Structure of spaces of rhombus tilings in the lexicograhic case.

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Abstract

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Keywords: rhombus tiling, flip, connectivity

Résumé

Nous étudions la classe des pavages rhomboèdriques *lexicographiques*, qui se trouvent être une généralisation des ordres de Bruhat supérieurs, obtenue en relâchant la condition d'unitarité. Nous prouvons que les espaces de pavages induits sont des ordres partiels gradués avec un élément minimal et un élément maximal.

Mots-clés: pavages par rhomboèddres, flips, connexité

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Abstract

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1 Introduction

Rhombus tilings are tilings of zonotopes with rhombohedra. They appear in physics as a classical model for quasicrystals [16]. We fix a sequence $(v_1, v_2, \ldots, v_D)$ of vectors of $\mathbb{R}^d$ (such that each subsequence of length d is a basis of $\mathbb{R}^d$) and a sequence $(m_1, m_2, \ldots, m_D)$ of positive integers (called *multiplicities*). The tiled zonotope Z is the set $\{\sum \alpha_i v_i \ 0 \leq \alpha_i \leq m_i\}$, and each prototile used for T is a rhombohedron constructed from a subsequence of vectors of length d. If a tiling T contains $d+1$ rhombic tiles which pairwise share a facet, then a new tiling T_{flip} of Z can be obtained just changing the position of those $d+1$ tiles. This operation is called a flip. The space of tilings of Z is the graph whose vertices are tilings of Z and two tilings are linked by an edge if they differ by a single flip.

Before this paper, study has been done by Ziegler [18], about *higher Bruhat orders*. Those combinatorial structures can be interpreted (via the Bohne-Dress theorem [14]) as tilings of some specific unitary zonotopes (i.e. all multiplicities are equal to 1). Ziegler proves that, this case in this the space of tilings can be directed so as to get a graded poset (with single maximal and minimal element).

In the present paper, we extend the previous result relaxing the unitarity condition. We first recall how ideas (deletion, minors) issued from matroid theory to get a decomposition method for tilings, and a representation of tilings by black or white points organized in arrows and lines (see [2, 3] for details). Afterwards, we use this representation to study that we call lexicographic tilings (extensions with non unitary multiplicities of tilings correspondings to *higher Bruhat orders*) We show how each space of lexicographic tilings can be directed so as to get a graded poset (with single maximal and minimal element), which implies the connectivity of the space.

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About related works, Felsner and Weil [8] prove the same result, when $d = 2$. To our knowledge, the connectivity problem is still open for the other kinds of zonotopes. We mention that R. Kenyon [11] has proved the connectivity in dimension 2, for any simply connected domain.

We have chosen to make an exposition which can be accessible for people who have no knowledge in oriented matroid theory. This gives some longer proofs, but difficulties are not hidden in strong theorems about matroids. Nevertheless, since tilings are strongly related to oriented matroids by the Bohne-Dress Theorem [14], we have also given the matroid interpretation of the material presented. We also have tried to give precise references in [1], the most classical book about oriented matroids.

2 Tilings of Zonotopes and Minors

We deal in this paper with a particular case of tilings in $\mathbb{R}^d$, called *zonotopal rhombus tilings* (or tight zonotopal tilings). Let us now define the fundamental elements studied in the following.

The canonical basis of $\mathbb{R}^d$ will be noticed $(e_1, e_2, ..., e_d)$. Let $V = (v_1, ..., v_D)$ be a sequence of D vectors in $\mathbb{R}^d$ such that $D \geq d$ and each subsequence $(v_{i_1}, v_{i_2}, \ldots, v_{i_d})$ is a basis of $\mathbb{R}^d$. The parameter $c = D - d$ is called the *codimension*.

Let $M = (m_1, ..., m_D)$ be a sequence of D nonnegative integers. m_i is associated with the vector v_i and called the *multiplicity* of v_i. The *zonotope* $Z(V, M)$ associated with the pair (V, M) is the region of $\mathbb{R}^d$ defined by: $\left\{v \in \mathbb{R}^d, v = \sum_{i=1}^{D} \lambda_i v_i, \lambda_i \in [-m_i, m_i]\right\}$. Thus, $Z(V, M)$ is the convex hull of the finite set $\left\{v \in \mathbb{R}^d, v = \sum_{i=1}^{D} \lambda_i m_i v_i, \lambda_i \in \{-1, 1\}\right\}$.

One can define classically (see for example [17] p. 51-52) its faces, vertices (faces of dimension 0), edges (faces of dimension 1), and facets (faces of dimension $d-1$). The number: $s = \sum_{i=1}^{D} m_i$ is the *size* of the zonotope $Z(V, M)$; we say that $Z(V, M)$ is an *s-zonotope*. The zonotope $Z(V, M)$ is said to be *unitary* if all the multiplicities are equal to 1 (see Figure 1 for examples).

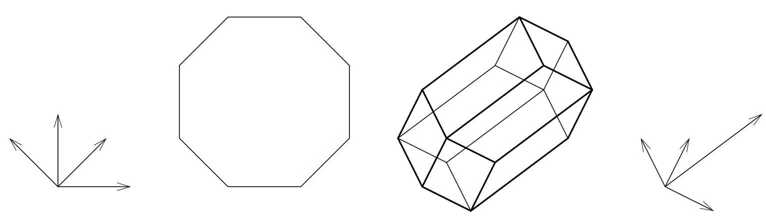

Figure 1: A 2-dimensional zonotope and a 3-dimensional zonotope both defined on 4 vectors

Let $Z(V, M)$ be a zonotope. The sequence V of vectors is called the *type* of $Z(V, M)$. A *prototile* of V is a unitary zonotope constructed with a subsequence V' of d distinct vectors taken in V. A sequence V of D vectors of $\mathbb{R}^d$ induces $\binom{D}{d}$ different prototiles. A *tile* t is a translated prototile, i.e. it is defined by a pair (p, w), where p is a prototile and w a translation vector (formally, we have: $t = w + p$). Since tiles are some particular polytopes, their vertices, edges and facets are defined as well. The type of a tile is the type of the corresponding prototile.

A *tiling* T of a zonotope $Z(V, M)$ is a set of tiles constructed with vectors in V, such that each intersection between tiles is a face of the tiles (i.e. there is no overlap) and the union of tiles is equal to $Z(V, M)$ (i.e. there is no gap). Two tiles are *adjacent* if they share a whole facet. We say that $Z(V, M)$ is the *support* of the tiling T. If $Z(V, M)$ is an s-zonotope, we say that T is an s-tiling.

2.1 De Bruijn zones

Definition 1 (lifting, height function) *Let $Z = (V, M)$ be a zonotope, with $V = (v_1, ..., v_D)$. A* lifting *of V is a sequence $U = (u_1, ..., u_D)$ of vectors of $\mathbb{R}^{d+1}$, for each integer j such that $1 \leq j \leq D$, there exists an real α_j such that: $u_j = (v_j, \alpha_j)$.*

Let T be a tiling of $Z(V, M)$. An associated lifting *is a function $f_{T,U}$ which associates to each vertex of T_Z a vector in $\mathbb{R}^{d+1}$ and satisfies the following property: for any pair (x, x') of vertices of T such that $x' = x + 2v_i$ and $[x, x']$ is an edge of T, we have: $f_{T,U}(x') = f_{T,U}(x) + 2u_i$. See Figure 2.*

The height function *$h_{T,U}$ associated with a lifted tiling $f_{T,U}$, is the component upon e_{d+1} of $f_{T,U}$.*

One easily proves that the definition of lifting of a tiling is consistent since a zonotope is homeomorphic to a closed disk of $\mathbb{R}^d$.

If $f_{T,U}$ is a given lifting, then each lifting $f'_{T,U}$ of T is such that $f'_{T,U} = f_{T,U} + ke_{d+1}$, where k denotes a fixed real number. Thus, if $h_{T,U}$ is a given height function, then each height function $h'_{T,U}$ of T is such that $h'_{T,U} = h_{T,U} + k$. For convention, in this paper, the real k is chosen in such a way that the height function does not take negative values and there exists a vertex v such that $h'_{T,U}(v) = 0$.

Notice that $f_{T,U}$ is defined for the set of vertices of Z and, for each vertex v on the boundary of Z, $f_{T,U}$ does not depend on the tiling T chosen.

The two mostly used lifting functions are the *principal lifting function*, defined by: $\forall v_i \in V$, $u_i = (v_i, 1/2)$, and the *k-located function*, where for a fixed integer k, $u_k = (v_k, 1/2)$ and $\forall i \neq k$, $u_i = (v_i, 0)$. The k-located function has the same value on all vertices of a tile whose type does not contain v_k, and differs by 1 at the endpoints of an edge of type v_k. Therefore, the principal function differs by 1 at the endpoints of each edge of the tiling.

Now, since height functions have been defined, one may introduce the important concept of de Bruijn families and zones, widely used in the core of the paper (See [6] for details). This is the main tool for inductions on tilings.

Definition 2 (de Bruijn zone, family) *Let T be a tiling of a zonotope $Z(V, M)$, and h_i the i-located function. The* de Bruijn family *associated with the vector v_i is the set of tiles having v_i in their type. Moreover, the j-th* de Bruijn zone *is the set of tiles whose i-located function is $j - 1$ on one facet, and j on the opposite facet. This zone will be noted $S_{\{v_i,j\}}$ (see Figure 2).*

One sees that a de Bruijn zone $S_{\{v_i,j\}}$ disconnects the tiling into two parts, $T^+_{\{v_i,j\}}$ and $T^-_{\{v_i,j\}}$. The first one is composed of tiles whose vertices have i-located function larger

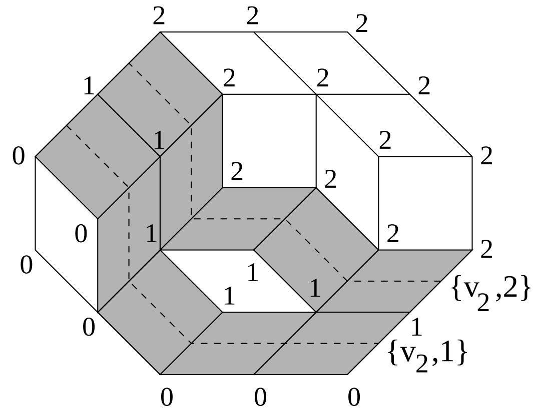

Figure 2: The 2-located height function and two de Bruijn zones.

than j, and the second corresponds to the tiles whose height function is smaller. Hence, for $j < j'$, we have: $T^-_{\{v_i,j\}} \subseteq T^-_{\{v_i,j'\}}$.

We say that two de Bruijn zones $S_{\{v_i,j\}}$ and $S_{\{v_k,l\}}$ are *parallel* if $v_i = v_k$. The intersection of a set of d de Bruijn zones of T which are pairwise not parallel is a tile of T. The intersection of a set of $d-1$ de Bruijn zones which are pairwise not parallel is a set of tiles which can be totally ordered in such a way that two consecutive tiles are adjacent. Such an intersection is called a *de Bruijn line*.

2.2 Flips

2.2.1 Tilings of a unitary d+1-zonotope

We first focus on a unitary zonotope of codimension 1. One easily checks that such a zonotope admits exactly two tilings: Let $V = (v_1, ..., v_{d+1})$ be the sequence of vectors and p_0 be the prototile constructed with the d first vectors: there is a tiling T with a tile t_0 of type p_0 such that $T^+_{\{v_{d+1},1\}}$ is empty and $T^-_{\{v_{d+1},1\}} = \{t_0\}$, and one tiling T' such that $T'^+_{\{v_{d+1},1\}} = \{t_0 + v_{d+1}\}$ and $T'^-_{\{v_{d+1},1\}}$ is empty.

Remark that T and T' are symmetrical. Any pair of tiles of T (or T') are adjacent, since they form a whole de Bruijn line of T. The orders in each de Bruijn line are opposite in T and T'.

2.2.2 Space of tilings

Those tilings of unitary zonotope of codimension 1 can appear, translated, in tilings of a larger zonotope Z. Assume that the tiling T_z of a unitary d+1-zonotope z of codimension 1 appears in a tiling T of Z, translated by a vector v (i. e. formally: $v + T_z \subset T$). We say that the tiling T' of Z, defined by: $T' = (T \setminus (v + T_z)) \cup (v + T'_z)$, is obtained from T by a *geometric flip*.

The *type* of the flip is the type of z. It will be denoted by the set of indexes of vectors of its support. We have $\binom{D}{d+1}$ types of flips; in particular, for $D = d+2$, we have D types of flips.

The *space of tilings* of a zonotope Z is the symmetric labeled graph whose vertices are the tilings of Z, and two tilings are linked by an edge if they differ by a geometric flip. The label of the edge is the type of the corresponding flip.

An important result is that flips induce connectivity between all tilings of zonotopes for $d = 2$, i.e. every tiling of a given dimension 2 zonotope Z can be deduced from another tiling of Z by a sequence of flips (see [4, 7, 11] for details). This is an open question in the case of larger dimensions.

The point is now to study spaces of zonotopal tilings. Despite the fact that rhombic tilings are defined for any dimension, the figures are in dimension 2, for convenience.

2.3 Connections with oriented matroid theory

Tilings are connected with oriented matroids via the Bohne-Dress Theorem [14]. In oriented matroids, we work with sign vectors, i. e. elements of $\{+, -, 0\}^n$, where n is a fixed positive integer. Let X be such a sign vector. For $1 \leq k \leq n$, the k^{th} component of X is denoted by X_k, and we state $X^+ = \{k, X_k = +\}$, $X^- = \{k, X_k = -\}$ and $X^0 = \{k, X_k = 0\}$.

We do not recall classical definitions, which can be found in the reference book [1] A motivating example for the matroid notion is given by *realisable* matroids: Any sequence $V = (v_1, v_2, ..., v_k)$ of vectors of $(\mathbb{R}^d)^*$ induces the set of covectors $\mathcal{L}(V) = \{(sign(c.v_1), sign(c.v_2), ..., sign(c.v_k)), c \in \mathbb{R}^d\}$ (the product used is the classical scalar product in $\mathbb{R}^d$).

2.3.1 Oriented matroid induced by a zonotope

For the zonotope $Z(V, M)$, we use the sequence V_M of length s, obtained by first repeating m_1 times the vector v_1, then repeating m_2 times the vector v_2, and so on until repeating m_D times the vector v_D. The oriented matroid $\mathcal{L}(Z(V, M))$ is the realizable matroid $\mathcal{L}(V_M)$. It is a matroid of rank d.

We have a direct interpretation of $\mathcal{L}(Z(V, M))$: for each element $X = v'_1, v'_2, ..., v'_s)$ of $\mathcal{L}(Z(V, M))$ the set $Z_X = \sum_{i \in X_0} [-v'_i, +v'_i] + \sum_{i \in X_+} v'_i - \sum_{i \in X_-} v'_i$ is a face of $Z(V, M)$ and, conversely, for each face f of $Z(V, M)$ there exists a unique X of $\mathcal{L}(Z(V, M))$ such that $Z_x = f$.

2.3.2 Oriented matroid induced by a tiling

Given a tiling T, a sign vector of length s can be associated with each face of a tile of T. For each pair (v_i, j) such that $1 \leq j \leq m_i$, we state $k(i, j) = \sum_{1 \leq i' < i} m_{i'} + j$. Let f be a face and X^f denote the sign vector associated with the face f. We have

- $X^f_{k(i,j)} = +$ when f is contained in $T^+_{\{v_i,j\}}$,
- $X^f_{k(i,j)} = -$ when f is contained in $T^-_{\{v_i,j\}}$,
- $X^f_{k(i,j)} = 0$ otherwise.

The set of all these sign vectors is denoted by $\mathcal{O}(T)$.

The set $\mathcal{L}(T) = \{(X,+), X \in \mathcal{O}(T)\} \cup \{(-X,-), X \in \mathcal{O}(T)\} \cup \{(X,0), X \in \mathcal{L}(Z(V,M)\}$ is the set of covectors of an oriented matroid of rank $d+1$, which will be called the *tiling matroid*.

In matroid language, a flip is a specific mutation (see [1] sect. 7. 3) of the tiling matroids which only involves cocircuits corresponding with tiles (i. e. cocircuits whose last component is not null).

3 Decomposition and reconstruction

3.1 Deletions

3.1.1 Geometrical deletion

The deletion is a basic operation in matroid theory. We first present it in a geometrical point of view.

Let T be a tiling of support $Z(V,M)$, and $S_{\{v_i,j\}}$ be a de Bruijn zone of T. One can remove the tiles of $S_{\{v_i,j\}}$ and translate all the tiles of $T^+_{\{v_i,j\}}$ by the vector $-v_i$. For $D > d$, the configuration obtained is a tiling of $Z' = (V, M')$ where M' is defined by: $m'_i = m_i - 1$ and $\forall k \neq i, m'_k = m_k$ (except in the special case when $m_i = 1$, in such a case, we have $Z' = (V', M')$ with V' and M' respectively obtained from V and M removing the i^{th} component). Such an operation defines a *deletion* relation on zonotope tilings.

The tiling obtained is denoted by $D_{\{v_i,j\}}(T)$, and for each tile t of T, we state: $D_{\{v_i,j\}}(t) = t$ for t in $T^+_{\{v_i,j\}}$, and $D_{\{v_i,j\}}(t) = t - v_i$ for t in $T^+_{\{v_i,j\}}$.

For consistence, the de Bruijn zones of $D_{\{v_i,j\}}(T)$ according to v_i are assumed to be numbered $1, 2, \dots, j-1, j+1, \dots, m_i$. By this way, $D_{\{v_i,j\}}(t)$ and t both are in de Bruijn zones with the same label. We also need this convention for the commutativity below, when $v_i = v_k$.

3.1.2 Deletions in matroid theory

As it has been said above, deletions are classical in matroid theory. If $\mathcal{L}$ is a set of covectors of length n, then for each integer i such that $1 \leq i \leq n$, the set $\mathcal{L}/i = \{(X_1, X_2, ..., X_{i-1}, X_{i+1}, ..., X_n), X \in \mathcal{L}, X_i = 0\}$ i s also a set of covectors.

Thus, in matroid language, the geometrical deletion above consists in constructing $\mathcal{L}(T)/k(i,j)$ from $\mathcal{L}(T)$. Notice that $\mathcal{L}(T)/(s+1) = \mathcal{L}(Z(V,M))$. The Bohne-Dress theorem [14] claims that the only sets $\mathcal{L}$ of covectors of uniform matroids such that $\mathcal{L}/(s+1) = \mathcal{L}(Z(V,M))$ are tiling matroids.

3.2 Decomposition

Proposition 3.1 (commutativity of deletions) *Let T be a tiling of a zonotope Z , and two deletions $D_{\{v_i,j\}}$ and $D_{\{v_k,l\}}$. We have:*

$$D_{\{v_i,j\}}(D_{\{v_k,l\}}(T)) = D_{\{v_k,l\}}(D_{\{v_i,j\}}(T))$$

This proposition is obvious in the matroid framework, but following the principle of the paper, we give a geometrical proof.

Proof: The tiling T can be partitioned into the five parts below:

- $T^-_{\{v_i,j\}} \cap T^-_{\{v_k,l\}}$: the tiles of this part remain unchanged by the successive deletions, taken in any order,

- $T^+_{\{v_i,j\}} \cap T^-_{\{v_k,l\}}$: the tiles of this part are translated by $-v_i$ during the successive deletions, taken in any order,

- $T^-_{\{v_i,j\}} \cap T^+_{\{v_k,l\}}$: the tiles of this part are translated by $-v_k$ during the successive deletions, taken in any order,

- $T^+_{\{v_i,j\}} \cap T^+_{\{v_k,l\}}$: the tiles of this part are translated by $-(v_i+v_k)$ during the successive deletions, taken in any order.

- $S_{\{v_i,j\}} \cup S_{\{v_k,l\}}$: the tiles of this part are removed during the successive deletions, taken in any order.

Thus the order of deletions does not give any change. This gives the commutativity result. □

A tiling obtained from T by a sequence of p deletions is called a *$s-p$-minor* of T.

The pairs $\{v_i, j\}$ can be totally ordered (using the integer $k(i, j)$, for example). From this order, the sets $\{\{v_{i_1}, j_1\}, \{v_{i_2}, j_2\}, \ldots, \{v_{i_p}, j_p\}\}$ formed by p elements of the type $\{v_i, j\}$ can also be totally ordered. Therefore, the $s-p$-minors of T can be totally ordered. The *sequence of $s-p$-minors* of T is given by this order.

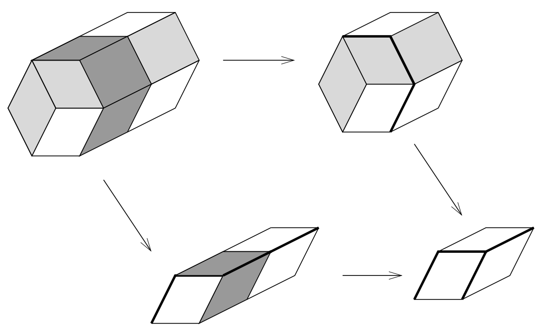

Figure 3: Commutativity of deletions.

Proposition 3.2 ((encoding)) *Assuming $s \geq d+2$, every tiling is defined by the sequence of its $s-1$-minors.*

Proof: Let Z be a zonotope, T one of its tilings. Notice that one can easily compute the sequence of $s-2$-minors of T from the sequence of its $s-1$-minors.

Let $\{v_i, j\}$ and $\{v_{i'}, j'\}$ be two distinct pairs, D_1 and D_2 respectively denote the corresponding deletions, and $D_{1,2}$ denote the corresponding double deletion.

For each tile t' of $D_{1,2}(T)$, one can easily compute the tiles t_1 such that t_1 is in $D_2(T)$ and $D_1(t_1) = t'$, and t_2 such that t_2 is in $D_1(T)$ and $D_2(t_2) = t'$. Precisely, one can compute the pair (ϵ_1, ϵ_2) of $\{0,1\}^2$ such that $t_1 = t' + \epsilon_1 v_i$ and $t_2 = t' + \epsilon_2 v_{i'}$.

Let t_0 be the tile of T such that $t_1 = D_2(t_0)$. From the commutativity, we also have: $t_2 = D_1(t_0)$ (see Figure 4), in such a way that $t' = D_2(D_1(t_0))$. Thus there exists a pair

(ϵ_3, ϵ_4) of $\{0,1\}^2$ such that $t_0 = t_1 + \epsilon_4 v_{i'}$ and $t_0 = t_2 + \epsilon_3 v_i$. thus by composition, we obtain: $t_0 = t' + \epsilon_1 v_i + \epsilon_4 v_{i'}$ and $t_0 = t' + \epsilon_2 v_{i'} + \epsilon_3 v_i$.

These equalities imply that $\epsilon_1 = \epsilon_3$ and $\epsilon_2 = \epsilon_{4)}$. when $v_i \neq v_{i'}$. When $v_i = v_{i'}$, these equalities are obtained by a case by case analysis according to the relative position of t_0 and the considered de Bruijn zones, as in the proof of Proposition 3.1 Thus we have the equality : $t_0 = t_2 + \epsilon_1 v_i = t' + \epsilon_1 v_i + \epsilon_2 v_{i'}$, which proves that t_0 can be computed from the triple (t', t_1, t_2).

This gives the result, since for each tile t of T, there are two distinct pairs $\{v_i, j\}$ and $\{v_{i'}, j'\}$ such that t is out of $S_{\{v_i,j\}} \cup S_{\{v_{i'},j'\}}$ (from the hypothesis: $s \geq d+2$). □

The same result can easily be obtained with chirotopes ([1] 3. 5) using matroid theory (especially the theorem from Lawrence ([1] 3. 5. 5) which characterizes oriented matroids by chirotopes): the knowledge of minors allows to compute the whole chirotope of the matroid whose set of covectors is $\mathcal{L}(T)$.

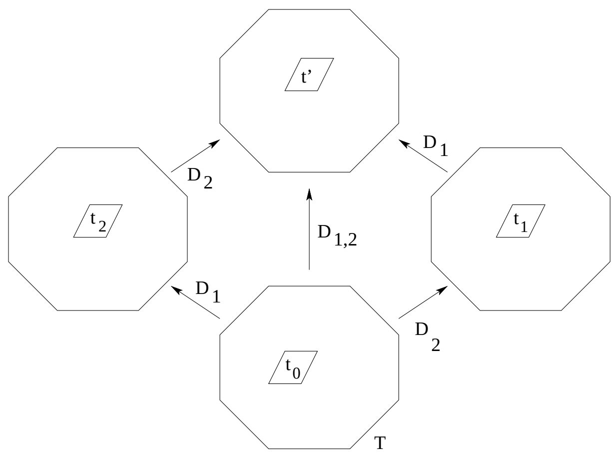

Figure 4: Proof of Proposition 3.2: computation of some tiles of T from $D_2(T)$ and $D_1(T)$.

Notice that the result is false for: $s = d+1$. Each d-minor is reduced to a single tile, thus the information about the arrangement of tiles is lost.

Iterating the proof for $(s-1)$-deletion, one obtains the following result as a corollary for proposition 3.2 (see Figure 3).

Corollary 3.3 *Let s' be a integer such that $d+1 \leq s' \leq s$. Assuming $s \geq d+2$, every tiling T of zonotope is defined by the sequence of its s'-minors.*

In particular, this is true for d+1-minors.

Proof: Obvious by induction. □

Remark that there are two kinds of d+1-minors: those of codimension 0, the *forced minors*, which are defined by the tiled zonotopes, and those of codimension 1, the *free minors*, which are tilings of unitary zonotopes. Only the free ones contain some information, useful to compute T. The encoded by a free minor information can be reduced to a single bit, corresponding to the tiling chosen (we recall that a $d+1$-unitary zonotope admits two tilings)

This gives an encoding of zonotope tilings by a word on the alphabet $\{0,1\}$ of length $\sum_{1\leq i_1<i_2<\ldots<i_d\leq D} m_{i_1}m_{i_2}\ldots m_{i_d}$ (see Figure 5 for an example).

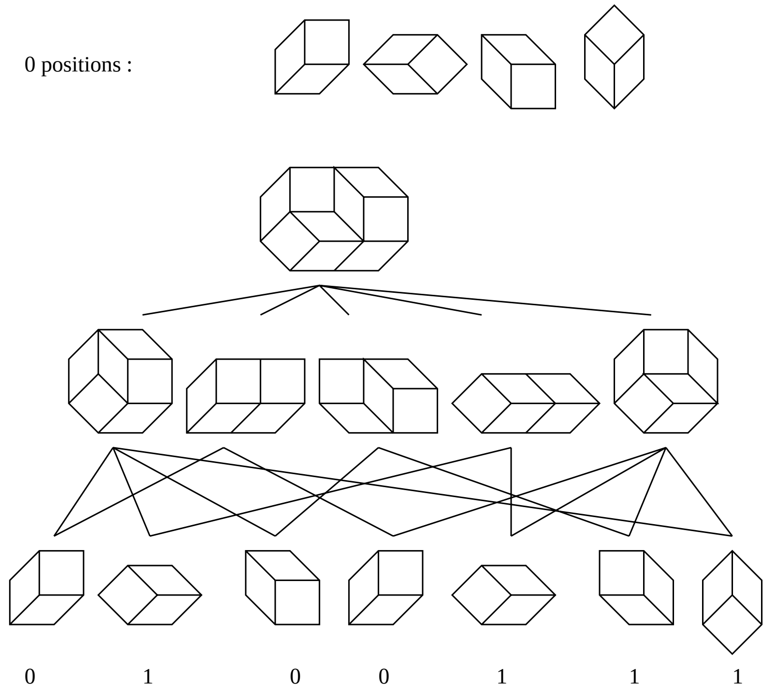

Figure 5: Coding of tilings with d+1-minors: each tiling of a $d+1$-zonotope is given a bit (above tilings corresponding to 0 bits) and the tiling is given by the sequence of bits of its minors

We define a *set flip* as follows: let T and T' be two tilings of a same zonotope such that all their d+1-minors are the same, except one. we say that T and T' differ by a set flip.

Proposition 3.4 *let T and T' be two tilings of a zonotope Z. T differs from T' by an set flip if and only if T differs from T' by a geometric flip.*

Proof: It is clear that a geometric flip is a set flip, because it changes locally the positions of $d+1$ tiles. Since only one d+1-minor contains all these tiles, their positions are changed only in this minor.

For the converse part, we first study how a deletion and a set flip act on a fixed de Bruijn line dBL. A deletion (which does not remove the whole de Bruijn line dBL) only removes one tile of dBL and does not change the order in this line for the other tiles. Thus a set flip changes the order on dBL if and only if dBL contains a pair $\{t,t'\}$ of tiles which appear in the flip. Moreover, in this case, the comparison order in dBL is changed only for the pair $\{t,t'\}$, since any other pair of tiles appears in a d+1-minor unchanged by the flip. Thus, for consistence of the order, the tiles t and t' necessarily share a whole facet. Thus the flip is actually geometric. □

This property is a special case of the equivalence between *mutations* in oriented matroids and *flippings* in arrangements of pseudospheres ([1] 7. 3) (we recall that the combinatorial structure of oriented matroids is equivalent to the topological structure of arrangement of pseudo sphere from the topological representation theorem from Folkman and Lawrence ([1] 1.4. 1)).

3.3 Reconstruction

We are interested in the following problem: given a zonotope Z and a sequence of d+1-tilings (with the good length, and the good vectors), does there exist a tiling T of Z such that the given sequence is the sequence of its d+1-minors ?

We can obviously solve the problem by constructing the (potential) d+2-minors, then the d+3-minors, and so on until the searched tiling is found. If there is a contradiction, the reconstruction is impossible, otherwise the tiling is obtained. But this can give the answer faster, and the following proposition states that the first step is enough to obtain the answer.

Proposition 3.5 *Let $(Zm)_m$ be a sequence of d+1-tilings. There exists a tiling T of a zonotope Z whose sequence of d+1-minors is exactly $(Zm)_m$ if and only if the d+2-minors are compatible, i.e. the sequence of d+2-minors can be correctly constructed.*

Proof: We do the proof by induction on the size s of the zonotope. The case $s = d + 2$ is obvious.

Let $s > d + 2$. Consider the prefix of the sequence $(Zm)_m$ formed by d+1-tilings where the tiles of the (potential) de Bruijn zone $\{v_D, m_D\}$ do not appear (since it is assumed that the deletion $D_{\{v_D,m_D\}}$ has been done). This subsequence is, by assumption, the sequence of d+1-minors of a tiling T' of size $s - 1$.

On the other hand, for each tile t of T', there exists a d+1-tiling T_t containing t and d tiles of the de Bruijn zone $\{v_D, m_D\}$. Hence t can be assigned a + or − sign, depending on its position in T_t, relatively to $S_{\{v_D,m_D\}}$ (+ if the D-located height function of t is 1, − if it is 0). Let T'^+ be the part of T' formed by the tiles marked + and T'^- the part formed by tiles marked −.

Let us now consider a straight line l directed by v_D. We claim that, following l in the sense of v_D, one first meets tiles marked −, then tiles marked +. This means that T'^+ and T'^- are convex along v_D, i.e. that the new de Bruijn zone can be inserted correctly in T', thus leading to a new tiling T. Two cases may occur:

- l only meets facets and interior parts tiles of T'. Consider two tiles of T', say t_1 and t_2, which share a facet, and such that t_2 follows t_1 in the succession of tiles crossed by l in the direction of v_D. There exists a d+2-minor T_{d+2} containing (tiles corresponding to) tiles of $\{v_D, m_D\}$ and tiles t_1 and t_2. There are only three possible sign assignment for (t_1, t_2), since the assignment: + for t_1 and − for t_2, is impossible ; otherwise the tile t_3 of type $\{\tau\} \cup \{v_D\}$ (where τ denotes the set of common vectors in the types of t_1 and t_2) cannot be placed in the d+2-minor T_{d+2} (see Figure 6).

- l meets a face f of the tiling T' of dimension lower than $d - 1$. Then, there are two tiles t_1 and t_2 with the same hypothesis as in the previous case, but sharing only the face f. There exists a parallel line l', arbitrarily close to l, satisfying the hypothesis of the previous case, and crossing both t_1 and t_2 (but t_1 and t_2 are not

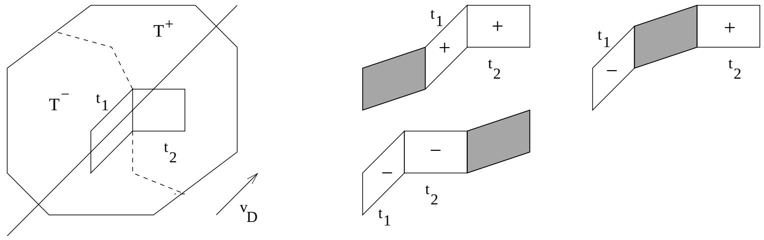

Figure 6: The three possible sign assignments for t_1 and t_2, from the possible positions in the de Bruijn line of T_{d+2}.

necessarily consecutive along l'. See Figure 7). Thus the assignment: $+$ for t_1 and $-$ for t_2, is impossible.

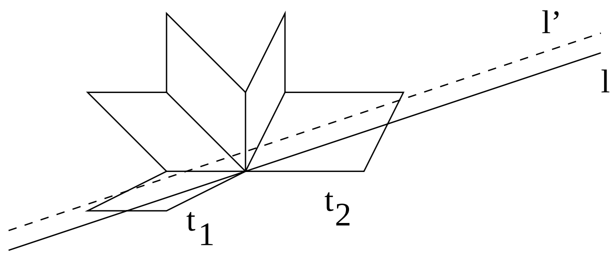

Figure 7: A line l crossing a vertex, and the auxiliary line l'

.

Hence T'^+ and T'^- are consistent according to $\{v_D, m_D\}$, allowing to translate the part T'^+ by v_D, in order to insert the de Bruijn zone $S_{\{v_D,m_D\}}$. The tiling T obtained (such that $T'^+ = T^+_{\{v_D,m_D\}} + v_D$ and $T'^- = T^-_{\{v_D,m_D\}}$) is the one searched, which ends the proof. □

The proposition above is very linked with the main axiom of the definition of chirotopes ([1] 3. 5. 3), which involves sets of $d+2$ indexes. An alternative strategy for proving the proposition is to check that chirotope axioms are satisfied, or, in a more direct way, to check the local realizability property ([1] 3. 6. 3)

4 Orders and representation

As seen formerly, the zonotopal tilings can be easily encoded by considering their minors. More precisely, one tiling is defined by the sequence of its (free) d+1-minors. We will now describe a representation tool for zonotope tilings based on the minor structure and the reconstruction proposition. But, before doing it, we have to defines the orders induced by flips, and we need some more knowledge about space of tilings of d+2-zonotopes.

4.1 Orders on tilings

If we have D vectors, we arbitrarily fix a basic tiling T_0 of the unitary D-zonotope. For each $d+1$-unitary zonotope, we define the *low position* as the d+1-tiling of this zonotope which is a d+1-minor of T_0. The other d+1-tiling is the *high position*.

Usually, the tiling T_0 chosen is constructed inductively as follows: given a sequence $(v_1, v_2, ...v_D)$ of vectors, we define for each integer i such that $d \leq i \leq D$ the tiling T_i of the unitary zonotope Z_i constructed on the i first vectors by :

- the tiling T_d is the unique tiling of Z_d
- for $d \leq i < D$, T_{i+1} is the unique tiling of Z_{i+1} such that $T^+_{\{v_{i+1},1\}}$ is empty and $D_{\{v_{i+1},1\}}(T_{i+1}) = T_i$.

With this definition, flips can be canonically directed: a flip is going upwards if it transforms a low position in a high position. The *directed* space of tilings is the space of tilings whose edges are directed as above. Obviously, it it is acyclic and defines a partial order relation, denoted by $<_{flip}$. Given a pair (T, T') of tilings, we have $T \leq_{flip} T'$ if one can pass from T to T' by a sequence of upwards flips.

We can also define another order relation, denoted by $<_{set}$. For each tiling T, we denote the set of its low $d+1$-minors by $low(T)$. Given a pair (T, T') of tilings, we have $T \leq_{set} T'$ if $low(T) \leq_{set} low(T')$. Obviously, $T \leq_{flip} T'$ yields $T \leq_{set} T'$.

We recall that tilings are single elements liftings of the zonotope matroid. By duality, there is a one-to-one correspondence between these liftings, and single element extensions of the dual matroid ([1] 3. 4). These extensions are studied in ([1] 7. 1) and can be canonically ordered. here, we introduce this order with a geometrical point of view.

The higher Bruhat orders are special cases of these orders for unitary zonotopes with a specific sequence V of vectors, such that, for each subsequence $(v_{i_1}, v_{i_2}, ..., v_{i_d})$, the determinant $det(v_{i_1}, v_{i_2}, ..., v_{i_d})$ is positive.

4.2 The basic d+2-zonotopes

In dimension d, there exists two basic kinds of d+2-zonotopes of dimension d whose tiling is not forced: either all vectors have multiplicity 1 (codimension 2), or there is one vector of multiplicity 2 (codimension 1). We first precisely study these cases. This should have been done easily in the madroid framework studying single element extensions of very basic matroids.

4.2.1 The d+2-zonotope of codimension 1

Proposition 4.1 *The space of tilings of the zonotope Z_i of codimension 1 with the vector v_i of multiplicity 2 (and the d other ones of multiplicity 1) contains three tilings and is a chain of length 2.*

Proof: In each tiling, there exists a unique tile t which is not element of $S_{\{v_i,1\}} \cup S_{\{v_i,2\}}$. Since $T^-_{\{v_i,1\}} \subseteq T^-_{\{v_i,2\}}$, we have three tilings:

- one tiling with $t \in T^-_{\{v_i,1\}}$,

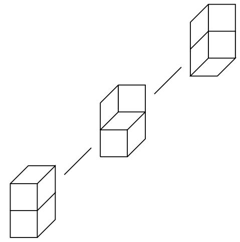

Figure 8: Space of tilings of a codimension 1 zonotope with one duplicated vector.

- one tiling with $t \in T^{+}_{\{v_i,2\}}$,
- one tiling with $t \in T^{-}_{\{v_i,2\}} \setminus T^{-}_{\{v_i,1\}}$.

The directed edges corresponding to flips are obvious (remark that both the free $d{+}1$-minors of Z_i are of the same type, which gives the chain) (see Figure 8). $\square$

4.2.2 The unitary $d{+}2$-zonotope

We first need more information about the structure of tilings of unitary $d{+}1$-zonotopes. This is given by the lemma below.

Lemma 4.2 *Let T be a tiling of unitary $d{+}1$-zonotope, and v be a vector not in the type of Z. We define a tournament $G_{(T,v)}$ on the tiles of T saying that (t_1, t_2) is an arc of $G_{(T,v)}$ if the vector v crosses their common facet passing from t_1 to t_2 (see Figure 9).*

The tournament $G_{(T,v)}$ is actually a total order.

Proof: Since all pairs of tiles are linked, we only have to prove that $G_{(T,v)}$ has no cycle of length 3. We prove it reducing the problem to the case: $d = 2$, for which the proof is easy by a case by case analysis.

In higher dimension, notice that, since $D = d + 1$, the types of three given tiles t_1, t_2 and t_3 contain $(d + 1) - 3 = d - 2$ common vectors. Let p denote the orthogonal projection on the 2-dimensional space which is orthogonal to the $d - 2$ common vectors. The projections $p(t_i)$ are parallelograms, and we have (t_i, t_j) in $G_{(T,v)}$ if and only if $(p(t_i), p(t_j))$ is in $G_{H,p(v)}$; where H denotes the hexagon covered by the parallelograms $p(t_i)$ (H is really a hexagon, since otherwise the tiles t_i cannot be pairwise adjacent). This gives the result, since $G_{p(v)}$ is not a cycle. $\square$

Proposition 4.3 *The space of tilings of a unitary $d{+}2$-zonotope is a cycle of length $2(d + 2)$, and each possible label is given to a pair of edges, which are opposite in the cycle.*

Proof: let T be a tiling of the unitary zonotope $Z = Z((v_1, v_2, \ldots, v_{d+2}), (1, 1, \ldots, 1))$. From the above lemma applied on the support Z' of $D_{\{v_{d+2},1\}}(T)$, $T^{-}_{\{v_{d+2},1\}}$ is an initial segment of the order induced by v_{d+2} on tiles of $D_{\{v_{d+2},1\}}(T)$.

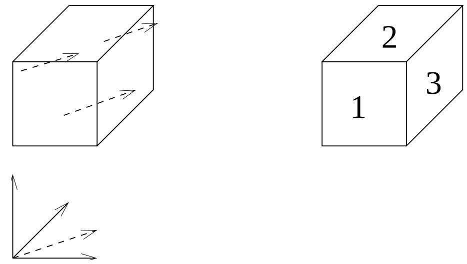

Figure 9: A codimension 1 tiling, the added vector (dashed), and the ordering of tiles according to this vector.

Conversely, given a tiling T' whose support is Z', and an initial segment T' (according to the order induced by v_{d+2}), one easily constructs a tiling of Z: tiles which are not in the initial segment are translated by v_{d+2}, and tiles of $S_{\{v_{d+2},1\}}$ are inserted in the remaining space. There exists $d+2$ possible initial segments for a set of $d+1$ elements, thus, since Z' admits two tilings, there exists $2(d+2)$ tilings of Z.

Now, take a tiling of T, i. e. a tiling T' of the zonotope Z' and I one initial segment of it. What are the possible flips from T ? First assume that the initial segment is proper (i. e. neither empty nor equal to T'). There are two possible flips, which correspond to adding or removing one tile in I. No other flip is possible because of the relative position of tiles given by the order on tiles of T' (the flip only using tiles of T' is not possible because of the cut by $S_{\{v_{d+2},1\}}$).

A similar argument holds for the other case. If I is empty, two flips are possible, one which corresponds to adding the first tile in I, the other one only uses tiles of T'. If $I = T'$, two flips are possible, one which corresponds to removing the last tile in I, the other one only uses tiles of T'. This gives the result, using the symmetry of both tilings of Z' to get the labels of opposite edges. $\square$

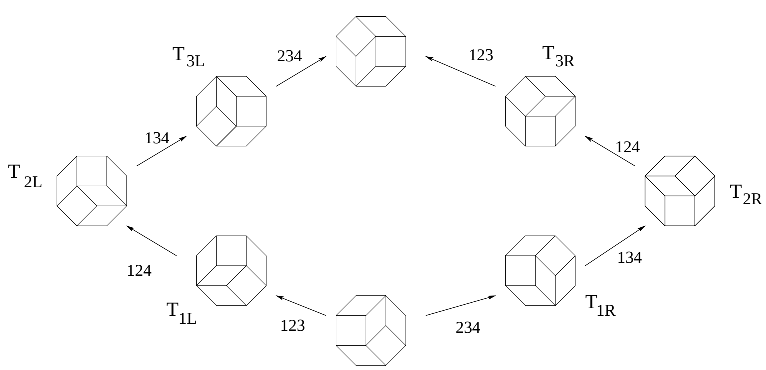

Figure 10: The order associated with a unitary octagon.

4.3 Tiling diagrams

We can now precisely explain how we represent a fixed tiling T. There exists some related representations in literature [18] about matroids, which are consequences of the topological representation theorem for matroids ([1] 1. 4. 1) and of the use of signatures of single element extensions [1] 7. 1). A specific originality of our representation consists in the use of arrows, to treat parallelism.

4.3.1 Points

In our representation, each d+1-minor is associated to a *point*. Each point p is defined by two parameters. We first have a *coordinate vector*, element of $\mathbb{Z}^D$, which indicates the position of the d+1-minor in the sequence of minors: the i^{th} component, denoted by $i(p)$, of this vector is equal to j if the deletion according to the pair $\{v_i, j\}$ has not been done to obtain the corresponding d+1-minor; the component $i(p)$ is null if, for each integer j such that $1 \leq j \leq m_i$, the deletion according to $\{v_i, j\}$ have been done (thus there exists exactly $d+1$ non-null component). The type of the corresponding flip is the support of the coordinate vector.

Remark that a similar coordinate vector will also be given to each d'-minor whose support is a unitary zonotope: the only difference is that there are d' non-null components.

For each such a unitary zonotope $Z_{d'}$, we define the *space* of points associated with $Z_{d'}$ as the set of points which corresponds to minors of $Z_{d'}$. We say that $d'-d-1$ is the dimension of this space. In particular, for $d' = d+2$, we speak of the *line* associated with Z_{d+2}. In order to justify our vocabulary, we can see that each line is defined by two points (but, unfortunately, each pair of points does not always define a line).

The other parameter is a *color*, which is white if the d+1-minor is in low position, or black if in high position.

The important thing for reconstructing a tiling T is the set of coloring constraints which are given by the sequence of d+2-minors. We now explain how coloring constraints are expressed.

4.3.2 Arrows

Two points correspond to the pair of minors of a same d+2-minor of T (the support of this d+2-minor is a d+2-zonotope of codimension 1) if and only if they only differ by one non-null coordinate.

From what has been seen about these d+2-tilings, there exists exactly three allowed colorings of such a pair of points, corresponding to tilings of a d+2-zonotope, and a forbidden coloring corresponding with no tiling.

The forbidden coloring uses both colors. In the diagram, an arrow is placed, linking these two points, in such a way that the origin of the arrow is white (and the tail is black) in the forbidden coloring. Thus, the three allowed colorings of the tiled d+2-zonotope are the fully black one, the fully white one, and the coloring with the origin of the arrow being black and the tail being white. This gives the *first constraint*: there is no edge from a white point to a black point.

Each arrow is labeled by the index of the coordinate which is different for the points linked.

The arrows of the diagram give the *covering relation*: a point p is covered by a point p' if there exists an arrow such that p is the origin of the arrow, and p' its endpoint.

4.3.3 Lines

Now, consider a $d{+}2$-minor of T whose support is a unitary $d{+}2$-zonotope. A point corresponds to a $d{+}1$-minor of this $d{+}2$-tiling if and only if its coordinate vector is obtained replacing one non-null coordinate of the $d{+}2$-minor by 0.

Such points form a line which can be totally ordered in the same way as flips are ordered in a path between the lowest tiling of the unitary $d{+}2$-zonotope to its largest tiling. (there are two possible opposite possible sequences, but this fact is not relevant, one of these can be chosen arbitrarily. We will choose later a specific order in a special case). From what has been seen about tilings of unitary $d{+}2$-zonotopes, with the order convention, the black points have to form a final or initial segment (i. e. a suffix or a prefix) of the line. This is the *second constraint*.

Hence, tilings of zonotopes are presented as *diagrams* on which lines represent unitary $d{+}2$-zonotopes, and arrows represent $d{+}2$-zonotopes of codimension 1 (see Figure 11). Notice that arrows and lines only depend on the support of the tiling, i. e. two tilings with the same support induce the same sets of lines and arrows. The translation of the reconstruction theorem gives the following result:

Proposition 4.4 *A coloring of points of a diagram induces a tiling if and only if it respects the two constraints stated above.*

Notice that the highest diagram (i. e. with all points black) and the lowest one (i. e. with all points white) are tilings. We denote the set of black points of the diagram of T by B_T. Hence, for each pair (T, T') of tilings, we have $T \leq_{set} T'$ if and only if $B_T \subseteq B_{T'}$.

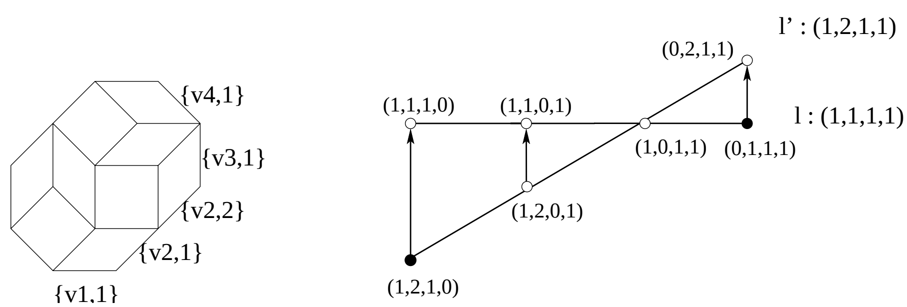

Figure 11: A tiling and the associated diagram (notice the orientation of the arrows, according to the inversion property).

4.4 Recursivity properties of diagrams

Diagrams have some interesting recursivity properties, which are very useful for induction arguments.

4.4.1 Deletions

Let T be a tiling of a zonotope Z, (v_i, j) be a pair defining a de Bruijn section of T, and T' be the tiling such that $T' = D_{\{v_i,j\}}(T)$. The diagram of T' is obtained from the diagram of T by removing all the points whose i^{th} coordinate is equal to j. The remaining line and arrow relations are preserved (we take the convention that the tiling $T'_0 = D_{\{v_i,1\}}(T_O)$ is used to define the sense of flips).

4.4.2 Contractions

Let T be a tiling of a zonotope Z , (v_i, j) be a pair defining a de Bruijn section of T. Let q_i be the orthogonal projection on the hyperplane $\{v_i\}^{\perp}$.

For $d > 3$, we define the contracted tiling $T' = C_{\{v_i,j\}}(T)$ as follows: T' is the tiling of formed by the set $\{q_i(t), t \in S_{\{v_i,j\}}\}$. The support of this tiling is the zonotope $Z' = (V', M')$ with $V' = (q_i(v_1), q_i(v_2), ..., q_i(v_{i-1}), q_i(v_{i+1}), q_i(v_{i+2}), ..., q_i(v_D))$ and $M' = (m_1, m_2, ..., m_{i-1}, m_{i+1}, m_{i+2}, ..., m_D)$. Hence, T' is a tiling of dimension $d-1$, codimension $D - d$, and size $s - m_i$. The contraction geometrically defined above is also a classical operation on matroids.

How can we find the diagram of T' from the diagram of T ? We have to keep only all points whose i^{th} coordinate is j. Each line either disappears (if its i^{th} coordinate different from j) or looses a point. Arrow relations are preserved between all the points which are kept (for orientation convention, the lowest tiling which defines the low positions is $T''_0 = C_{\{v_i,1\}}(T_0)$).

4.5 Consistence properties of arrows

We now give two propositions about the relative positions of arrows.

4.5.1 Same type property

Proposition 4.5 *Let p, p', p_1 and p'_1 be four points of a diagram, such that:*

- *there exists an integer i such that $0 < i(p) < i(p')$ and $0 < i(p_1) < i(p'_1)$*
- *for each integer j of $\{1, 2, \ldots, D\}$ such that $j \neq i$, we have: $j(p) = j(p')$ and $j(p_1) = j(p'_1)$. Moreover, if $j(p) = 0$, we have $j(p_1) = 0$.*

There exists an arrow from p to p' if and only if there exists an arrow from p_1 to p'_1.

Proof: This is obvious: the support of the tiling whose minors are p and p' is a $d + 2$-zonotope with mulitiplicity 2 for v_i, and it is also the tiling whose minors are p_1 and p'_1.

The inequalities: $i(p) < i(p')$ and $i(p_1) < i(p'_1)$ indicate that p and p_1 both give tilings of minors constructed by the same deletion, as p' and p'_1 also do. □

4.5.2 Inversion property

Proposition 4.6 (inversion property) *Let $l = (p_1, p_2, \ldots, p_{d+2})$ and $l' = (p'_1, p'_2, \ldots, p'_{d+2})$ be two distinct lines such that there exists a unique integer k such that $p_k = p'_k$.*

Assume p_1 is covered by p'_1. For any integer j such that $1 \leq j < k$, p_j is covered by p'_j, and for any integer j such that $k < j \leq d+2$, p'_j is covered by p_j (see Figure 11 for an illustration of this property).

Of course, a similar property holds when it is assumed that p_{d+2} is covered by p'_{d+2}.

Proof: With the notations above, the coordinates vectors of l and l' have the same positions for non-null coordinates, that we denote by $i_1, i_2, \ldots..$ and i_{d+2}. Moreover, all these non-null coordinates are equal except for one position. We denote this unique position is i_0.

Consider the d+2-minor whose sequence of minors corresponds to points p_j and p'_j. Its support Z_0 has codimension 2, its multiplicity is 2 according to the vector v_{i_0}, and its multiplicity is 1 according to any vector other vector appearing in the type of at least a point.

Consider the tiling $T_{wh.}$ of Z_0 corresponding to the fully white coloring. This tiling has two minors of codimension 2 which are obtained by a deletion according to v_i. By definition, both these minors are equal to T_0. That means that there is no tile between both de Bruijn sections according to v_{i_0} of $T_{wh.}$.

By a sequence of flips, one can move the de Bruijn sections in such a way that each tile (whose type does not contain v_i) of the resulting tiling T_{inside} is between both de Bruijn sections according to v_{i_0} (see Figure 12).

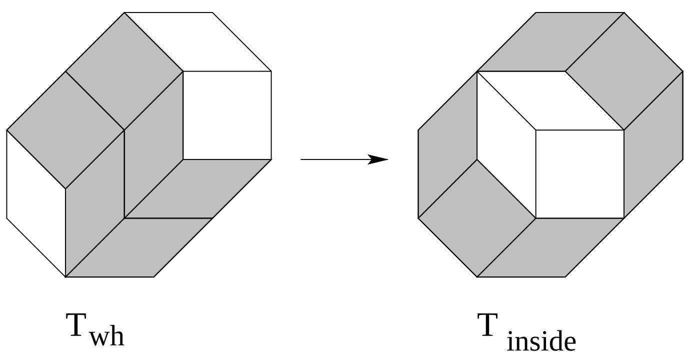

Figure 12: The tilings T_{wh} and T_{inside}.

The set of black points of T_{inside} is necessarily one of the sets: $\{p_1, p_2, \ldots, p_{i-1}, p'_{i+1}, p'_{i+2}, \ldots, p'_{d+2}\}$ or $\{p'_1, p'_2, \ldots, p'_{i-1}, p_{i+1}, p_{i+2}, \ldots, p_{d+2}\}$. But the second set is not possible, from our assumption about the arrow from p_1 to p'_1 (if p'_1 is black, then p_1 is necessarily black).

Thus the set of black points is $\{p_1, p_2, \ldots, p_{i-1}, p'_{d+2}, p'_{d+1}, \ldots, p'_{i+1}\}$, which forces the sense of arrows, and gives the result. □

5 Structures for lexicographic sequences of vectors

We say that sequence $(v_1, v_2, ..., v_D)$ is *lexicographic* if for each line l, defined by a subsequence of $d+2$ vectors, the set of points of l is ordered according to the lexicographic order of types.

Notice that the operations of contraction and deletion both preserve the lexicographic property.

We have seen that in codimension 2 , one can assume without loss of generality that the given sequence is lexicographic. This is not true in the general case.

Nevertheless, we can prove there exists a lexicographic sequence for any value of the parameters d and D. It suffices to take a sequence $(v_1, v_2, ..., v_D)$ such that, for each subsequence $(v_{i_1}, v_{i_2}, ..., v_{i_d})$, the determinant $det(v_{i_1}, v_{i_2}, ..., v_{i_d})$ is positive (see details in Annex). For example, this can be done fixing an increasing sequence $(x_1, x_2, ..., x_D)$ of positive distinct reals numbers. We define v_i by $v_i = (1, x_i, x_i^2, ..., x_i^{d-1})$. Thus for each subsequence $(v_{i_1}, v_{i_2}, ..., v_{i_d})$, the determinant $det(v_{i_1}, v_{i_2}, ..., v_{i_d})$ is a Vandermonde determinant, which is equal to $\Pi_{1 \leq i < j \leq d}(x_j - x_i)$, and, therefore, is positive. In the unitary case, the order of the directed space of tilings is isomorphic to a higher Bruhat order [18].

Our main theorem about lexicographic sequences of vectors is stated below. It is a generalization of a theorem of Ziegler about higher Bruhat orders [18]. The idea is that, even if Theorem **??** is false in the lexicographic case (Ziegler gives a counterexample in [18]), there still remains a lot of possibilities to pass from the lowest tiling to the highest one by a sequence of upwards flips.

Theorem 5.1 *Let T be a tiling of a zonotope constructed on an lexicographic sequence of vectors. There exists a sequence of downwards flips which can be done from T, satisfying the following properties:*

- *after the sequence is done, all points are white*
- *the sequence can be decomposed into two parts, the first part only contains flips whose type contains v_D, and the second part only contains flips whose type does not contain v_D.*

5.1 Arrow properties in the lexicographic case

In order to prove the theorem above, we first need the proposition below.

Proposition 5.2 *Let p and p' be two points linked by an arrow from p to p', labeled by D. Assume that the last coordinate $D(p)$ of p is lower than the last coordinate $D(p')$ of p'. Then, for each pair (p_1, p'_1) such that there exists an arrow from p_1 to p'_1 labeled by D, we have $D(p_1) < D(p'_1)$.*

Proof:. For each pair (q, q') of points, we denote by $\Delta(q, q')$ the number of coordinates which are null for q and non-null for q'. We use an induction on $\Delta(p, p_1)$. If $\Delta(p, p_1) = 0$, the result is just Proposition 4.5.

Otherwise, there exists a coordinate i which is null for p and not for p_1, and, in a symmetric way, a coordinate j which is null for p_1 and not for p.

Let p_2 (respectively p'_2) be the point whose i^{th} coordinate is null, j^{th} coordinate is the j^{th} coordinate of p, and other coordinates are equal to coordinates of p_1 (respectively p'_1). We have $\Delta(p, p_2) = \Delta(p, p_1) - 1$

Let l (respectively l') be the line passing through p_1 and p_2 (respectively p'_1 and p'_2). These two lines meet each other in a point p_0, whose last component is null. Since the sequence of vectors is acyclic, the point p_0 is an endpoint of lines p and p'. Applying the inversion property, we obtain that there exists an arrow from p_2 to p'_2 labeled by D, Thus, by induction hypothesis, we have $D(p_2) < D(p'_2)$, i. e. $D(p_1) < D(p'_1)$ □

From the above proposition, up to symmetry, it can be assumed that, for each arrow labeled by D, the last coordinate of the origin point of the arrow is lower than the last coordinate of its tail point.

5.2 Secondary arrows

We now introduce some other arrows, which will be called *secondary arrows*, as follows: let $l = (p_1, p_2, ..., p_{D+2})$ be a line. We assume that the point p_1 is the point of lowest label in l. This point is called the *directing point* of l.

- if p_1 is black in T, then we have a secondary arrow from p_i to p_{i+1}, for each integer i such that $1 \leq i < D + 2$,
- if p_1 is white in T, then we have a secondary arrow from p_{i+1} to p_i, for each integer i such that $1 \leq i < D + 2$.

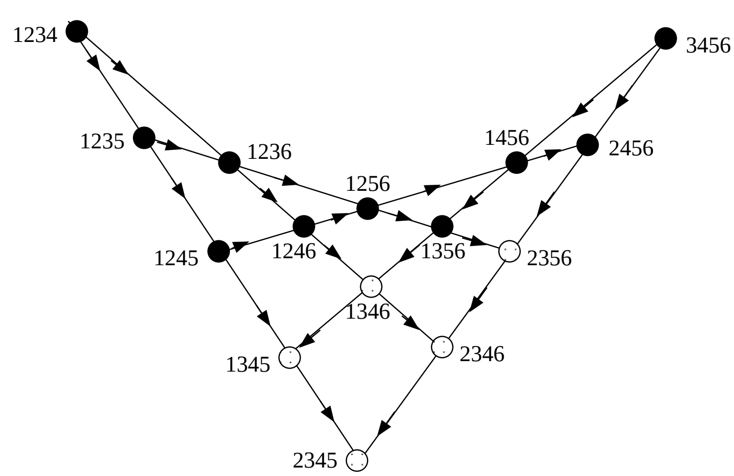

Figure 13: orientation of lines for a tiling of a lexicographic unitary zonotope of codimension 3 and dimension 3 (non-null coordinates are given for each point).

Secondary arrows are not labeled. They strongly depend on the tiling T, at the opposite of the primary arrows introduced previously. They encode a sense for each line. Notice that the arrow constraint is still satisfied: there is no (secondary) arrow starting in a white point and finishing in a black point. A primary arrow links two points corresponding to tilings of the same zonotope, while a secondary arrow links two points corresponding to tilings of different zonotopes.

To avoid confusion, we say that the diagram with secondary arrows is the *enriched diagram.*

Proposition 5.3 *Let T be a tiling. Each directed cycle of black points in the enriched diagram of T is reduced to a single point.*

Proof The proof is done by induction on the size s of the zonotope. The result is obvious for $s \leq d$: there is no line in the diagram, thus the result directly comes from Proposition 4.5. This gives the initialization. Now, we assume that the size of the zonotope is at least $d+1$. We have two cases. In each of them, we introduce a partition of points according to the two last de Bruijn zones.

1) if $m_D \geq 2$, we define:

- $A_0 = \{p \,|\, D(p) \leq m_D - 2\}$
- $A_1 = \{p \,|\, D(p) = m_D - 1\}$
- $A_2 = \{p \,|\, D(p) = m_D\}$

If a black directed cycle remains in $A_0 \cup A_1$, then it can be seen as a cycle of the diagram of $D_{\{v_D, m_D\}}(T)$ which gives the result by induction. Otherwise, Let p_2 be a point in A_2 : there is no primary arrow from p_2 to any point p outside of A_2, since such an arrow should be labeled by D, this would contradict Proposition 5.2. On the other hand, each line passing though p_2 has its directing point in A_0. Thus, if a black directed cycle has a point in A_2, then the cycle is necessarily contained in A_2 (since there is no possibility to have an arrow, in the cycle, starting in A_2 and finishing out of A_2). Thus this cycle can be seen as a cycle of the diagram of $D_{\{v_D, m_D-1\}}(T)$ which gives the result by induction.

2) if $m_D = 1$, we define:

- $A_0 = \{p \,|\, (D-1)(p) < m_{D-1} \text{ and } D(p) = 0\}$
- $A_1 = \{p \,|\, (D-1)(p) = m_{D-1} \text{ and } D(p) = 0\}$
- $A_2 = \{p \,|\, (D-1)(p) < m_{D-1} \text{ and } D(p) = 1\}$
- $A_3 = \{p \,|\, (D-1)(p) = m_{D-1} \text{ and } D(p) = 1\}$

Let p_2 be a point in A_2: there exists exactly one line passing though p_2 whose directing point is a point p_1 of A_1. The other lines passing through p_2 all have their directing points in A_0. Let p_3 be a point in A_3: all the line passing though p_3 have their directing points in A_1.

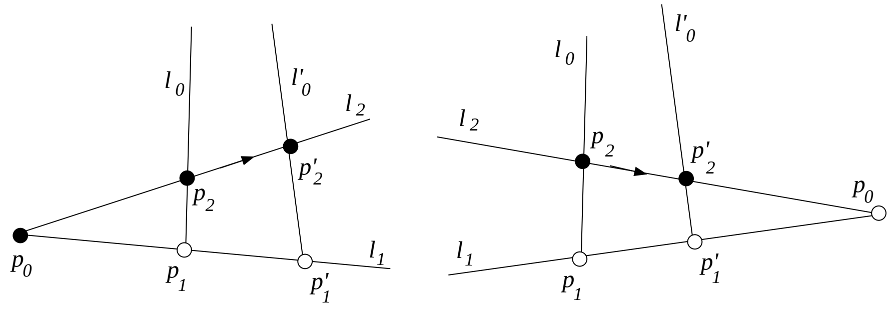

Figure 14: The two possible cases, induced by the choice of the color of p_0. The color of p_1' is forced by colors of other points

Thus, If a black directed cycle has a point in $A_0 \cup A_1$ then it remains in $A_0 \cup A_1$ (since there is no way for the cycle to pass from $A_2 \cup A_3$ to $A_0 \cup A_1$). Thus this cycle it can be seen as a cycle of cycle of the diagram of $D_{\{v_D,1\}}(T)$ which gives the result by induction.

Now take a directed black cycle contained in $A_2 \cup A_3$. let. p_2 be a point of A_2 in this cycle. Assume that the point p_1 of A_1, such that there exists a line l_0 passing through p_1 and p_2, is white. Le p_2' the successor of p_2 in the cycle. By definition of line orientations, p_2' cannot be a point of l_0, thus p_2' is element of A_2. Let p_1' be the point of p_1 of A_1 such that there exists a line l_0' passing through p_1' and p_2'. There exists a line l_2 passing through p_2 and p_2', thus, by an elementary counting of same coordinates, we see that there exists a line l_1 passing through p_1 and p_1'. Moreover, l_1 and l_2 have a common directing point p_0 in A_0, and p_1 is between p_0 and p_1' if and only if p_2 is between p_0 and p_2' (see figure 14). If this last condition is realized, then p_0 is necessarily black; otherwise, p_0 is white. The disposition above ensures that, in any case, the point p_1' is white (there are two cases according to the color of p_0).

Assume that the cycle has (at least) a point in A_2 and a point in A_3. The cycle has to pass at least once from A_3 to A_2, which forces that there exists a point p_2 satisfying the hypothesis above. Thus, by induction, the following points in the cycle all are in A_2, which is is a contradiction. Thus we have two alternatives, described below :

- either the cycle is contained in A_2. Thus it is a cycle of the enriched diagram of $D_{\{v_{D-1},m_{D-1}\}}(T)$ (notice that all the lines used by the cycle are contained in $A_2 \cup A_0$, thus they are lines of $D_{\{v_{D-1},m_{D-1}\}}(T)$),

- or the cycle is contained A_3, Thus it is a cycle of the enriched diagram of $C_{\{v_{D-1},m_{D-1}\}}(T)$ (notice that all the lines used by the cycle are contained in $A_3 \cup A_1$, thus, for such a line, a line of $C_{\{v_{D-1},m_{D-1}\}}(T)$) with the same directing point is obtained by removing the second (in the lexicographic order) point).

In any case, we can apply an induction process. $\square$

5.3 End of the proof

We now have the tools to easily prove Theorem 5.1

Proof: (Th. 5.1) Let (p, p') be a pair of points, on the same line l, such that the type of p does not contain v_D and the type of p' contains v_D. Assume that p' is black. The point p is necessarily the directing point of l, thus if p is white, then we have a secondary arrow from p' to p, and if p is black, then we have a secondary arrow from p to p'. This yields that, in the enriched diagram, if a path of black points starts in a point whose type contains v_D, then all the types of points of this path contain v_D. Moreover, such a path is finite from Proposition 5.3. Therefore, if the enriched diagram of T contains a black point whose type contain v_D, then the enriched diagram contains a black point whose type contain v_D which is covered by no other black point. This last point can be turned in white to get another tiling. This operation can be repeated until there is no more black point whose type contain v_D.

Afterwards, one can select a black point which is not covered by another black point. This point can be turned in white to get another tiling. This operation can be repeated until there is no more black point . $\square$

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Annex

Proposition 5.4 *let $(v_1, v_2, ..., v_D)$ be a sequence of vectors of dimension d such that, for each subsequence $(v_{i_1}, v_{i_2}, ..., v_{i_d})$, the determinant $det(v_{i_1}, v_{i_2}, ..., v_{i_d})$ is positive. This sequence is lexicographic.*

Proof:. For each subsequence $(v_{i_1}, v_{i_2}, ..., v_{i_d}, v_{i_{d+1}}, v_{i_{d+2}})$, We will study the tournament (introduced in 4.2.2) induced by $v_{i_{d+2}}$ on tiles on the lowest tiling T of the unitary zonotope constructed on the sequence $(v_{i_1}, v_{i_2}, ..., v_{i_d}, v_{i_{d+1}})$. There is no loss of generality in only studying the tournament induced by v_{d+2} on the low tiling of the unitary zonotope Z constructed on $(v_1, v_2, ..., v_d, v_{d+1})$.

Each tile of Z can be denoted by $t_{\bar{i}}$, where i is the unique integer of $\{1, 2, ..., d+1\}$ such that v_i does not appear in the type of $t_{\bar{i}}$.

For each pair (i, j) of distinct integers, we denote by $\{f^+_{(\bar{i},\bar{j})}, f^-_{(\bar{i},\bar{j})}\}$ the pair of of facets of $t_{\bar{i}}$ such that $f^-_{(\bar{i},\bar{j})} + v_j = f^+_{(\bar{i},\bar{j})}$. This common facet $t_{\bar{i}} \cap t_{\bar{j}}$ is denoted by $f_{\bar{i}\cap\bar{j}}$. Hence, either $f^+_{(\bar{i},\bar{j})} = f_{\bar{i}\cap\bar{j}}$ or $f^-_{(\bar{i},\bar{j})} = f_{\bar{i}\cap\bar{j}}$.

From the definition of the lowest tiling , $t_{\overline{d+1}}$ is in $T^-_{\{v_{d+1},1\}}$. Thus the facet $f_{\overline{d+1}\cap\bar{i}}$ is $f^-_{(\bar{i},\overline{d+1})}$.

For any facet f of T and any vector v, we introduce the value $sign(f, v)$ as the the sign (seen as an element of $\{-1, 1\}$ for convenience) of the determinant $det(S, v)$ where S denotes the sequence of vectors of the type of f (sorted according to the increasing indexes). Informally, $sign(f, v)$ is a tool to know in what sense the vector v passes through the facet f. For each vertex v and each pair (i, j), we have $sign(f^+_{(\bar{i},\bar{j})}, v) = sign(f^-_{(\bar{i},\bar{j})}, v)$, thus we can canonically define $sign(f_{(\bar{i},\bar{j})}, v)$.

From our hypothesis on determinants, we have: $sign(f, v_{d+1}) = sign(f, v_{d+2}) = 1$ for any facet of $t_{\overline{d+1}}$. Moreover, $sign(f_{(\overline{d+1},\bar{i})}, v_i) = (-1)^{d-i}$ since vectors of the corresponding determinant can be ordered in a lexicographic way by moving v_i leftwards by a sequence of $d - i$ transpositions.

We now study the case when $d - i$ is odd, (the case when $d - i$ is even is treated in a similar way). For $d - i$ odd, $sign(f_{(\overline{d+1},\bar{i})}, v_i) = -sign(f_{(\overline{d+1},\bar{i})}, v_{d+1})$, which means that v_i and v_{d+1} pass through the face $f_{\overline{d+1}\cap\bar{i}}$ in opposite senses. thus, since $f_{\overline{d+1}\cap\bar{i}} = f^-_{(\bar{i},\overline{d+1})}$, we necessarily have $f_{\overline{d+1}\cap\bar{i}} = f^-_{(\overline{d+1},\bar{i})}$. For $i < d$, we also have $f_{\overline{d+1}\cap\overline{i+1}}$ is $f^+_{(\overline{d+1},\overline{i+1})}$, by the same argument (using that $sign(f_{(\overline{d+1},\overline{i+1})}, v_{i+1}) = sign(f_{(\overline{d+1},\overline{i+1})}, v_{d+1})$.

The facet $f_{\bar{i}\cap\overline{i+1}}$ contains the $(d-2)$-face $f_{\overline{d+1}\cap\bar{i}} \cap f_{\overline{d+1}\cap\overline{i+1}} = f^-_{(\overline{d+1},\bar{i})} \cap f^+_{(\overline{d+1},\overline{i+1})}$. Thus $f_{\bar{i}\cap\overline{i+1}}$ is necessarily $f^+_{(\bar{i},\overline{i+1})} = f^-_{(\overline{i+1},\bar{i})}$:

let v be a vector in $f_{\bar{i}\cap\overline{i+1}}$; the vector v is $f^+_{(\bar{i},\overline{i+1})}$, i . e. there exists a vector v' in $t_{(\bar{i}}$ such that $v' + v_{i+1} = v$. This enforces that v is in $f^+_{(\bar{i},\overline{i+1})}$ (one proves by a symmetric way that v is in $f^-_{(\overline{i+1},\bar{i})}$).

Moreover, $sign(f_{(\bar{i},\overline{i+1})}, v_i) = sign(f_{(\bar{i},\overline{i+1})}, v_{i+1}) = (-1)^{d-i}$ since the corresponding determinant can be ordered in a lexicographic way by moving v_i (or v_{i+1}) leftwards by a sequence of $d-i$ transpositions. Thus, $sign(f_{(\bar{i},\overline{i+1})}, v_i) = -sign(f_{(\bar{i},\overline{i+1})}, v_{d+2})$ which yields that v_{d+2} and v_i pass through the facet $f_{\bar{i}\cap\overline{i+1}}$ in opposite senses. Thus the vector v_{d+2} passes through $f_{\bar{i}\cap\overline{i+1}}$ from $t_{\overline{i+1}}$ to $t_{\bar{i}}$.

In other words, we have: $t_{\overline{i+1}} < t_{\bar{i}}$ if the order induced by v_{d+2}. One can prove the same result for i even. Thus, we have $t_{\overline{d+1}} < t_{\bar{d}} < .... < t_{\bar{2}} < t_{\bar{1}}$. This yields that the sequence of flips successively labeled by $(\bar{1}, \bar{2}, ..., , \overline{d+2})$ can be done, starting from the lowest tiling of the unitary zonotope induced by $(v_{i_1}, v_{i_2}, ..., v_{i_{d+2}})$, which is the result. $\square$