))}$. So, as $\Gamma \vdash N : A$, one obtains with (cut) that $\Gamma \vdash B\langle x{:=}N\rangle : \sigma\langle x{:=}N\rangle \ {\scriptstyle ((k))}$. As $\sigma\langle x{:=}N\rangle \overset{\beta x}{\equiv} \sigma$ and, by the initialization lemma (4.3), $\Gamma \vdash \sigma : \int \ {\scriptstyle [0]}$, one can apply the (conversion) to obtain $\Gamma \vdash B\langle x{:=}N\rangle : \sigma \ {\scriptstyle ((k))}$.

- (cut): $\dfrac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle \ {\scriptstyle [k+1]}}$
  By induction hypothesis, there exists $\overline{\sigma}$ in $\overline{\mathcal{S}}$ such that $\Gamma,\, x : A \vdash B : \overline{\sigma} \ {\scriptstyle ((k))}$. Moreover, $\Gamma \vdash N : A$, so, with (cut), $\Gamma \vdash B\langle x{:=}N\rangle : \overline{\sigma}\langle x{:=}N\rangle \ {\scriptstyle ((k))}$. As $\overline{\sigma}\langle x{:=}N\rangle \overset{\beta x}{\equiv} \overline{\sigma} \in \mathcal{S}$ and, by the initialization lemma (4.3), $\Gamma \vdash \overline{\sigma} : \int \ {\scriptstyle [0]}$, one can apply the (conversion) to obtain $\Gamma \vdash B\langle x{:=}N\rangle : \overline{\sigma} \ {\scriptstyle ((k))}$.

- (xpand): $\dfrac{\Gamma \vdash M : B \ \ \Delta \vdash N : A \ \ P\langle x{:=}N\rangle \xrightarrow{x} M}{\Gamma \vdash P\langle x{:=}N\rangle : B \ {\scriptstyle [k]}}$
  Obvious by induction hypothesis.

- (conversion): $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \overline{\sigma} \qquad A \overset{\beta x}{\equiv} B}{\Gamma \vdash M : B \ {\scriptstyle ((k+1))}}$
  Obvious.

4.8 & 4.6 For these we are going to show the following stronger property: let $k \in \mathbb{N}$, Σ, Σ', Θ, Ξ, v, W, X, X', Y, Z be such that $\Sigma \vdash X : Y \ {\scriptstyle ((k))}$, $\Sigma \xrightarrow{\beta x} \Sigma'$, $X \xrightarrow{\beta x} X'$, $\Sigma = \Theta,\, v : W,\, \Xi$, $\Theta \vdash Z : W \ {\scriptstyle ((k-1))}$; then:

(*i*) $\Sigma \vdash X' : Y \ {\scriptstyle ((k))}$;

(*ii*) $\Sigma' \vdash X : Y \ {\scriptstyle ((k))}$;

(*iii*) $\Theta,\, \Xi\langle v{:=}Z\rangle \vdash X\langle v{:=}Z\rangle : Y\langle v{:=}Z\rangle \ {\scriptstyle ((k))}$.

This property shall be shown by induction on the pair $(k,\, n)$, where n is the number of inferences in the derivation of $\Sigma \vdash X : Y \ {\scriptstyle ((k))}$, with the lexicographic sorting.

The three properties are obvious for all derivations of pair $(0,\, 1)$: The derivation is bound to be (axiom):
$\dfrac{(\overline{\sigma}, \overline{\tau}) \in \overline{\mathcal{A}}}{\vdash \overline{\sigma} : \overline{\tau} \ {\scriptstyle [0]}}$ and $\overline{\sigma}$, $()$ are βx-irreductible, $v \notin \mathrm{dom}(\Sigma)$.

One assumes the properties be true for all derivations the pair of which is strictly less than $(k,\, n)$.
The last applied rule is considered:

- (rule): $\dfrac{\Gamma\vdash A{:}\rho \ \ \Gamma, x{:}A\vdash B{:}\sigma \ \ (\rho,\sigma,\tau)\in\mathcal{R}; x\notin\mathrm{dom}(\Gamma)}{\Gamma \vdash \Pi x{:}A.B : \tau \ {\scriptstyle ((k))}}$
  For (i): $X = \Pi x{:}A.B$ so $X' = \Pi x{:}A'.B'$ with either $A = A'$ and $B \xrightarrow{\beta x} B'$, or $A \xrightarrow{\beta x} A'$ et $B = B'$. In both cases, $\Gamma,\, x : A' \vdash B' : \sigma \ {\scriptstyle ((k-1))}$ (by induction hypothesis (i) in the first case, (ii) in the second). Moreover, $\Gamma \vdash A' : \rho \ {\scriptstyle ((k-1))}$ (by induction hypothesis (i) in the first case, obvious in the second). So, by application of (rule), $\Gamma \vdash X' : \tau \ {\scriptstyle ((k))}$.
  For (ii): by induction hypothesis (ii), $\Gamma' \vdash A : \rho \ {\scriptstyle ((k-1))}$ and $\Gamma',\, x : A \vdash B : \sigma \ {\scriptstyle ((k-1))}$ so, by

application of (rule), $\Gamma' \vdash \Pi x{:}A.B : \tau$ $((k))$.

For (iii): by induction hypothesis (iii), $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash A\langle v{:=}Z\rangle : \rho\langle v{:=}z\rangle$ $((k-1))$ and $\Theta, \Xi\langle v{:=}Z\rangle, x{:}A\langle v{:=}Z\rangle \vdash B{:}\langle v{:=}Z\rangle{:}\sigma\langle v{:=}z\rangle$ $((k-1))$. As by the initialization lemma (4.3) $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash \rho : \int [0]$ and $\Theta, \Xi\langle v{:=}Z\rangle, x{:}A\langle v{:=}Z\rangle \vdash \sigma{:}\int [0]$, one gets by (conversion) that $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash A\langle v{:=}Z\rangle : \rho$ $((k-1))$ et $\Theta,\ \Xi\langle v{:=}Z\rangle, x : A\langle v{:=}Z\rangle \vdash B : \langle v{:=}Z\rangle : \sigma$ $((k-1))$. So, by application of (rule), one gets $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash \Pi x{:}A\langle v{:=}Z\rangle.B\langle v{:=}Z\rangle : \tau$ $((k))$. As $\Theta \vdash Z : W$ $((k))$, by application of (xpand), $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash (\Pi x{:}A.B)\langle v{:=}Z\rangle : \tau$ $((k))$.

- (hypothesis): $\dfrac{\Gamma \vdash A : \sigma\ ((k-1)) \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma,\ x : A \vdash x : A\ ((k))}$

  For (i): $x \xrightarrow{\beta x}\!\!\!\!\!\times$ so the property is obvious.

  For (ii): $\Sigma' = \Gamma', x : A$ ou $\Gamma, x : A'$. In the first case, by induction hypothesis (ii), $\Gamma' \vdash A : \sigma$ $((k-1))$ so, by application of (hypothesis), $\Gamma', x : A \vdash x : A$ $((k))$. In the second case, by induction hypothesis (ii), $\Gamma \vdash A' : \sigma$ $((k-1))$, so with (hypothesis), $\Gamma,\ x : A' \vdash x : A'$ $((k))$. Moreover, by induction hypothesis (i), $\Gamma \vdash A' : \sigma$ $((k-1))$; so by (weakening), $\Gamma,\ x : A' \vdash A : \sigma$ $((k))$. So the (conversion) rule can be applied, as $A \overset{\beta x}{\equiv} A'$. The result is $\Gamma,\ x : A' \vdash x : A$ $((k))$.

  For (iii): If $\Xi = ()$, then $v = x$ and the property is obvious by (cut). Else $\Xi = \Xi_1,\ x : A$. Then by induction hypothesis (iii) $\Theta,\ \Xi_1\langle v{:=}Z\rangle \vdash A\langle v{:=}Z\rangle : \sigma\langle v{:=}Z\rangle$ $((k-1))$, and by (conversion), $\Theta,\ \Xi_1\langle v{:=}Z\rangle \vdash A\langle v{:=}Z\rangle : \sigma$ $((k-1))$, so by application of (hypothesis), $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash x : A\langle v{:=}Z\rangle$ $((k))$.

- (weakening): $\dfrac{\Gamma \vdash A{:}B\ ((k))\ \ \Gamma \vdash C{:}\sigma\ ((h))\ \ x \notin \mathrm{dom}(\Gamma)}{\Gamma,\ x : C \vdash A : B\ ((k))}$

  For (i): let A' be such that $A \xrightarrow{\beta x} A'$. By induction hypothesis (i) $\Gamma \vdash A' : B$ $((k))$. Moreover, $\Gamma \vdash C : \sigma$. So by (weakening) $\Gamma,\ x : C \vdash A' : B$ $((k))$.

  For (ii): let Δ be such that $\Gamma,\ x : C \xrightarrow{\beta x} \Delta$. Either $\Delta = \Gamma',\ x : C$ with $\Gamma \xrightarrow{\beta x} \Gamma'$, or $\Delta = \Gamma,\ x : C'$ with $C \xrightarrow{\beta x} C'$. In the first case, by induction hypothesis (ii) $\Gamma' \vdash A : B$ $((k))$ and $\Gamma' \vdash C : \sigma$. So by (weakening) $\Gamma',\ x : C \vdash A : B$ $((k))$. In the second case, $\Gamma \vdash A : B$ $((k))$ and by induction hypothesis (i) $\Gamma \vdash C' : \sigma$. So by (weakening), $\Gamma, x : C' \vdash A : B$ $((k))$.

- (Π - I): $\dfrac{\Gamma \vdash (\Pi x{:}A.B){:}\sigma\ ((k-1))\ \ \Gamma, x{:}A \vdash M{:}B\ ((k-1))}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B)\ ((k))}$

  For (i): $X = \lambda x{:}A.B$ so $X' = \lambda x{:}A' : B'$ with either $A = A'$ et $B \xrightarrow{\beta x} B'$ or $A \xrightarrow{\beta x} A'$ and $B = B'$. In both cases, $\Gamma \vdash \Pi x{:}A'.B' : \sigma$ $((k-1))$ by induction hypothesis (i) and $\Gamma,\ x : A' \vdash M : B$ $((k-1))$ (Obvious in the first case, by induction hypothesis (ii) in the second). So, by application of (Π - I), $\Gamma \vdash X' : \Pi x{:}A'.B$ $((k))$, which we want to prove in the first case. Moreover, in the second case, $\Gamma \vdash \Pi x{:}A.B : \sigma$ $((k-1))$ and $\Pi x{:}A'.B \overset{\beta x}{\equiv} \Pi x{:}A.B$ so one can apply (conversion). In both cases, $\Gamma \vdash X' : \Pi x{:}A.B$ $((k))$.

  For (ii): the property is obvious by induction hypothesis (ii) – as $x \notin \mathrm{dom}(\Gamma)$ – and use of (Π-I). For (iii): by induction hypothesis (iii), $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash (\Pi x{:}A.B)\langle v{:=}Z\rangle : \sigma\langle v{:=}Z\rangle$ $((k-1))$ and $\Theta,\ \Xi\langle v{:=}Z\rangle,\ x : A\langle v{:=}Z\rangle \vdash M\langle v{:=}Z\rangle : B\langle v{:=}Z\rangle$ $((k-1))$. Applying the induction hypothesis (i) and (conversion) to the first judgement yields the judgement $\Theta, \Xi\langle v{:=}Z\rangle \vdash (\Pi x{:}A\langle v{:=}Z\rangle.B\langle v{:=}Z\rangle){:}\sigma\langle v{:=}Z\rangle$ $((k-1))$. So by (Π - I), one obtains that $\Theta, \Xi\langle v{:=}Z\rangle \vdash \lambda x{:}A\langle v{:=}Z\rangle.M\langle v{:=}Z\rangle : (\Pi x{:}A\langle v{:=}Z\rangle.B\langle v{:=}Z\rangle)$ $((k))$. The use of rules (xpand) and (conversion), as by (conversion) $\Theta, \Xi\langle v{:=}Z\rangle \vdash (\Pi x{:}A.B)\langle v{:=}Z\rangle : \sigma$ $((k-1))$, yields the expected result $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash (\lambda x{:}A.M)\langle v{:=}Z\rangle : (\Pi x{:}A.B)\langle v{:=}Z\rangle$ $((k))$.

- (Π - E): $\dfrac{\Gamma \vdash M{:}(\Pi x{:}A.B)\ _{((k-1))} \quad \Gamma \vdash N{:}A\ _{((k-1))}}{\Gamma \vdash MN : B\langle x{:=}N\rangle\ _{((k))}}$

  For (i): $X{=}MN$ so X' may take the following forms:

  (a) $X' = M'N$ with $M \xrightarrow{\beta x} M'$;
  (b) $X' = MN'$ with $N \xrightarrow{\beta x} N'$;
  (c) $X' = Q\langle x{:=}N\rangle$ where $M = \lambda x{:}P.Q$.

  Case (a) and (b) are analogous to (Π - I).
  In case (c), by the generation lemma (4.5), as $\Gamma \vdash \lambda x{:}P.Q : (\Pi x{:}A.B)\ _{((k-1))}$, there exist τ, R such that $\Gamma \vdash (\Pi x{:}P.R) : \tau\ _{((k-2))}$, $\Gamma,\ x : P \vdash Q : R\ _{((k-2))}$ and $(\Pi x{:}P.R) \overset{\beta x}{\equiv} (\Pi x{:}A.B)$. This last assertion implies $A \overset{\beta x}{\equiv} P$ and $B \overset{\beta x}{\equiv} R$. $\Gamma \vdash (\Pi x{:}P.R) : \tau\ _{((k-2))}$, so by the generation lemma (4.5), there exist ρ and σ such that $\Gamma \vdash P : \rho\ _{((k-3))}$ and $\Gamma,\ x : P \vdash R : \sigma\ _{((k-3))}$. As $A \overset{\beta x}{\equiv} P$, this is enough to apply (conversion) to $\Gamma \vdash N : A$, ce qui donne $\Gamma \vdash N : P\ _{((k-1))}$. $\Gamma,\ x : P \vdash Q : R\ _{((k-2))}$ and $\Gamma \vdash N : P\ _{((k-1))}$ so with (cut) $\Gamma \vdash Q\langle x{:=}N\rangle : R\langle x{:=}N\rangle\ _{((k))}$. Moreover, $\Gamma,\ x : P \vdash R : \sigma\ _{((k-3))}$ and $\Gamma \vdash N : P\ _{((k-1))}$ so by (cut) and (conversion), $\Gamma \vdash R\langle x{:=}N\rangle : \sigma\ _{((k))}$. As $B\langle x{:=}N\rangle \overset{\beta x}{\equiv} R\langle x{:=}N\rangle$ (because $B \overset{\beta x}{\equiv} R$), this allows to apply (conversion) to $\Gamma \vdash Q\langle x{:=}N\rangle : R\langle x{:=}N\rangle\ _{((k))}$, and hence $\Gamma \vdash Q\langle x{:=}N\rangle : B\langle x{:=}N\rangle\ _{((k))}$.
  For (ii): the property is obvious by induction hypothesis (ii) and use of (Π - I).
  For (iii): by induction hypothesis (iii), $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash M\langle v{:=}Z\rangle : (\Pi x{:}A.B)\langle v{:=}Z\rangle\ _{((k-1))}$ and $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash N\langle v{:=}Z\rangle : A\langle v{:=}Z\rangle\ _{((k-1))}$. Moreover, $\Gamma \vdash M : (\Pi x{:}A.B)\ _{((k-1))}$ so, by the type correction theorem (4.7), there exists τ such that $\Gamma \vdash (\Pi x{:}A.B) : \tau\ _{((k-1))}$ – this type cannot be of type $\int$ because it is no sort. So by the generation lemma (4.5), there exist a rule $(\rho,\ \sigma,\ \tau)$ such that $\Gamma \vdash A : \rho\ _{((k-2))}$, $\Gamma,\ x : A \vdash B : \sigma\ _{((k-2))}$. By induction hypothesis (iii) and (conversion), one gets that $\Theta, \Xi\langle v{:=}Z\rangle \vdash A\langle v{:=}Z\rangle{:}\rho\ _{((k-2))}$ and $\Theta, \Xi\langle v{:=}Z\rangle, x{:}A\langle v{:=}Z\rangle \vdash B\langle v{:=}Z\rangle{:}\sigma\ _{((k-2))}$ and so by the application of (rule) one obtains that $\Theta, \Xi\langle v{:=}Z\rangle \vdash \Pi x{:}A\langle v{:=}Z\rangle.B\langle v{:=}Z\rangle{:}\tau\ _{((k-1))}$. So using (conversion) on the first of the two former results; hence $\Theta, \Xi\langle v{:=}Z\rangle \vdash M\langle v{:=}Z\rangle{:}\Pi x{:}A\langle v{:=}Z\rangle.B\langle v{:=}Z\rangle\ _{((k-1))}$. Moreover using (Π - E) on the second of those two result yields the expected judgement $\Theta, \Xi\langle v{:=}Z\rangle \vdash M\langle v{:=}Z\rangle N\langle v{:=}Z\rangle{:}B\langle v{:=}Z\rangle\ _{((k))}$.

- (cut): $\dfrac{\Gamma,\ x : A \vdash M : B\ _{((k-1))} \qquad \Gamma \vdash N : A\ _{((k-2))}}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle\ _{((k))}}$

  For (i): $X{=}M\langle x{:=}N\rangle$ so X' may take several forms:

  (a) $X'{=}M'\langle x{:=}N\rangle$ with $M \xrightarrow{\beta x} M'$;
  (b) $X'{=}M\langle x{:=}N'\rangle$ with $N \xrightarrow{\beta x} N'$;
  (c) $X'{=}\Pi y{:}P\langle x{:=}N\rangle.Q\langle x{:=}N\rangle$ where $M{=}\Pi y{:}P.Q$;
  (d) $X'{=}\lambda y{:}P\langle x{:=}N\rangle.Q\langle x{:=}N\rangle$ where $M{=}\lambda y{:}P.Q$;
  (e) $X'{=}P\langle x{:=}N\rangle\, Q\langle x{:=}N\rangle$ where $M{=}PQ$;
  (f) $X'{=}N$ where $M{=}x$;
  (g) $X'{=}y$ where $M{=}y{\neq}x$.

  (We have here made use of the (var) rule instead of (gc): in fact, the theorems obtained with either rules are equivalent and the use of (gc) would make the demonstration much more technical but unchanged in the idea.)

  Case (a) and (b) are analogous to (Π - I).

In case (c), by the generation lemma (4.5), as $\Gamma,\ x : A \vdash \Pi y{:}P.Q : B$ $((k{-}1))$, there exist a rule $(\rho,\ \sigma,\ \tau)$ such that $\Gamma,\ x : A, \vdash P : \rho$ $((k{-}2))$, $\Gamma,\ x : A,\ y : P \vdash Q : \sigma$ $((k{-}2))$ and $B \overset{\beta x}{\equiv} \tau$. $\Gamma,\ x : A, \vdash P : \rho$ $((k{-}2))$ and $\Gamma \vdash N : A$ $((k{-}2))$ so by (cut) $\Gamma \vdash P\langle x{:=}N\rangle : \rho\langle x{:=}N\rangle$ $((k))$ and by (conversion) – as $\Gamma \vdash \rho : \int$ $[0]$ by the initialization lemma (4.3) – $\Gamma \vdash P\langle x{:=}N\rangle : \rho$ $((k))$.

In case (d), by the generation lemma (4.5), as $\Gamma, x{:}A \vdash \lambda y{:}P.Q : B$ $((k{-}1))$, there exist a σ and a term $R \in LTx$ such that $\Gamma, x{:}A \vdash (\Pi y{:}P.R) : \sigma$ $((k{-}2))$, $\Gamma, x{:}A, y{:}P \vdash Q : R$ $((k{-}2))$ and $B \overset{\beta x}{\equiv} \Pi y{:}P.R$. Now $\Gamma \vdash N : A$ $((k{-}2))$ so by induction hypothesis (iii), $\Gamma \vdash (\Pi y{:}P.R)\langle x{:=}N\rangle : \sigma$ $((k{-}1))$ et $\Gamma, y : P\langle x{:=}N\rangle \vdash Q\langle x{:=}N\rangle : R\langle x{:=}N\rangle$ $((k{-}1))$. The application of induction hypothesis (i) to the first of these results yields the judgement $\Gamma \vdash \Pi y{:}P\langle x{:=}N\rangle.R\langle x{:=}N\rangle : \sigma$ $((k{-}1))$. With these two judgements, one gets by (Π - I) $\Gamma \vdash X' : \Pi y{:}P\langle x{:=}N\rangle.R\langle x{:=}N\rangle$ $((k))$, and, by (conversion) – as by the type correction theorem (4.7) $\Gamma \vdash B\langle x{:=}N\rangle : \overline{v}$ $((k))$ – $\Gamma \vdash X' : B\langle x{:=}N\rangle$ $((k))$.

In case (e), by the generation lemma (4.5), as $\Gamma, x{:}A \vdash PQ : B$ $((k{-}1))$, there exist R and S such that $\Gamma, x{:}A \vdash P : (\Pi y{:}R.S)$ $((k{-}2))$ and $\Gamma, x{:}A \vdash Q : R$ $((k{-}2))$ and $B \overset{\beta x}{\equiv} S\langle y{:=}Q\rangle$. In order to respect Barendregt's convention, one chooses for y a fresh variable, in particular y must neither belong to $av(N)$ nor to $av(N)$. As $\Gamma \vdash N{:}A$ $((k{-}2))$, induction hypothesis (iii) yields $\Gamma \vdash P\langle x{:=}N\rangle{:}(\Pi y{:}R.S)\langle x{:=}N\rangle$ $((k{-}1))$ and $\Gamma \vdash Q\langle x{:=}N\rangle{:}R\langle x{:=}N\rangle$ $((k{-}1))$. By the type correction theorem (4.7), $\Gamma \vdash (\Pi y{:}R.S)\langle x{:=}N\rangle{:}\overline{\sigma}$ $((k{-}1))$ so by induction hypothesis (i), $\Gamma \vdash \Pi y{:}R\langle x{:=}N\rangle.S\langle x{:=}N\rangle : \overline{\sigma}$ $((k{-}1))$. Therefore with rule (conversion), $\Gamma \vdash P\langle x{:=}N\rangle : \Pi y{:}R\langle x{:=}N\rangle.S\langle x{:=}N\rangle$ $((k{-}1))$. And so applying rule (Π - E) yields $\Gamma \vdash P\langle x{:=}N\rangle Q\langle x{:=}N\rangle : S\langle x{:=}N\rangle\langle y{:=}Q\langle x{:=}N\rangle\rangle$ $((k))$. Besides, as $y \notin av(N) \cup av(Q)$ because of Barendregt's convention, one can apply the compose lemma (4.2) to get $S\langle x{:=}N\rangle\langle y{:=}Q\langle x{:=}N\rangle\rangle \overset{\beta x}{\equiv} S\langle y{:=}Q\rangle\langle x{:=}N\rangle \overset{\beta x}{\equiv} B\langle x{:=}N\rangle$. As by the type correction theorem (4.7), $\Gamma \vdash B\langle x{:=}N\rangle : \overline{\tau}$ $((k))$, one can apply (conversion), which yields the expected result: $\Gamma \vdash P\langle x{:=}N\rangle Q\langle x{:=}N\rangle : B\langle x{:=}N\rangle$ $((k))$.

In case (f), $M = x$ so $(x : B) \in \Gamma, x : A$. So $A = B$. Moreover, $av(A) \subseteq \mathrm{dom}(\Gamma)$, so $x \notin av(A)$, so $B\langle x{:=}N\rangle \overset{\beta x}{\equiv} B = A$. As $\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle$ $((k))$, by the type correction theorem (4.7), (conversion) can be applied with $B\langle x{:=}N\rangle \overset{\beta x}{\equiv} A$ to $\Gamma \vdash N : A$ $((k{-}1))$, and hence $\Gamma \vdash N : B\langle x{:=}N\rangle$ $((k))$.

In case (g), $M = y$ so $(y : B) \in \Gamma, x : A$ so $(y : B) \in \Gamma$. So $av(B) \subseteq \mathrm{dom}(\Gamma)$ so $x \notin av(B)$ so $B \overset{\beta x}{\equiv} B\langle x{:=}N\rangle$. Besides, by the type correction theorem (4.7), $\Gamma \vdash B : \overline{\sigma}$ $((k{-}1))$. So, applying (conversion) to $\Gamma \vdash M : B$ $((k{-}1))$, $\Gamma \vdash M : B\langle x{:=}N\rangle$ $((k{-}1))$.

For (ii): the property is obvious by induction hypothesis (ii).

For (iii): by induction hypothesis (iii), $\Theta, \Xi\langle v{:=}Z\rangle, x{:}A\langle v{:=}Z\rangle \vdash M\langle v{:=}Z\rangle{:}B\langle v{:=}Z\rangle$ $((k{-}1))$ and $\Theta, \Xi\langle v{:=}Z\rangle \vdash N\langle v{:=}Z\rangle{:}A\langle v{:=}Z\rangle$ $((k{-}2))$. Therefore using induction hypothesis (iii), one gets $\Theta,\ \Xi\langle v{:=}Z\rangle \vdash M\langle v{:=}Z\rangle\langle x{:=}N\langle v{:=}Z\rangle\rangle : B\langle v{:=}Z\rangle\langle x{:=}N\langle v{:=}Z\rangle\rangle$ $((k{-}1))$.
By the compose lemma (4.2), $M\langle v{:=}Z\rangle\langle x{:=}N\langle v{:=}Z\rangle\rangle \overset{x}{\equiv} M\langle x{:=}N\rangle\langle v{:=}Z\rangle$ and hence $\Theta, \Xi\langle v{:=}Z\rangle \vdash M\langle x{:=}N\rangle\langle v{:=}Z\rangle{:}B\langle v{:=}Z\rangle\langle x{:=}N\langle v{:=}Z\rangle\rangle$ $((k{-}1))$ by (xpand) and induction hypothesis (i). By the compose lemma (4.2), $B\langle v{:=}Z\rangle\langle x{:=}N\langle v{:=}Z\rangle\rangle \overset{\beta x}{\equiv} B\langle x{:=}N\rangle\langle v{:=}Z\rangle$. Moreover, by the type correction theorem (4.7), $\Gamma, x{:}A \vdash B{:}\overline{\tau}$ $((k{-}1))$ so by use of induction hypothesis (iii) twice and (conversion), $\Theta, \Xi\langle v{:=}Z\rangle \vdash B\langle x{:=}N\rangle\langle v{:=}Z\rangle : \overline{\tau}$ $((k{-}1))$. And so, by (conversion), $\Theta, \Xi\langle v{:=}Z\rangle \vdash M\langle x{:=}N\rangle\langle v{:=}Z\rangle : B\langle x{:=}N\rangle\langle v{:=}Z\rangle$ $((k))$.

- (xpand): $\dfrac{\Gamma\vdash M{:}B_{\,((k-1))}\ \Delta\vdash N{:}A_{\,((k-2))}\ P\langle x{:=}N\rangle\xrightarrow{\mathsf{x}}M}{\Gamma\vdash P\langle x{:=}N\rangle : B_{\,((k))}}$
  For (i): $X' = M$ and the property is obvious.
  For (ii): obvious by induction hypothesis (ii).
  For (iii): obvious by induction hypothesis (iii).
- (conversion): $\dfrac{\Gamma\vdash M{:}A_{\,((k-1))}\ \Gamma\vdash B{:}\overline{\sigma}_{\,((k-2))}\ A\overset{\beta\mathsf{x}}{\equiv}B}{\Gamma\vdash M : B_{\,((k))}}$
  All properties are obvious by induction hypothesis.

V - Strong normalization for the explicit λ-cube

5.1 Obvious with the definition of the cube.

5.2
(i) By easy induction.
(ii) By easy induction.
(iii), (iv): By simultaneous induction upon the derivation of $\Gamma\vdash A : B$:

- (axiom): $\dfrac{(\sigma,\,\tau)\ \in\ \mathcal{A}}{\vdash\sigma : \tau}$
  Then $\sigma = *$ and $\tau = \Box$ and (iii) is obvious. Besides (iv) is vacuously true.
- (rule): $\dfrac{\Gamma\vdash A{:}\rho\ \Gamma, x{:}A\vdash B{:}\sigma\ (\rho,\sigma,\tau)\in\mathcal{R}; x\notin\mathrm{dom}(\Gamma)}{\Gamma\vdash\Pi x{:}A.B : \tau}$
  We are in the cube, so $\tau = \sigma$. By induction hypothesis, $\delta(\tau) = \delta(\sigma) = \delta(B) + 1 = \delta(\Pi x{:}A.B) + 1$.
  Besides (iv) is vacuously true.
- (hypothesis): $\dfrac{\Gamma\vdash A : \sigma \qquad x\notin\mathrm{dom}(\Gamma)}{\Gamma,\, x : A\vdash x : A}$
  $\delta({}^{\sigma}x) + 2 = \delta(\sigma) = \delta(A) + 1$ by induction hypothesis.
  Besides (iv) is vacuously true.
- (weakening): $\dfrac{\Gamma\vdash A : B\ \ \Gamma\vdash C : \sigma\ \ x\notin\mathrm{dom}(\Gamma)}{\Gamma,\, x : C\vdash A : B}$
  (iii) is clear – similarly to (hypothesis).
  (iv) is obvious by induction hypothesis.
- (Π - I): $\dfrac{\Gamma\vdash(\Pi x{:}A.B){:}\sigma\ \ \Gamma, x{:}A\vdash M{:}B\ \ x\notin\mathrm{dom}(\Gamma)}{\Gamma\vdash\lambda x{:}A.M : (\Pi x{:}A.B)}$
  By induction hypothesis, $\delta(\lambda x{:}A.M) + 1 = \delta(M) + 1 = \delta(B) = \delta(\Pi x{:}A.B)$.
  For (v) the subject does not have an appropriate form.
- (Π - E): $\dfrac{\Gamma\vdash M : (\Pi x{:}A.B) \qquad \Gamma\vdash N : A}{\Gamma\vdash MN : B\langle x{:=}N\rangle}$
  By induction hypothesis, $\delta(MN) + 1 = \delta(M) + 1 = \delta(\Pi x{:}A.B) = \delta(B) = \delta(B\langle x{:=}N\rangle)$.
  For (iv), by induction hypothesis (iii), substatement $x{:}A$ and statement $N{:}A$ are hereditarily compatible, so $\delta(x) + 1 = \delta(A)$ and $\delta(N) + 1 = \delta(A)$, so $\delta(x) = \delta(N)$.

- (cut): $\dfrac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle}$
  $\delta(M\langle x{:=}N\rangle) + 1 = \delta(M) + 1 = \delta(B) = \delta(B\langle x{:=}N\rangle)$ by induction hypothesis.

  Besides (iv) is vacuously true.

- (xpand): $\dfrac{\Gamma \vdash M : B \quad \Delta \vdash N : A \quad P\langle x{:=}N\rangle \xrightarrow{\mathsf{x}} M}{\Gamma \vdash P\langle x{:=}N\rangle : B}$
  (iii) is obvious by induction hypothesis (iv).

  Besides (iv) is vacuously true.

- (conversion): $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \overline{\sigma} \qquad A \overset{\beta\mathsf{x}}{\equiv} B}{\Gamma \vdash M : B}$
  (iii) is obvious by induction hypothesis (iv). (iv) is obvious by induction hypothesis.

(v) by easy induction upon the derivation of $\Gamma \vdash_{\lambda\mathsf{C}\mathsf{x}} B\langle x{:=}A\rangle : T$: somewhere in Γ there is an $x : U$ and somewhere in the derivation tree there is an $A : U$ – because as $x \in av(B)$ there must be a (cut) – and one can then conclude by (iii).

5.3 (i) by easy induction upon the derivation of $\Gamma \vdash_{\lambda\mathsf{C}\mathsf{x}} A : U$.
(ii) by easy induction upon relation $\xrightarrow{\beta\mathsf{x}}$.

5.4 If $\sigma = *$, then $Þ_M = þM$ and the property clearly holds. If $\sigma = \Box$, then the result follows by induction upon M.

5.5 (i) by easy induction upon the relation $\xrightarrow{\beta\mathsf{x}}$.
(ii) by induction upon the derivation of $\Gamma \vdash_{\lambda\mathsf{C}\mathsf{x}} A : B$:

- (axiom): $\dfrac{(\overline{\sigma}, \overline{\tau}) \in \mathcal{A}}{\vdash \overline{\sigma} : \overline{\tau}}$
  True because $\sharp\sigma = \emptyset : * = \flat\tau$ belongs to Θ.

- (rule): $\dfrac{\Gamma \vdash A{:}\rho \; \Gamma, x{:}A \vdash B{:}\sigma \; (\rho,\sigma,\tau) \in \mathcal{R}; x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \Pi x{:}A.B : \tau}$
  True by induction hypothesis for all possible rules.

- (hypothesis): $\dfrac{\Gamma \vdash A : \sigma \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \vdash x : A}$
  Obvious by induction hypothesis (necessarily $x =^{\Box} x$ because of $\sharp$ is not defined on terms of degree 0).

- (weakening): $\dfrac{\Gamma \vdash A : B \;\; \Gamma \vdash C : \sigma \;\; x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : C \vdash A : B}$
  By induction hypothesis, $\sharp(\Gamma) \vdash \sharp A : \flat B$ and $\sharp(\Gamma) \vdash \sharp C : *$.

  If $x =^{*} x$, the property is obvious with (weakening). If $x =^{\Box} x$, the property follows from two uses of (weakening).

- (Π - I): $\dfrac{\Gamma \vdash (\Pi x{:}A.B){:}\sigma \;\; \Gamma, x{:}A \vdash M{:}B \;\; x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B)}$
  By assumption of the theorem, $\sigma = \Box$.

  1st case: $\delta(A) = 2$. Then by induction hypothesis: $\tau\Gamma \vdash (\Pi x{:}\flat A.\sharp A \to \sharp B) : *$ and

$\tau\Gamma,\ ^{\square}x{:}\flat A,\ ^{*}x{:}\sharp A \vdash \sharp M : \flat B$. $^{*}x$ occurs neither in $\flat B$ nor in $\sharp M$ so, using the former two equation with the substitution lemma (4.6) using $Þ_{\sharp A}$ and (conversion) and the subject reduction thorem, one gets: $\sharp\Gamma,\ ^{\square}x{:}\flat A \vdash \sharp M : \flat B$, and hence $\sharp\Gamma \vdash (\lambda x{:}\flat A.\sharp M) : (\Pi x{:}\flat A.\flat B) = \flat(\Pi x{:}A.B)$ since as $\flat B$ is closed, $(\Pi x{:}\flat A.\flat B) = \flat A \to \flat B$.
2nd case: $\delta(A) = 1$. Similarly.

- (Π - E): $\dfrac{\Gamma \vdash M : (\Pi x{:}A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B\langle x{:=}N\rangle}$
If $\delta(N) = 0$ the proposition is obvious by induction hypothesis.
$\delta(N) = \delta(A) - 1 = \delta(\sigma) - 2$ where $\Gamma \vdash A : \sigma$ by the generation lemma (4.5). So if $\delta(N) \neq 0$, $\delta(N) = 1$ and $\sigma = \square$. Then the induction hypothesis can be applied to $\Gamma \vdash N : A$ and the result follows from the use of (Π - E).

- (cut): $\dfrac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle}$
1st case: $x = {}^{*}x$. By induction hypothesis, $\sharp\Gamma,\ ^{*}x{:}\sharp A \vdash \sharp M : \flat B$. $^{*}x$ occurs neither in $\flat B$ nor in $\sharp M$ so, using the former equation with the substitution lemma (4.6) using $Þ_{\sharp A}$ and (conversion) and the subject reduction thorem, one gets: $\sharp\Gamma \vdash \sharp M : \flat B$.
2nd case: $x = {}^{\square}x$. By the type correction theorem (4.7) $\Gamma \vdash A : \sigma$ and by considerations of degree $\sigma = \square$ or $A \overset{\beta\mathsf{x}}{\equiv} \square$ so, by induction hypothesis, $\sharp\Gamma \vdash \sharp N : \flat A$. Moreover, by induction hypothesis, $\sharp\Gamma,\ ^{\square}x{:}\flat A,\ ^{*}x{:}\sharp A \vdash \sharp M : \flat B$. $^{*}x$ occurs neither in $\flat B$ nor in $\sharp M$ so, using the former equation with the substitution lemma (4.6) using $Þ_{\sharp A}$ and (conversion) and the subject reduction thorem, one gets: $\sharp\Gamma,\ ^{\square}x{:}\flat A, \vdash \sharp M : \flat B$ and by (cut) it follows that $\sharp\Gamma \vdash \sharp M\langle x{:=}\sharp N\rangle : \flat B$.

- ($\mathsf{x}$pand): $\dfrac{\Gamma \vdash M : B \quad \Delta \vdash N : A \quad P\langle x{:=}N\rangle \xrightarrow{\mathsf{x}} M}{\Gamma \vdash P\langle x{:=}N\rangle : B}$
One discriminates upon the form of P.
In all cases it is easy to show that $\sharp(P\langle x{:=}N\rangle) \xrightarrow{\mathsf{x}}\!\!\!\!\rightarrow \sharp M$, and the result follows.

- (conversion): $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \overline{\sigma} \qquad A \overset{\beta\mathsf{x}}{\equiv} B}{\Gamma \vdash M : B}$
Obvious because $\flat A = \flat B$.

5.6 (i) By induction upon the generation of $A \xrightarrow{\beta\mathsf{x}} B$.
The only non-obvious case is the case of the (B) rule: $(\lambda x{:}A.M)N \xrightarrow{\beta\mathsf{x}} M\langle x{:=}N\rangle$.
1st case: $x = {}^{*}x$. Then:

$$\begin{aligned}\llbracket(\lambda x{:}A.M)N\rrbracket &= (\lambda^{*}ð : ø.\lambda^{*}x{:}\sharp A.\llbracket M\rrbracket)\llbracket A\rrbracket\llbracket N\rrbracket \\ &\xrightarrow{\beta\mathsf{x}}{}^{+} \llbracket M\rrbracket\langle^{*}x{:=}\llbracket N\rrbracket\rangle = \llbracket M\langle^{*}x{:=}N\rangle\rrbracket\end{aligned}$$

2nd case: $x = {}^{\square}x$. Then:

$$\begin{aligned}\llbracket(\lambda^{\square}x{:}A.M)N\rrbracket &= (\lambda^{*}ð : ø.\lambda^{\square}x{:}\flat A.\lambda^{*}x{:}\sharp A.\llbracket M\rrbracket)\llbracket A\rrbracket\sharp N\llbracket N\rrbracket \\ &\xrightarrow{\beta\mathsf{x}}{}^{+} \llbracket M\rrbracket\langle^{\square}x{:=}\sharp N\rangle\langle^{*}x{:=}\llbracket N\rrbracket\rangle \quad \text{(since } {}^{*}ð \notin fv(\llbracket M\rrbracket)\text{)} \\ &= \llbracket M\langle x^{\square}{:=}N\rangle\rrbracket\end{aligned}$$

(ii) By induction upon the derivation of $\Gamma \vdash_{\lambda\mathsf{Cx}} A : B$.

- (axiom): $\dfrac{(\overline{\sigma}, \overline{\tau}) \in \overline{\mathcal{A}}}{\vdash \overline{\sigma} : \overline{\tau}}$
$\delta\sigma \leqslant 2$ so $(\sigma, \tau) = (*, \square)$ so * $(\llbracket\sigma\rrbracket, \sharp\tau) = (Þ_{ø}, ø)$. By lemma 5.4, $\sharp[] = \Theta \vdash_{\lambda\omega\mathsf{x}} Þ_{ø} : ø$.

- (rule): $\dfrac{\Gamma\vdash A{:}\rho\ \Gamma, x{:}A\vdash B{:}\sigma\ (\rho,\sigma,\tau)\in\mathcal{R}; x\notin\mathrm{dom}(\Gamma)}{\Gamma\vdash \Pi x{:}A.B : \tau}$
  We are in the cube so $\tau=\sigma$.
  1st case: $\rho = *$. Then $x =^* x$ and $\sharp(\Gamma,\ x{:}A) = \sharp\Gamma,\ x{:}\sharp A$.
  So with the substitution lemma (4.6) $\sharp\Gamma\vdash_{\lambda\omega\mathtt{x}}[\![B]\!]\langle x{:=}Þ_{\sharp A}\rangle : ø$, so by (Π - E) twice: $\sharp\Gamma\vdash_{\lambda\omega\mathtt{x}} Þ_{ø\to ø\to ø}[\![A]\!]([\![B]\!]\langle x{:=}Þ_{\sharp A}\rangle) : ø$.
  2nd case: $\rho = \square$. Then $x =^{\square} x$ and $\sharp(\Gamma,\ x:A) = \sharp\Gamma,\ {}^{\square}x : \flat A,\ {}^{*}x : \sharp A$.
  So with the substitution lemma (4.6) $\sharp\Gamma\vdash_{\lambda\omega\mathtt{x}}[\![B]\!]\langle {}^{\square}x{:=}Þ_{\flat A}\rangle\langle {}^{*}x{:=}Þ_{\sharp A}\rangle : ø$, hence by applying rule (Π - E) twice: $\sharp\Gamma\vdash_{\lambda\omega\mathtt{x}} Þ_{ø\to ø\to ø}[\![A]\!]([\![B]\!]\langle {}^{\square}x{:=}Þ_{\flat A}\rangle\langle {}^{*}x{:=}Þ_{\sharp A}\rangle) : ø$.

- (hypothesis): $\dfrac{\Gamma\vdash A:\sigma \qquad x\notin \mathrm{dom}(\Gamma)}{\Gamma,\ x:A\vdash x:A}$
  Obvious by lemma 5.5.

- (weakening): $\dfrac{\Gamma\vdash A:B\ \ \Gamma\vdash C:\sigma\ \ x\notin \mathrm{dom}(\Gamma)}{\Gamma,\ x:C\vdash A:B}$
  Similar to (hypothesis).

- (Π - I): $\dfrac{\Gamma\vdash(\Pi x{:}A.B){:}\sigma\ \ \Gamma, x{:}A\vdash M{:}B\ \ x\notin\mathrm{dom}(\Gamma)}{\Gamma\vdash \lambda x{:}A.M : (\Pi x{:}A.B)}$
  By the generation lemma (4.5) there is some such sort τ that $\Gamma\vdash_{\lambda\mathsf{Cx}} A : \tau$ and $\Gamma, x : A\vdash_{\lambda\mathsf{Cx}} B : \sigma$. By induction hypothesis, one has: $\sharp(\Gamma, x : A)\vdash_{\lambda\omega\mathtt{x}}[\![M]\!] : \sharp B$ and $\sharp(\Gamma)\vdash_{\lambda\omega\mathtt{x}}[\![A]\!] : ø$. By lemma 5.5, one has $\sharp(\Gamma)\vdash_{\lambda\omega\mathtt{x}}\sharp A : *$. and $\sharp(\Gamma,\ x : A)\vdash_{\lambda\omega\mathtt{x}}\sharp B : *$. Then one can treat two cases: $\tau = *$ and $\tau=\square$. By the use of appropriate inference rules, one obtains in both cases that $\sharp\Gamma\vdash_{\lambda\omega\mathtt{x}}[\![\lambda x{:}A.M]\!] : \sharp(\Pi x{:}A.B)$.

- (Π - E): $\dfrac{\Gamma\vdash M : (\Pi x{:}A.B) \qquad \Gamma\vdash N : A}{\Gamma\vdash MN : B\langle x{:=}N\rangle}$
  By induction hypothesis, $\sharp\Gamma\vdash[\![M]\!] : \sharp(\Pi x{:}A.B)$ and $\sharp\Gamma\vdash[\![N]\!] : \sharp A$.
  1st case: $\delta(N)+1=\delta(A)=2$ (and so by degree argument $x =^{\square} x$). Then the two equations above become $\sharp\Gamma\vdash[\![M]\!] : \Pi x{:}\flat A.\sharp A{\to}\sharp B$ and $\sharp\Gamma\vdash[\![N]\!] : \sharp A$. Moreover $\sharp\Gamma\vdash\sharp N : \flat A$. So with (Π - E) twice: $\sharp\Gamma\vdash[\![M]\!]\sharp N[\![N]\!] : \sharp B\langle x{:=}\sharp N\rangle\langle {}^{*}ð{:=}[\![N]\!]\rangle$, and with (conversion) $\sharp\Gamma\vdash[\![M]\!]\sharp N[\![N]\!] : \sharp B\langle x{:=}\sharp N\rangle$, since ${}^{*}ð \notin av(\sharp B\langle x{:=}\sharp N\rangle)$.
  2nd case: $\delta(N)+1=\delta(A)=1$. Then the first of the two equations above becomes $\sharp\Gamma\vdash[\![M]\!] : \Pi x{:}\sharp A.trucB$. Moreover $\sharp\Gamma\vdash\sharp N : \flat A$. And one can conclude with a similar argument.

- (cut): $\dfrac{\Gamma,\ x:A\vdash M:B \qquad \Gamma\vdash N:A}{\Gamma\vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle}$
  1st case: $x =^* x$. By induction hypothesis $\sharp\Gamma,\ {}^{*}x : \sharp A\vdash[\![M]\!] : \sharp B$ and $\sharp\Gamma\vdash[\![N]\!] : \sharp A$ so with (cut) $\sharp\Gamma\vdash[\![M]\!]\langle {}^{*}x{:=}[\![N]\!]\rangle : \sharp B\langle {}^{*}x{:=}\sharp N\rangle$, and by (conversion), since ${}^{*}x\notin av(\sharp B)$, $\sharp\Gamma\vdash[\![M]\!]\langle {}^{*}x{:=}[\![N]\!]\rangle : \sharp B$.
  2nd case: $x =^{\square} x$. By induction hypothesis $\sharp\Gamma,\ {}^{\square}x : \flat A,\ {}^{*}x : \sharp A\vdash[\![M]\!] : \sharp B$ and $\sharp\Gamma\vdash[\![N]\!] : \sharp A$. As by lemma 5.5 – as for degree reason A is not typable by sort $*$ –, $\sharp\Gamma\vdash\sharp N : \flat A$, with (cut) twice $\sharp\Gamma\vdash[\![M]\!]\langle {}^{*}x{:=}[\![N]\!]\rangle\langle {}^{\square}x{:=}\sharp N\rangle : \sharp B\langle {}^{*}x{:=}\sharp N\rangle\langle {}^{\square}x{:=}\sharp N\rangle$, and as ${}^{*}x\notin av(\sharp B)$, by (conversion) $\sharp\Gamma\vdash[\![M]\!]\langle {}^{*}x{:=}[\![N]\!]\rangle\langle {}^{\square}x{:=}\sharp N\rangle : \sharp B\langle {}^{\square}x{:=}\sharp N\rangle$.

- (xpand): $\dfrac{\Gamma\vdash M:B\ \ \ \Delta\vdash N:A\ \ \ P\langle x{:=}N\rangle\xrightarrow{\mathtt{x}}M}{\Gamma\vdash P\langle x{:=}N\rangle : B}$
  One discriminates upon the form of P.

In all cases it is easy to show that $[\![P\langle x{:=}N\rangle]\!] \xrightarrow{\mathsf{x}}\!\!\!\!\rightarrow [\![M]\!]$, and the result follows.

- (conversion): $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \overline{\sigma} \qquad A \overset{\beta\mathsf{x}}{\equiv} B}{\Gamma \vdash M : B}$
  Obvious because $[\![A]\!] \overset{\beta\mathsf{x}}{\equiv} [\![B]\!]$.

5.7 With the type correction theorem (4.7) it is enough to show that if $\Gamma \vdash \lambda\mathsf{C}\mathsf{x}M : T$ then M is strongly normalizing. Consider a reduction starting at M: $M \xrightarrow{\beta\mathsf{x}} M' \xrightarrow{\beta\mathsf{x}} M'' \xrightarrow{\beta\mathsf{x}} \ldots$. We have $\sharp\,\Gamma \vdash \lambda\omega\mathsf{x}[\![M]\!] : \sharp T$ and $[\![M]\!] \xrightarrow{\beta\mathsf{x}}{}^{+} [\![M']\!] \xrightarrow{\beta\mathsf{x}} [\![M'']\!] \xrightarrow{\beta\mathsf{x}} \ldots$. Assuming that $\lambda\omega\mathsf{x}$ is strongly normalizing, this sequence must be finite, and hence the reduction of M also.

5.8 A proof of this is given in [17] (in French; also translated in English as Appendix B, available at http://perso.ens-lyon.fr/romain.kervarc/Articles/LICS04).

5.9 By induction upon M having a $\beta\mathsf{x}$-redex:

- $M{=}\Pi^{\sigma}x{:}A.B$. Then $M'{=}\Pi^{\sigma}x{:}A'.B'$ – either $B{=}B'$, $A \xrightarrow{\beta\mathsf{x}} A'$ or $A{=}A'$, $B \xrightarrow{\beta\mathsf{x}} B'$. By induction hypothesis $|M| = (\lambda ß.|B|)|A| \xrightarrow{\beta\mathsf{x}}{}^{+} (\lambda ß.|B'|)|A'| = |M'|$.
- $M = \lambda^{\sigma}x{:}A.B$. Analoguous to the former case.
- $M = BA$. Either the reduction takes place in B or in A or $B = \lambda x{:}P.Q$ and $M' = Q\langle x{:=}A\rangle$. In the first subcase, then the result is obvious by induction hypothesis as in the two former cases. In the second subcase, $|M| = (\lambda ß.\lambda x_{\sigma}.|Q|)|P||A| \xrightarrow{\beta\mathsf{x}} (\lambda x_{\sigma}.|Q|)|A| \xrightarrow{\beta\mathsf{x}} Q\langle x_{\sigma}{:=}|A|\rangle = |M'|$.
- $M = B\langle^{\sigma} x{:=}A\rangle$. Obvious by induction hypothesis.

5.10 Easy by induction upon $A \xrightarrow{\beta\mathsf{x}} B$.

5.11 It is an obvious application of rule ($\forall$ - E).

5.12 By induction upon the pair (k, n) where n and k are respectively the number of inferences and the complexity of the derivation of $\Gamma \vdash_{\lambda\omega\mathsf{x}} A : B$.

- (axiom): $\dfrac{(\overline{\sigma}, \overline{\tau}) \in \overline{\mathcal{A}}}{\vdash \overline{\sigma} : \overline{\tau}\ [0]}$
  It is obvious by (hypothesis) that $\Xi \vdash ß : \varsigma$.

- (rule): $\dfrac{\Gamma \vdash A{:}\rho\ \Gamma, x{:}A \vdash B{:}\sigma\ (\rho,\sigma,\tau) \in \mathcal{R}; x \notin \mathrm{dom}(\Gamma)}{\Gamma \vdash \Pi x{:}A.B : \tau}$
  By induction hypothesis, $\natural\Gamma, x : \natural A \vdash |B| : \varsigma$ so by ($\rightarrow$ - I), $\natural\Gamma \vdash \lambda x.|B| : \natural A {\rightarrow} \varsigma$. As by weakening lemma (4.4) and lemma 5.11, $\natural\Gamma \vdash æ : \natural A$, by ($\rightarrow$ - E), one obtains that $\natural\Gamma \vdash (\lambda x.|B|)æ : \varsigma$. So with ($\rightarrow$ - I) and the weakening lemma (4.4), $\natural\Gamma \vdash \lambda ß.(\lambda x.|B|)æ : \varsigma{\rightarrow}\varsigma$, and with ($\rightarrow$ - E), as by induction hypothesis, $\natural\Gamma \vdash |A| : \varsigma$, one can conclude that $\natural\Gamma \vdash (\lambda ß.(\lambda x.|B|)æ)|A| : \varsigma$.

- (hypothesis): $\dfrac{\Gamma \vdash A : \sigma\ ((k{-}1)) \qquad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : A \vdash x : A\ ((k))}$
  $\natural(\Gamma, x{:}A) {=} \natural\Gamma, x{:}\natural A$ and by (hyp) $\natural\Gamma, x{:}\natural A \vdash x{:}\natural A$.

- (weakening): $\dfrac{\Gamma \vdash A : B \;\; \Gamma \vdash C : \sigma \;\; x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : C \vdash A : B}$
  By induction hypothesis, $\natural\Gamma \vdash |A| : \natural B$ and the result holds with the weakening lemma for $\mathcal{F}$x.

- (Π - I): $\dfrac{\Gamma\vdash(\Pi x{:}A.B){:}\sigma \;\; \Gamma, x{:}A\vdash M{:}B \;\; x\notin\mathrm{dom}(\Gamma)}{\Gamma \vdash \lambda x{:}A.M : (\Pi x{:}A.B)}$
  By induction hypothesis, $\natural\Gamma,\, x : \natural A \vdash |M| : \natural B$. We distinguish several cases depending on the degrees:

  a. $\delta(x) = \delta(M) = \delta(B) - 1 = \delta(A) - 1$. Then by ($\rightarrow$ - I), $\natural\Gamma \vdash \lambda x.|M| : \natural A{\rightarrow}\natural B$. By ($\rightarrow$ - I) and the weakening lemma (4.4), $\natural\Gamma \vdash \lambda ß.\lambda x.|M| : \varsigma{\rightarrow}\natural A{\rightarrow}\natural B$.
  b. $\delta(x) = \delta(M) + 1 = \delta(B) = \delta(A) - 1$. Then one can show by easy induction that since $\delta(x) > \delta(M)$, x does not occur available in M, and therefore it need not appear in the typing context. So $\natural\Gamma \vdash |M| : \natural B$; moreover since contexts are ordered in pure type systems, ξ may not occur free in $\natural\Gamma$, so by ($\forall$ - I), $\natural\Gamma \vdash |M| : \forall\xi.\natural B$. Then by ($\rightarrow$ - I) and the weakening lemma (4.4), $\natural\Gamma \vdash \lambda ß.|M| : \varsigma{\rightarrow}\natural B$.

  As $\Gamma \vdash (\Pi x{:}A.B) : \sigma_{\;((k-1))}$, by the generation lemma (4.5) there is some sort ρ such that $\Gamma \vdash A : \rho_{\;((k-2))}$. Therefore by induction hypothesis, $\natural\Gamma \vdash |A| : \varsigma$. So by ($\rightarrow$ - E), in both cases $\natural\Gamma \vdash |\lambda x{:}A.M| : \natural(\Pi x{:}A.B)$.

- (Π - E): $\dfrac{\Gamma \vdash M : (\Pi^\sigma x : A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B\langle x^\sigma{:=}N\rangle}$
  By induction hypothesis, $\natural\Gamma \vdash |M| : \natural(\Pi^\sigma x{:}A.B)$ and $\natural\Gamma \vdash |N| : \natural A$. We distinguish several cases depending on the nature of $(\Pi^\sigma x{:}A.B)$:

  a. $\delta(A) - 1 = \delta(x) < \delta(B)$. Then $\natural(\Pi^\sigma x{:}A.B) = \natural A{\rightarrow}\natural B$ and $|MN| = |M|\;|N|$. So $\natural\Gamma \vdash |M| : \natural A{\rightarrow}\natural B$ and $\natural\Gamma \vdash |N| : \natural A$ so by ($\rightarrow$ - I) $\natural\Gamma \vdash |M||N| : \natural B$. One can show by easy induction that if $\delta(x) = \delta(B)$ then $\xi \notin fv(\natural B)$. So $\natural\Gamma \vdash |M||N| : \natural B[\natural N/\xi]$
  b. $\delta(A) - 1 = \delta(x) = \delta(B)$. Then $\natural\Pi^{\Box}x{:}A.B = \forall\xi.\natural B$ and $|MN| = (\lambda ß.|M|)|N|$. So $\natural\Gamma \vdash |M| : \forall\xi.\natural B$ and so by ($\forall$ - E), $\natural\Gamma \vdash |M| : \natural B[\natural N/\xi]$. So by ($\rightarrow$ - I) and the weakening lemma (4.4), $\natural\Gamma \vdash \lambda ß.|M| : \varsigma{\rightarrow}\natural B[\natural N/\xi]$. Moreover, one has then that $\delta(N) = 2$, and one can show by induction that if $\Gamma \vdash_{\lambda C} X : Y$ and $\delta(X) = 2$ then $Y \overset{\beta x}{\equiv} \Box$ or $\int$. So $\natural\Gamma \vdash |N| : \natural A = \varsigma$. So by ($\rightarrow$ - E), $\natural\Gamma \vdash (\lambda ß.|M|)|N| : \natural B[\natural N/\xi]$.

  In both cases $\natural\Gamma \vdash |MN| : \natural B[\natural N/\xi]$.

- (cut): $\dfrac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\langle x{:=}N\rangle : B\langle x{:=}N\rangle}$
  By induction hypothesis, $\natural\Gamma, x : \natural A \vdash |M| : \natural B$ and $\natural\Gamma \vdash |N| : \natural A$. We distinguish the same cases (a. and b.) as for the two former rules, and the same proof holds – since up to βx-reduction, a (cut) is a ($\rightarrow$ - I) followed by a ($\rightarrow$ - E).

- (xpand): $\dfrac{\Gamma \vdash M : B \;\; \Delta \vdash N : A \;\; P\langle x{:=}N\rangle \xrightarrow{x} M}{\Gamma \vdash P\langle x{:=}N\rangle : B}$
  This result clearly holds by easy structural induction upon P.

- (conversion): $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : \overline{\sigma} \qquad A \overset{\beta\mathtt{x}}{\equiv} B}{\Gamma \vdash M : B}$

  By induction hypothesis, $\natural\Gamma \vdash |M| : \natural A$. As by the type correction theorem (4.7) and the second premise of the inference both A and B are predicate terms, as by the third premise $A \overset{\beta\mathtt{x}}{\equiv} B$, we have that $\natural A = \natural B$ by lemma 5.10, and hence $\natural\Gamma \vdash |M| : \natural B$.

5.13 With the type correction theorem (4.7) it is enough to show that if $\Gamma \vdash \lambda\omega\mathtt{x}M : T$ then M is strongly normalizing.
Consider an infinite reduction starting at M: $M \xrightarrow{\beta\mathtt{x}} M' \xrightarrow{\beta\mathtt{x}} M'' \xrightarrow{\beta\mathtt{x}} \ldots$. By lemma 5.12, we have that $\natural\Gamma \vdash \mathcal{F}\mathtt{x}|M| : \natural T$ and by lemma 5.9, we obtain that $[\![M]\!] \xrightarrow{\beta\mathtt{x}}{}^{+} [\![M']\!] \xrightarrow{\beta\mathtt{x}} [\![M'']\!] \xrightarrow{\beta\mathtt{x}} \ldots$. But by theorem 5.8, this sequence must be finite, and so must be the initial one.

VI - Conclusion

Part VIII

Appendix B: System $\mathcal{F}\mathtt{x}$

In this appendix translated from [17], we are going to study the explicit substitution calculus $\lambda\mathsf{x}$ (cf. [6]), and develop for it a type system named $\mathcal{F}\mathsf{x}$ – an explicit version of Girard's system $\mathcal{F}$ (cf. [2, 16]). We shall especially prove that the subject reduction and the strong normalization hold for our system.

1 λ-calculus with explicit substitutions

Definition B.1: $\lambda\mathtt{x}$*-calculus*
The set $\mathcal{L}\mathsf{x}$ of terms is inductively defined upon an infinite variable set $\mathcal{V}$ by the following grammar:

$$M ::= x \mid \lambda x.M \mid M\ M \mid M\ \langle x{:=}M\rangle \qquad (x \in \mathcal{V})$$

The relation $\stackrel{\alpha\mathtt{x}}{\equiv}$ of $\alpha\mathtt{x}$-*equivalence* is defined as follows upon $\mathcal{L}\mathsf{x}$, where usual implicit substitution is noted between square brackets:

- if $M = x \in \mathcal{V}$, $M \stackrel{\alpha\mathtt{x}}{\equiv} N$ if $N = x$;
- if $M = \lambda x.P$, $M \stackrel{\alpha\mathtt{x}}{\equiv} N$ if $N = \lambda y.R$ with $P[x{:=}z] \stackrel{\alpha\mathtt{x}}{\equiv} R[y{:=}z]$ for all z except a finite number;
- if $M = PQ$, $M \stackrel{\alpha\mathtt{x}}{\equiv} N$ if $N = RS$ with $P \stackrel{\alpha\mathtt{x}}{\equiv} R$ and $Q \stackrel{\alpha\mathtt{x}}{\equiv} S$;
- if $M = P\langle x{:=}Q\rangle$, $M \stackrel{\alpha\mathtt{x}}{\equiv} N$ if $N = R\langle y{:=}S\rangle$, with $Q \stackrel{\alpha\mathtt{x}}{\equiv} S$ and $P[x{:=}z] \stackrel{\alpha\mathtt{x}}{\equiv} R[y{:=}z]$ for all z except a finite number.

This relation is an equivalence relation, so one can define the quotient set $\Lambda\mathtt{x} = \mathcal{L}\mathsf{x}/\stackrel{\alpha\mathtt{x}}{\equiv}$, the elements of which are the $\lambda\mathtt{x}$-terms, to which abstraction, application and substitution are canonically extended.

Definition B.2: *Free, bound, available variables*
These notions are defined as in pure type systems.

In the following, we shall make no distinction between a type and its representants, and we shall apply for the choice of representants Barendregt's convention that a same variable does not occur both bound and free in a same term.

We shall adopt the following writing convention: latin small letters denote variables, latin capital letter, terms, fraktur letters, substitutions. We shall use the abbreviations $\overline{\mathfrak{S}}$ for a sequence of substitutions and $\overline{T}$ for a sequence of term applications.

Lemma B.1: *Variable Conservation*
Let M and M' be such that $M \xrightarrow{\beta\mathtt{x}} M'$.
Then $av(M') \subseteq av(M)$.
Proof.
Obvious by definition of available variables. Q.E.D.

2 Reduction

The notion of β-reduction is thus modified for the λx-calculus:

Definition B.3: *βx-reduction*

βx-reduction is induced by the following rules:

(B) $(\lambda x.B)\,A \xrightarrow{\beta\mathrm{x}} B\langle x{:=}A\rangle$;

(abs) $(\lambda y.B)\langle x{:=}A\rangle \xrightarrow{\beta\mathrm{x}} \lambda y.B\langle x{:=}A\rangle$;

(app) $(M\,N)\langle x{:=}A\rangle \xrightarrow{\beta\mathrm{x}} M\langle x{:=}A\rangle\,N\langle x{:=}A\rangle$;

(sbst) $x\langle x{:=}A\rangle \xrightarrow{\beta\mathrm{x}} A$;

(gc) $M\langle x{:=}A\rangle \xrightarrow{\beta\mathrm{x}} M \quad$ if $x \notin av(M)$.

The reduction presented here contains the (gc) rule of « garbage collection » rather than the more elementary (var) rule: $y\langle x{:=}A\rangle \xrightarrow{\beta\mathrm{x}} y \quad (y \neq x)$. Both systems are equivalent in most aspects.

Notation:

Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of Λx.

We set $\mathcal{A}{\to}\mathcal{B} = \{M \in \Lambda\mathrm{x} \,/\, \forall N \in \mathcal{A},\ MN \in \mathcal{B}\}$.

Remark:

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be such subsets of Λx that $\mathcal{A} \supseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{D}$. Then $(\mathcal{A}{\to}\mathcal{B}) \subseteq (\mathcal{C}{\to}\mathcal{D})$.

3 Saturated sets

Definition B.4: *$\mathcal{X}$-saturated subsets*

Let $\mathcal{X}$ be a subset of Λx.

A subset $\mathcal{A}$ of Λx is said to be *$\mathcal{X}$-saturated* if the following assertions are satisfied:

(sat-B) $B\langle x{:=}A\rangle\,\overline{T} \in \mathcal{A}$
$\Rightarrow (\lambda x.B)A\,\overline{T} \in \mathcal{A}$;

(sat-abs) $(\lambda y.B\langle x{:=}A\rangle)\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$
$\Rightarrow (\lambda y.B)\langle x{:=}A\rangle\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$;

(sat-comp) $B\langle y{:=}C\rangle\langle x{:=}A\langle y{:=}C\rangle\rangle\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$
$\Rightarrow B\langle x{:=}A\rangle\langle y{:=}C\rangle\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$;

(sat-var) $A\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$
$\Rightarrow x\langle x{:=}A\rangle\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$;

(sat-app) $(B\langle x{:=}A\rangle)(C\langle x{:=}A\rangle) \in \mathcal{A}$
$\Rightarrow (BC)\langle x{:=}A\rangle$;

(sat-gc) $B\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$, $A \in \mathcal{X}$, $x \notin av(B)$
$\Rightarrow B\langle x{:=}A\rangle\,\overline{\mathfrak{S}}\,\overline{T} \in \mathcal{A}$.

Let $sat_{\mathcal{X}}$ denote the set of all $\mathcal{X}$-saturated subsets of Λx and $\mathfrak{S}_{\mathcal{X}} = sat_{\mathcal{X}} \cap \mathfrak{P}(\mathcal{X})$ this of all $\mathcal{X}$-saturated subsets of $\mathcal{X}$.

Proposition B.2: *(Stability of $sat_{\mathcal{X}}$)*

The following assertions hold:

i. If $\mathcal{A} \in \mathfrak{P}(\Lambda\mathrm{x})$ and $\mathcal{B} \in sat_{\mathcal{X}}$, then $\mathcal{A}{\to}\mathcal{B} \in sat_{\mathcal{X}}$;

ii. $sat_{\mathcal{X}}$ is closed under intersection.

Proof.
This is a consequence of the definition of $\mathcal{X}$-saturated sets. Q.E.D.

Proposition B.3:
Let $\mathcal{N}$ be the set of all strongly normalizing terms.
$\mathcal{N}$ is $\mathcal{N}$-saturated.
Proof.
This lemma was proven in [7] (lemma 5, p. 16). The calculus and the notion of saturation are here the same. Q.E.D.

4 Type System

Definition B.5: *System $\mathcal{F}$*
Let Υ be an infinite set of type variables. The set Φ of formulae is defined as the closure of Υ by the binary connector $\rightarrow$ and the quantifier $\forall$. Upon Φ is defined as follows the relation $\stackrel{\alpha}{\equiv}$ of α-*equivalence*:

- if $\varphi = \alpha \in \Upsilon$, $\varphi \stackrel{\alpha}{\equiv} \psi$ if $\psi = \alpha$;
- if $\varphi = \rho{\rightarrow}\sigma$, $\varphi \stackrel{\alpha}{\equiv} \psi$ if $\psi = \tau{\rightarrow}\upsilon$ with $\rho \stackrel{\alpha}{\equiv} \tau$ and $\sigma \stackrel{\alpha}{\equiv} \upsilon$;
- if $\varphi = \forall\alpha.\sigma$, $\varphi \stackrel{\alpha}{\equiv} \psi$ if $\psi = \forall\beta.\tau$ and $\sigma[\alpha{:=}\gamma] \stackrel{\alpha}{\equiv} \tau[\beta{:=}\gamma]$ for all γ except a finite number.

Just as for terms one defines the quotient set $\mathcal{F} = \Phi/\stackrel{\alpha}{\equiv}$, the elements of which are the types of system $\mathcal{F}$, to which $\rightarrow$ and $\forall$ are canonically extended.

In the following we shall make no distinction between a type and its representants and apply also Barendregt's convention to types. We shall adopt the following writing convention: greek small letters denote types and type variables.

5 Typing

We can now define the usual typing notions:

Definition B.6: *Typing in $\mathcal{F}$x*
A *type assertion* is an expression of the form $M : \tau$ where M and τ are a term and a type, respectively called *subject* and predicate of the assertion. A *typing context* Γ is a set of type assertions, the subjects of which are distinct variables, which form its domain denoted $\mathrm{dom}(\Gamma)$. A *type judgement* is an expression of the form $\Gamma \vdash M : \tau$, obtained by derivation using the inference rules enounced in table 5. A terme is said to be *typable* if it is the subject of a valid judgement.

Notation:
$\Gamma, x : \sigma$ stands for $\Gamma \cup \{x : \sigma\}$ where $x \notin \mathrm{dom}(\Gamma)$.

The (drop) is not usual. It was introduced in [10], and allows to type more terms – e.g. $yz\langle x{:=}zy\rangle$.

Lemma B.4: *Extended weakening*
Let Γ, M, τ be such that $\Gamma \vdash M : \tau$. Let Δ be a context of disjoint domain with that of Γ. Then $\Gamma \cup \Delta \vdash M : \tau$.
Proof.

$$\frac{}{\Gamma,\, x:\sigma \vdash x:\sigma}\ \text{(hypothesis)}$$

$$\frac{\Gamma,\, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma{\rightarrow}\tau}\ (\rightarrow\text{ - I})$$

$$\frac{\Gamma \vdash M:\sigma{\rightarrow}\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}\ (\rightarrow\text{ - E})$$

$$\frac{\Gamma \vdash M:\tau \quad \alpha \in \Upsilon \text{ not free in } \Gamma}{\Gamma \vdash M:\forall\alpha.\tau}\ (\forall\text{ - I})$$

$$\frac{\Gamma \vdash M:\forall\alpha.\tau \quad \sigma \in \mathcal{F}}{\Gamma \vdash M:\tau[\alpha{:=}\sigma]}\ (\forall\text{ - E})$$

$$\frac{\Gamma,\, x:\sigma \vdash M:\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash M\langle x{:=}N\rangle:\tau}\ \text{(cut)}$$

$$\frac{\Gamma \vdash M:\tau \quad \Delta \vdash N:\sigma \quad x \notin av(M)}{\Gamma \vdash M\langle x{:=}N\rangle:\tau}\ \text{(drop)}$$

Table 5: Typing rules for $\mathcal{F}$x

Obvious by induction on the derivation of $\Gamma \vdash M:\tau$. Q.E.D.

Corollary B.5:
Let Γ, Δ, E, M, τ be such that $\Gamma \subseteq \Delta \subseteq$ E, $\Gamma \vdash M:\tau$, E $\vdash M:\tau$. Then $\Delta \vdash M:\tau$.

Lemma B.6: *Extended strengthening*
Let Γ, M, τ be such that $\Gamma \vdash M:\tau$.
Let $\Gamma_X = \{(x:\sigma) \in \Gamma \,/\, x \in av(X)\}$.
Then $\Gamma_M \vdash M:\tau$.

Proof.
By induction upon the derivation of $\Gamma \vdash M:\tau$.

- (hypothesis): $\frac{}{\Gamma,\, x:\tau \vdash x:\tau}$
  Let $\Delta = \Gamma,\, x:\tau$. $\Delta_M = \{x:\tau\}$, and hence the property holds.

- ($\rightarrow$ - I): $\frac{\Gamma,\, x:\rho \vdash N:\sigma}{\Gamma \vdash \lambda x.N:\rho{\rightarrow}\sigma}$
  $av(M){=}av(N)\backslash\{x\}$, so $\Gamma_M{=}\Gamma_N$ as $x{\notin}\text{dom}(\Gamma)$. By induction hypothesis (plus the extended weakening lemma B.4 if $x{\notin}av(N)$), $\Gamma_N, x{:}\rho \vdash N{:}\sigma$, so, with rule ($\rightarrow$ - I), one gets that $\Gamma_M \vdash \lambda x.N:\rho{\rightarrow}\sigma$.

- ($\rightarrow$ - E): $\frac{\Gamma \vdash N:\sigma{\rightarrow}\tau \quad \Gamma \vdash P:\sigma}{\Gamma \vdash NP:\tau}$
  $\Gamma_M = \Gamma_N \cup \Gamma_P$, so this is obvious by induction hypothesis with the corollary B.5 of the extended weakening lemma.

- ($\forall$ - I): $\frac{\Gamma \vdash M:\rho \quad \alpha \in \Upsilon \text{ not free in } \Gamma}{\Gamma \vdash M:\forall\alpha.\rho}$
  Obvious by induction hypothesis (if α is not free in Γ, *a fortiori* it is nor free in $\Gamma_M \subseteq \Gamma$).

- ($\forall$ - E): $\frac{\Gamma \vdash M:\forall\alpha.\rho \quad \sigma \in \mathcal{F}}{\Gamma \vdash M:\rho[\alpha{:=}\sigma]}$
  Obvious by induction hypothesis.

- (coupure): $\dfrac{\Gamma,\, x:\sigma \vdash N:\tau \quad \Gamma \vdash P:\sigma}{\Gamma \vdash N\langle x{:=}P\rangle : \tau}$
  $x{\notin}\mathrm{dom}(\Gamma)$, and either $x{\notin}av(N)$ or $x{\in}av(N)$.
  In the first case, $\Gamma_M = \Gamma_N$ and, as by induction hypothesis, $\Gamma_N \vdash N : \tau$, with (drop) one gets that $\Gamma_M \vdash M : \tau$.
  In the second case, $\Gamma_M = \Gamma_N \cup \Gamma_P$, and by induction hypothesis with the corollary of the extended weakening lemma B.4, by (cut) one gets that $\Gamma_M \vdash M : \tau$.

- (drop): $\dfrac{\Gamma \vdash N:\tau \quad \Delta \vdash P:\sigma \quad x \notin av(N)}{\Gamma \vdash N\langle x{:=}P\rangle : \tau}$
  Then $\Gamma_M = \Gamma_N$ and, by induction hypothesis, with (drop), one gets that $\Gamma_M \vdash M : \tau$.

Q.E.D.

6 Subject reduction

Here we will show that reduction preserves typing.

Theorem B.7: *Subject reduction*

Let Γ, M, M', τ be such that $\Gamma \vdash M : \tau$ and $M \xrightarrow{\beta\mathsf{x}} M'$. Then $\Gamma \vdash M' : \tau$.

Proof.

By induction upon the derivation of $\Gamma \vdash M : \tau$.

- (hypothesis): $\dfrac{}{\Gamma,\, x:\tau \vdash x:\tau}$
  $M = x \in \mathcal{V}$, so there is no M' such that $M \xrightarrow{\beta\mathsf{x}} M'$, and the property holds.

- ($\rightarrow$ - I): $\dfrac{\Gamma,\, x:\rho \vdash N:\sigma}{\Gamma \vdash \lambda x.N : \rho{\rightarrow}\sigma}$
  $M{=}\lambda x.N$ so if $M{\xrightarrow{\beta\mathsf{x}}}M'$, necessarily $M'{=}\lambda x.N'$ with $N{\xrightarrow{\beta\mathsf{x}}}N'$. Applying the induction hypothesis gives $\Gamma, x{:}\rho \vdash N'{:}\sigma$, so by ($\rightarrow$ - I) $\Gamma \vdash M'{:}\rho{\rightarrow}\sigma$.

- ($\rightarrow$ - E): $\dfrac{\Gamma \vdash N:\sigma{\rightarrow}\tau \quad \Gamma \vdash P:\sigma}{\Gamma \vdash NP:\tau}$
  1st case: $M' = N'P$ with $N \xrightarrow{\beta\mathsf{x}} N'$ or $M' = NP'$ with $P{\xrightarrow{\beta\mathsf{x}}}P'$, respectively. Then by induction hypothesis, $\Gamma \vdash N' : \sigma{\rightarrow}\tau$, respectively $\Gamma \vdash P' : \sigma$, and so, with rule ($\rightarrow$ - E), $\Gamma \vdash M' : \tau$.

  2nd case: $N = \lambda x.Q$ and $M' = Q\langle x{:=}P\rangle$. So, as $\Gamma \vdash \lambda x.Q : \sigma{\rightarrow}\tau$, necessarily $\Gamma, x : \sigma \vdash Q : \tau$. Moreover, $\Gamma \vdash P : \sigma$, and so, with rule (cut), $\Gamma \vdash M' : \tau$.

- ($\forall$ - I): $\dfrac{\Gamma \vdash M:\rho \quad \alpha \in \Upsilon \text{ not free in } \Gamma}{\Gamma \vdash M : \forall\alpha.\rho}$
  By induction hypothesis, $\Gamma \vdash M' : \rho$, and so, with rule ($\forall$ - I), $\Gamma \vdash M' : \forall\alpha.\rho$.

- ($\forall$ - E): $\dfrac{\Gamma \vdash M:\forall\alpha.\rho \quad \sigma \in \mathcal{F}}{\Gamma \vdash M : \rho[\alpha{:=}\sigma]}$
  By induction hypothesis, $\Gamma \vdash M' : \rho$, and so with rule ($\forall$ - E), $\Gamma \vdash M' : \rho[\alpha{:=}\sigma]$.

- (cut): $\dfrac{\Gamma, x:\sigma \vdash N:\tau \quad \Gamma \vdash P:\sigma}{\Gamma \vdash N\langle x{:=}P\rangle : \tau}$

  1st case: $M' = N'\langle x{:=}P\rangle$ with $N \xrightarrow{\beta x} N'$ or $M' = N\langle x{:=}P'\rangle$ with $P \xrightarrow{\beta x} P'$, respectively. Then by induction hypothesis, $\Gamma, x:\sigma \vdash N':\tau$, respectively $\Gamma \vdash P':\sigma$, and so with rule (cut), $\Gamma \vdash M':\tau$.

  2nd case: N may take several forms.
  Subcase a: $N = \lambda y.Q$ and $M' = \lambda y.Q\langle x{:=}P\rangle$. Then $\Gamma, x:\sigma \vdash \lambda y.Q:\tau$ and $\Gamma \vdash P:\sigma$. The last applied rule to obtain the judgement $\Gamma, x:\sigma \vdash \lambda y.Q:\tau$ is either ($\rightarrow$ - I), or ($\forall$ - I) or ($\forall$ - E). We will show by recurrence upon the number k of rules ($\forall$ - I) and ($\forall$ - E) that $\Gamma, x:\sigma \vdash \lambda y.Q\langle x{:=}P\rangle:\tau$.

  - $k{=}0$. Then $\tau{=}\xi{\rightarrow}\chi$ and $\dfrac{\Gamma, x{:}\sigma, y{:}\xi \vdash Q{:}\chi}{\Gamma \vdash \lambda y.Q{:}\xi{\rightarrow}\chi}$
    And so, as by the extended weakening lemma B.4 $\Gamma, y:\xi \vdash P:\sigma$, with (cut), one gets that $\Gamma, y:\xi \vdash Q\langle x{:=}P\rangle:\chi$, and so, with ($\rightarrow$ - I), $\Gamma \vdash \lambda y.Q\langle x{:=}P\rangle : \xi{\rightarrow}\chi = \tau$.
  - Assume that the property be true for k.
    For $k+1$: according to the last applied rule in the derivation, which can either be rule ($\forall$ - I): $\dfrac{\Gamma \vdash \lambda y.Q:\zeta \quad \alpha \in \Upsilon \text{ not free in } \Gamma}{\Gamma \vdash \lambda y.Q:\forall\alpha.\zeta}$ or, respectively, rule ($\forall$ - E): $\dfrac{\Gamma \vdash \lambda y.Q:\forall\alpha.\vartheta \quad \eta \in \mathcal{F}}{\Gamma \vdash \lambda y.Q:\vartheta[\alpha{:=}\eta]}$, $\tau = \forall\alpha.\zeta$ or, respectively, $\tau = \vartheta[\alpha{:=}\eta]$. By recurrence hypothesis, $\Gamma \vdash \lambda y.Q\langle x{:=}P\rangle : \zeta$ or, respectively, $\Gamma \vdash \lambda y.Q\langle x{:=}P\rangle : \forall\alpha.\vartheta$, and so, with ($\forall$ - I), respectively ($\forall$ - E), one derives $\Gamma \vdash \lambda y.Q\langle x{:=}P\rangle:\tau$.

  Subcase b: $N = QR$ and $M' = Q\langle x{:=}P\rangle R\langle x{:=}P\rangle$. As for the first subcase, one unfolds the derivation to the application of rule ($\rightarrow$ - E) by eliminating recursively the ($\forall$ - I) and ($\forall$ - E) rules, and one gets analoguously the expected result.
  Subcase c: $N = x$ and $M' = P$. Then $\Gamma, x{:}\sigma \vdash N{:}\tau$, so $\tau = \sigma$; now $\Gamma \vdash P:\sigma$.
  Subcase d: $x{\notin}av(N)$ and $M' = N$. Now $\Gamma, x{:}\sigma \vdash N{:}\tau$, so, as $x{\notin}av(N)$, $\Gamma \vdash N:\tau$.

- (drop): $\dfrac{\Gamma \vdash N:\tau \quad \Delta \vdash P:\sigma \quad x \notin av(N)}{\Gamma \vdash N\langle x{:=}P\rangle : \tau}$

  1st case: $M' = N'\langle x{:=}P\rangle$ with $N \xrightarrow{\beta x} N'$ or $M' = N\langle x{:=}P'\rangle$ with $P \xrightarrow{\beta x} P'$, respectively. Then by induction hypothesis, $\Gamma \vdash N':\tau$, respectively $\Delta \vdash P':\sigma$, and so with (drop), $\Gamma \vdash M':\tau$.

  2nd case: $M' = N$. Then by induction hypothesis $\Gamma \vdash M':\tau$.

  3rd case: analoguous to the second case of (cut).

Q.E.D.

7 Interpretations

Definition B.7: *$\mathcal{X}$-interpretation*
Let $\mathcal{X}$ be a subset of Λx.
An *$\mathcal{X}$-interpretation* $\mathcal{I}$ is a map from Υ into $\mathfrak{S}_{\mathcal{X}}$, the set of the $\mathcal{X}$-saturated subsets of $\mathcal{X}$.

Notation:
Let $\mathcal{I}$ be an $\mathcal{X}$-interpretation, $\alpha \in \Upsilon$ be a type variable and $\mathcal{A}$ an $\mathcal{X}$-saturated subset of $\mathcal{X}$. One denotes by $\mathcal{I}_{\alpha\leftarrow\mathcal{A}}$ the $\mathcal{X}$-interpretation defined by:

$$\mathcal{I}_{\alpha\leftarrow\mathcal{A}} : \begin{cases} \alpha & \mapsto \mathcal{A} \\ \beta \neq \alpha & \mapsto \mathcal{I}(\beta) \end{cases}$$

Definition B.8: *$\mathcal{X}$-interpretation of types*
An $\mathcal{X}$-interpretation can be canonically extended to a map from $\mathcal{F}$ onto $sat_{\mathcal{X}}$ on the following way:

- $\mathcal{I}(\rho{\rightarrow}\sigma) = \mathcal{I}(\rho){\rightarrow}\mathcal{I}(\sigma)$;
- $\mathcal{I}(\forall\alpha.\tau) = \bigcap_{\mathcal{Y}\in\mathfrak{S}_{\mathcal{X}}} \mathcal{I}_{\alpha\leftarrow\mathcal{Y}}(\tau)$.

By a notation abuse, one confuses an interpretation and its extension as a type application.

Lemma B.8: *Substitution*
Let $\mathcal{I}$ be an $\mathcal{X}$-interpretation, $\alpha \in \Upsilon$ a type variable and $\sigma, \tau \in \mathcal{F}$ two types. Then $\mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\tau) = \mathcal{I}(\tau[\alpha{:=}\sigma])$.
Proof.
By induction upon τ:

- $\tau = \alpha$. Then $\mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\alpha) = \mathcal{I}(\sigma) = \mathcal{I}(\alpha[\alpha{:=}\sigma])$.
- $\tau{=}\beta{\in}\Upsilon\backslash\{\alpha\}$. Then $\mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\beta){=}\mathcal{I}(\beta){=}\mathcal{I}(\beta[\alpha{:=}\sigma])$.
- $\tau = \rho{\rightarrow}\xi$. Then (by i.h. for the second equality):
$$\begin{aligned}\mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\rho{\rightarrow}\xi) &= \mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\rho){\rightarrow}\mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\xi)\\ &= \mathcal{I}(\rho[\alpha{:=}\sigma]){\rightarrow}\mathcal{I}(\xi[\alpha{:=}\sigma])\\ &= \mathcal{I}(\rho[\alpha{:=}\sigma]{\rightarrow}\xi[\alpha{:=}\sigma])\\ &= \mathcal{I}((\rho{\rightarrow}\xi)[\alpha{:=}\sigma]).\end{aligned}$$
- $\tau = \forall\beta.\rho$. One can assume that $\beta \neq \alpha$ because of Barendregt's convention.
Then
$$\begin{aligned}\mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\tau) &= \bigcap_{\mathcal{Y}\in\mathfrak{S}(\mathcal{X})} \mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma),\,\beta\leftarrow\mathcal{Y}}(\rho) && (1)\\ &= \bigcap_{\mathcal{Y}\in\mathfrak{S}(\mathcal{X})} \mathcal{I}_{\beta\leftarrow\mathcal{Y},\,\alpha\leftarrow\mathcal{I}(\sigma)}(\rho) && (2)\\ &= \bigcap_{\mathcal{Y}\in\mathfrak{S}(\mathcal{X})} \mathcal{I}_{\beta\leftarrow\mathcal{Y}}(\rho[\alpha{:=}\sigma]) && (3)\\ &= \mathcal{I}(\forall\beta.\rho[\alpha{:=}\sigma]) && (1)\\ &= \mathcal{I}((\forall\beta.\rho)[\alpha{:=}\sigma]) && (2)\end{aligned}$$
(1) by definition, (2) because $\alpha \neq \beta$, (3) by induction hypothesis.

Q.E.D.

Lemma B.9: *Inclusion*
Let $\mathcal{I}$ be an $\mathcal{N}$-interpretation.
Then for any type $\tau \in \mathcal{F}$, $\mathcal{I}(\tau) \subseteq \mathcal{N}$.
Proof.
Let $\mathcal{N}_0 = \{x\,\overline{T} \,/\, x \in \Upsilon, \overline{T} \in \mathcal{N}\}$.
We are going to show by induction upon τ the stronger following result: for all $\mathcal{N}$-interpretation $\mathcal{I}$, for all type $\tau \in \mathcal{F}$, $\mathcal{N}_0 \subseteq \mathcal{I}(\tau) \subseteq \mathcal{N}$.

In order to do that, we are going to show first some properties of $\mathcal{N}_0$ relatively to $\mathcal{N}$:

i. $\mathcal{N}_0 \subseteq \mathcal{N}$

ii. $\mathcal{N}_0 \subseteq (\mathcal{N}\rightarrow\mathcal{N}_0)$

iii. $(\mathcal{N}_0\rightarrow\mathcal{N}) \subseteq \mathcal{N}$

Proposition i. is obvious as $\mathcal{N}$ is closed under application and contains the variables.

Proposition ii. is also obvious by definition of $\mathcal{N}_0$.

Proposition iii. will be shown ad absurdum: let $T \in \mathcal{N}_0\rightarrow\mathcal{N}$, one assumes that $T \notin \mathcal{N}$. T admits an infinite derivation $T\xrightarrow{\beta\mathsf{x}}T_1\xrightarrow{\beta\mathsf{x}}T_2\xrightarrow{\beta\mathsf{x}}\ldots$. Let $x \in \mathcal{V} \subseteq \mathcal{N}_0$ be a variable; then $T\,x \in \mathcal{N}$. But $T\,x$ admits an infinite derivation $T\xrightarrow{\beta\mathsf{x}}T_1x\xrightarrow{\beta\mathsf{x}}T_2x\xrightarrow{\beta\mathsf{x}}\ldots$. The result of this is a contradiction.

If $\tau = \alpha \in \Upsilon$, it is obvious that $\mathcal{N}_0 \subseteq \mathcal{N}$.

If $\tau = \rho\rightarrow\sigma$, by induction hypothesis $\mathcal{N}_0 \subseteq \mathcal{I}(\rho)$ and $\mathcal{I}(\sigma) \subseteq \mathcal{N}$. So by iii., $\mathcal{I}(\rho)\rightarrow\mathcal{I}(\sigma) \subseteq (\mathcal{N}_0\rightarrow\mathcal{N}) \subseteq \mathcal{N}$. Similarly, by induction hypothesis, $\mathcal{I}(\rho) \subseteq \mathcal{N}$ and $\mathcal{N}_0 \subseteq \mathcal{I}(\sigma)$. So by ii., $\mathcal{I}(\rho)\rightarrow\mathcal{I}(\sigma) \supset (\mathcal{N}\rightarrow\mathcal{N}_0) \supset \mathcal{N}_0$.

If $\tau = \forall\alpha.\rho$, $\mathcal{I}(\tau) \subseteq \mathcal{I}(\rho) \subseteq \mathcal{N}$ by induction hypothesis, and, by induction hypothesis again, $\mathcal{N}_0 \subseteq \mathcal{J}(\rho)$ for all $\mathcal{N}$-interpretation $\mathcal{J}$, and especially for the $\mathcal{N}$-interpretations of the form $\mathcal{I}_{\alpha\leftarrow\mathcal{Y}}$, so $\mathcal{N}_0 \subseteq \mathcal{I}(\tau)$. Q.E.D.

8 Strong normalisation

The aim of this section is to show the strong normalization theorem:

Theorem B.10: *Strong normalization*
Any Λx-term typable in $\mathcal{F}\mathsf{x}$ is strongly normalizing.

In order to do this, we are going to show a theorem stating that the intuition that one has of an interpretation fits with the reality in the particular case of $\mathcal{N}$-interpretations:

Theorem B.11: *Adequation*
Let $\mathcal{I}$ be an $\mathcal{N}$-interpretation. Let Γ, M and τ be such that $\Gamma \vdash M : \tau$. Then $M \in \mathcal{I}(\tau)$.
Proof.
We are going to show by induction upon the derivation of $\Gamma \vdash M : \tau$ the stronger following result:
Set $\Gamma = \{x_1 : \rho_1, \ldots, x_n : \rho_n\}$. Let $A_i \in \mathcal{I}(\rho_i)$ (for $i \in [\![1, n]\!]$) be such that $\forall j \leqslant 0,\ x_{i+j} \notin av(\rho_i)$. Let $\mathcal{I}$ be an $\mathcal{N}$-interpretation. Then $M\langle x_1{:=}A_1\rangle \cdots \langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$.

There are several cases according to the last applied rule:

- (hypothesis): $\dfrac{}{\Gamma,\, x : \sigma \vdash x : \sigma}$
  There exists i such that $M \equiv x_i$ and $\tau \equiv \alpha_i$. One applies $(i-1)$ times the (sat-gc) rule to get back to $x_i\langle x_i{:=}A_i\rangle\langle x_{i+1}{:=}A_{i+1}\rangle \cdots \langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$, then once the (sat-var) rule to get back to $A_i\langle x_{i+1}{:=}A_{i+1}\rangle \cdots \langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$, then $(n-i)$ times the (sat-gc) rule to get back to $A_i \in \mathcal{I}(\tau)$, which is true by hypothesis.

- $(\rightarrow$ - E): $\dfrac{\Gamma \vdash U : \sigma\rightarrow\tau \quad \Gamma \vdash V : \sigma}{\Gamma \vdash UV : \tau}$
  To show that $(UV)\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$, it is enough with rule (sat-app), to

show that $(U\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle)\ (V\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle) \in \mathcal{I}(\tau)$. Then by induction hypothesis, $U\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\sigma\to\tau) = \mathcal{I}(\sigma)\to\mathcal{I}(\tau)$ and $V\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\sigma)$. So $(U\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle)\ (V\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle) \in \mathcal{I}(\tau)$.

- ($\to$ - I): $\dfrac{\Gamma,\, x:\rho \vdash N:\sigma}{\Gamma \vdash \lambda x.N : \rho\to\sigma}$
  Because of Barendregt's convention, one can assume that x is not free in any A_i. To show that $(\lambda x.N)\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\rho\to\sigma)$, it is enough, with (sat-abs), to show that $\lambda x.N\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\rho\to\sigma) = \mathcal{I}(\rho)\to\mathcal{I}(\sigma)$. So let $B \in \mathcal{I}(\rho)$ be a term, we want to show that $\lambda x.N\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle B \in \mathcal{I}(\sigma)$. For this, it is enough to show that $N\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle\langle x{:=}B\rangle \in \mathcal{I}(\sigma)$ with rule (sat-B). x been free in no A_i, this is a consequence of the induction hypothesis.

- ($\forall$ - E): $\dfrac{\Gamma \vdash M:\forall\alpha.\rho \quad \sigma \in \mathcal{F}}{\Gamma \vdash M:\rho[\alpha{:=}\sigma]}$
  Then by induction hypothesis
  $M\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\forall\alpha.\rho) = \bigcap_{\mathcal{Y}\in\mathfrak{S}_\mathcal{N}} \mathcal{I}_{\alpha\leftarrow\mathcal{Y}}(\rho) \subseteq \mathcal{I}_{\alpha\leftarrow\mathcal{I}(\sigma)}(\rho) = \mathcal{I}(\rho[\alpha{:=}\sigma])$.
  (Indeed, $\mathcal{I}(\sigma)$ is $\mathcal{N}$-saturated, and by the inclusion lemma B.9, it is included in $\mathcal{N}$. The last equality is got by the substitution lemma B.8.)

- ($\forall$ - I): $\dfrac{\Gamma \vdash M:\rho \quad \alpha \in \Upsilon \text{ non libre dans } \Gamma}{\Gamma \vdash M:\forall\alpha.\rho}$
  Then for all $Y \in \mathfrak{S}_\mathcal{X}$, by induction hypothesis applied to $\mathcal{I}_{\alpha:=\mathcal{Y}}$,
  $M\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}_{\alpha:=\mathcal{Y}}(\rho)$.
  And so $M\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}_(\forall\alpha.\rho)$.

- (cut): $\dfrac{\Gamma,\, x:\sigma \vdash U:\tau \quad \Gamma \vdash V:\sigma}{\Gamma \vdash U\langle x{:=}V\rangle : \tau}$
  Because of Barendregt's convention, one can assume that x is not free in any A_i. In order to show that $U\langle x{:=}V\rangle\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$, it is enough to show that $U\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle\langle x{:=}V\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle\rangle \in \mathcal{I}(\tau)$, with rule (sat-comp). Now $V\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\sigma)$, by induction hypothesis applied to $\Gamma \vdash V:\sigma$, and so with the induction hypothesis applied to $\Gamma,\, x:\sigma \vdash U:\tau$, one gets the wished result.

- (drop): $\dfrac{\Gamma \vdash U:\tau \quad \Delta \vdash V:\sigma \quad x \notin av(U)}{\Gamma \vdash U\langle x{:=}V\rangle : \tau}$
  By induction hypothesis $U\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$ so, as by induction hypothesis $V \in \mathcal{I}(\sigma) \subseteq \mathcal{N}$ by the inclusion lemma B.9 and $x \notin av(U)$, with rule (sat-gc), $U\langle x{:=}V\rangle\langle x_1{:=}A_1\rangle\cdots\langle x_n{:=}A_n\rangle \in \mathcal{I}(\tau)$.

Q.E.D.

Corollary B.12:
Any Λx-term typable in $\mathcal{F}$x is strongly normalizing.
Proof.
Just consider the $\mathcal{N}$-interpretation identically equal to $\mathcal{N}$ on all type variables. By the adequation lemma B.11, any typable term belongs to the image of its type through this interpretation, and by the inclusion lemma B.9 this image is contained in $\mathcal{N}$. Q.E.D.

Part IX

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