The Infinite Versions of LogSpace!=P Are Consistent with the Axioms of Set Theory.

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Abstract

We consider the infinite versions of the usual computational complexity questions LogSpace $\stackrel{?}{=}$ P, NLogSpace $\stackrel{?}{=}$ P by studying the comparison of their descriptive logics on infinite partially ordered structures rather than restricting ourselves to finite structures. We show that the infinite versions of those famous class separation questions are consistent with the axioms of set theory and we give a sufficient condition on the complexity classes in order to get other such relative consistency results.

Keywords: Computational complexity, Large cardinals, Relative consistency results

Résumé

Nous considérons les versions infinies des questions usuelles de complexité LogSpace $\stackrel{?}{=}$ P, NLogSpace $\stackrel{?}{=}$ P en étudiant, sur les structures infinies partiellement ordonnées, la comparaison de leurs logiques, les décrivants, au lieu de se limiter aux structures finies. Nous montrons que les versions infinies de ces fameuses questions de séparation sont consistantes avec la théorie des ensemble et nous donnons une condition suffisante sur les classes de complexité comparées pour obtenir les mêmes résultats avec d'autres classes de complexité.

Mots-clés: Complexité de calcul, Grands cardinaux, Résultats de consistance relative

THE INFINITE VERSIONS OF LogSpace $\neq$ P ARE CONSISTENT WITH THE AXIOMS OF SET THEORY

GRÉGORY LAFITTE AND JACQUES MAZOYER

Introduction

Looking at infinite versions of problems is an approach to solving problems in complexity theory : the infinite case might be easier to solve. It is then perhaps possible to apply the proof techniques from the infinite case to the finite complexity theory questions. In one of the best examples of this technique, Sipser [15] showed that an infinite version of parity does not have bounded depth countable-size circuits using ideas from descriptive set theory. By making an analogy between *polynomial* and *countable*, Furst, Saxe and Sipser [5] used the techniques of the infinite case from Sipser's paper to show that parity does not have constant-depth polynomial-size circuits.

Recall that in descriptive complexity, two logically characterizable complexity classes $\mathcal{C}$, $\mathcal{C}'$ are equal ($\mathcal{C} = \mathcal{C}'$) if and only if the corresponding logics $\mathcal{L}_{\mathcal{C}}$ and $\mathcal{L}_{\mathcal{C}'}$ correspond[1] on ordered finite structures. Our study focuses on the comparison of the logics on partially ordered infinite structures. This is what we call the infinite version of complexity class separation questions.

We settle for the infinite case the usual computational complexity question in an unusual way : "the infinite version of (N)LogSpace $\neq$ P" is consistent with the standard axioms of set theory. Apart from the trivial separation of Σ^1_1 and Π^1_1 (NP and co-NP) on structures of cardinality $\kappa \geq \omega$, little was known. Fortnow, Kurtz and Whang [4] pointed out an open communication complexity problem whose infinite version Miller [13] had proved to be independent. As far as we know, our results are the first known relative consistency results for infinite versions of complexity questions, for which the infinite versions were not directly connected to already known relative consistent propositions in set theory.

Note that the relative consistency of the infinite case does not imply anything about the provability of the usual computational complexity questions. What it does tell us, is that any proof that LogSpace and P correspond (in the usual finite case meaning) will not carry over to the infinite case.

As indicated above, the infinite separation of NP and co-NP is straight forward : it is a known fact from set-theoretical absoluteness study that Σ^1_1 and Π^1_1 separate on infinite structures, e.g. "α is an ordinal". Of course, the separation of NP and co-NP on infinite structures also implies, because P and PSpace are closed by complementary, the separation of P and NP, P and co-NP, NP and PSpace, co-NP and PSpace, and obviously also the separation of classes contained in P and of classes containing NP or co-NP such as LogSpace and NP (co-NP), P and PSpace. Thus the only non-trivial case (when considering the combinations of the above mentioned complexity classes) on infinite structures is in comparing (N)LogSpace and P.

The complexity classes that are appropriate for our method are those that verify certain conditions, named ($\star$), which are given in the following section. The two complexity classes $\mathcal{C}$ and $\mathcal{C}'$ must be inclusion-wise comparable (hence NP and co-NP are not appropriate), the lower complexity class $\mathcal{C}$ must be logically characterizable on ordered finite structures and $\mathcal{C}'$ must be characterizable in a certain precise way (as monotone-FO[Operator] also on ordered finite structures) , that most of the usual computational complexity classes verify. Moreover, the two fixed point logics on first order definable functions must separate on finite structures (that are not necessarily linearly ordered). (N)LogSpace and P verify those ($\star$) conditions as well as all of the above complexity classes (apart from co-NP for which we do not know). Of course, there is nothing surprising in obtaining the relative consistency of a provable proposition. In the rest of the paper,

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[1]Each sentence in $\mathcal{L}_{\mathcal{C}}$ has an equivalent sentence (same models) in $\mathcal{L}_{\mathcal{C}'}$ and vice versa.

we strive to show the relative consistency of the separation for classes verifying the conditions detailed above and not only for (N)LogSpace and P.

Note that the result is indeed about the consistency of the separation of *all* infinite versions of the complexity classes that verify certain precise conditions. One of those is that the greater complexity class be logically characterizable by some fixed point operator on functions that are monotone for inclusion. Any infinite version of that fixed point operator is suitable as long as we keep the restriction to monotone functions. And so, we not only prove the consistency of the separation for fixed infinite versions of suitable complexity classes but mainly the consistency of the separation of all the suitable infinite interpretations. When we fix the infinite interpretation and obtain the consistency result, most of the time (at least for easy interpretations that naturally come to mind) the separation is provable and not only relatively consistent. Nevertheless it is quite surprising that in *all* cases, the separation is consistent. Those results are thus particularly interesting in the prospect of obtaining some transfer theorems between certain infinite and finite propositions, which could then give us hints on the finite usual case.

For the proof, we fix $\mathcal{C}, \mathcal{C}'$ that verify our conditions. We then compare monotone[Operator] and $\mathcal{C}$. We show that if the two logics correspond on certain partially ordered infinite structures, then the cardinality of those structures is an inaccessible[2] cardinal. This means that on structures of any infinite cardinality apart (perhaps) from inaccessible cardinals, the logics separate. We thus compare logics on structures of a certain cardinality or greater. This is known to be the same[3] as comparing them on any structure.

Forcing arguments tell us that it is consistent with set theory, adjoined with some set-theoretical assumptions and "there is no inaccessible cardinal", that every monotone (according to a particular convenient partial order) function be definable in first order logic. Hence we get the consistency of the separation of our complexity classes on infinite partially ordered structures with the axioms of set theory.

Considering large cardinals in this context is the crucial point in reaching consistency results. It had already been considered in the study of precise fixed point operators on arbitrary structure. When one has an operator function Γ on subsets of a *countable* set X, the closure ordinal $|\Gamma|$ is the smallest ordinal α such that $A_{\alpha+1} = \Gamma(A_\alpha)$ where A starts with $\emptyset$, on which we transfinitely iterate Γ. If we have C, a set of operators of a certain form on $P(X)$, then $|C| = \sup\{|\Gamma| \ : \Gamma \in C\}$. Gandy (unpublished) has first shown that $|\Pi^0_1| = \omega_1$ (operators that are definable by a Π^0_1 formula). Then Richter [14] obtained characterizations of certain natural extensions of Π^0_1 in terms of recursive analogues of large cardinals. In particular, it was shown that even $|\Pi^0_2|$ is much larger than the first recursively Mahlo ordinal, the first recursively hyper-Mahlo ordinal, etc. Aczel and Richter proved some recursive analogues of large cardinal characterizations of $|\Pi^0_n|$ and of $|\Delta^1_1|$, $|\Pi^1_1|$ and $|\Sigma^1_1|$. So it was predictable that we could perhaps obtain large cardinals when considering fixed point operators on subsets of *non-countable* sets. But again, the fact that we obtain large cardinals comes mostly from considering *all* fixed point operators with little requirements on their forms. The more important requirement of being definable is taken care of afterwards, by forcing techniques.

1. Appropriate complexity classes

The following definitions remind us to which logic a complexity class corresponds and vice versa. To get a much better overview of the descriptive study of complexity through finite model theory, the reader is invited to consult [2].

Let us recall some descriptive complexity results. The computational complexity class P is logically characterized by first order logic enhanced by a fixed point operator. A fixed point operator takes a first-order definable[4] function, called *operator function*, F on 2^{domain} and gives a new relation [Op F] such that[5] [Op $F]\overline{x}$ if and only if $\overline{x}$ belongs to the fixed point (if it exists, otherwise $\emptyset$) of the iteration of F starting from $\emptyset$ $(\emptyset, F(\emptyset), F(F(\emptyset)), \ldots)$. To ensure that we have a fixed point, we can oblige the fixed point operator function to be inductive (that is $X \subseteq F(X)$) and this can easily be done by transforming the formula φ used

[2]This is one example of a large cardinal. Large cardinals are not only unbelievably greater than any cardinal you could possibly think of, but their very existence is also not provable in (independent of) set theory. See Kanamori and Magidor [12, 11] for a background on large cardinals, their properties and relative consistency strengths.

[3]Let $\mathcal{K}$ be a class of ordered structures. $\mathcal{K}_m = \{A \in \mathcal{K} \mid |A| \geq m\}$. For any usual complexity class $\mathcal{C}$, $\mathcal{K} \in \mathcal{C}$ iff $\mathcal{K}_m \in \mathcal{C}$.

[4]F is definable if there is a first-order formula φ, with free first-order variables $x_1, x_2, \ldots, x_k$ (noted $\overline{x}$) and second-order variable X, such that $F(X)$ is the set of $\overline{x}$ which verify $\varphi(\overline{x}, X)$.

[5]$\overline{x}$ is $x_1, \ldots, x_k$ for a certain k.

to define the operator function to $X\overline{x} \vee \varphi(\overline{x}, X)$. First order logic enhanced by this operator is called the inductive fixed point (IFP) logic, which characterizes P on finite linearly ordered structures.

Gurevich and Shelah [8] have shown that this logic is equivalent to first order logic enhanced by least fixed points, which gives for an F (derived again from φ as above) its least (according to the subset order) fixed point starting from $\emptyset$. It has been shown that those fixed points are obtained by monotone functions (for all X,Y subsets of the domain, $X \subseteq Y$ implies $F(X) \subseteq F(Y)$), which are described by formulas *positive* in X (loosely speaking, there is an even number of $\neg$ before each occurrence of X). This is the logical description of P that we use.

Abiteboul, Vardi and Vianu [1, 2.2] found similar logical descriptions for NP and PSpace with the use of inflationary fixed point operators. By using Gurevich's and Shelah's techniques in [8], it seems straight forward to show from those later logical descriptions that NP and PSpace are characterizable by fixed point logics on functions that are monotone (for $\subset$) and definable in first order logic.

Another fixed point operator that can be defined is the (non-deterministic) *transitive closure* operator, which (over first order logic) captures the complexity class (N)LogSpace (see [2]).

P, NP and PSpace are thus characterizable by fixed point logics on functions that are monotone (for $\subset$) and definable in first order logic. We use the notation Monotone $-$ FO[Operator] for such logics, where Operator is different depending on the complexity class we are characterizing. There may be multiple fixed point operators *extending* Monotone $-$ FO (as it is the case for NP and PSpace), but it does not affect our study.

1.1. Definition. Working on an ordered structure $\mathcal{A}$, we consider functions from $2^{\mathcal{A}^k}$ to $2^{\mathcal{A}^k}$. We say that such a function F is :

- *monotone* if for all $X \subseteq Y$, $F(X) \subseteq F(Y)$;
- *definable* if there is a first order formula $\varphi(x_1, \ldots, x_k, \overline{u}, X, \overline{Y})$ such that $F(R) = \{(a_1, \ldots, a_k) \mid \mathcal{A} \models \varphi[a_1, \ldots, a_k, \overline{b}, R, \overline{S}]\}$, where $\overline{b}$ and $\overline{S}$ are interpretations of $\overline{u}$ and $\overline{Y}$.

We can now define the following logics : (Monotone$-$)FO[Operator] contains first order logic and is closed under operation Operator (which to every function F assigns Operator$[F]$, a k-ary relation on $\mathcal{A}$) on (monotone) definable functions : $\mathcal{A} \models [\text{Operator } F]\overline{t}[\overline{b}]$ if and only if $(t_1[\overline{b}], \ldots, t_k[\overline{b}]) \in \text{Operator}[F]$.

To be able to go through the following section, the two complexity classes, $\mathcal{C}$ and $\mathcal{C}$' containing $\mathcal{C}$, that we compare should verify the three following conditions :

- $\mathcal{C}$ is characterizable by a FO[Operator^{①}] logic;
- ($\star$) $\mathcal{C}$' is characterizable by a Monotone $-$ FO[Operator^{②}] logic;
- FO[Operator^{①}] $<$ FO[Operator^{②}] on finite (not necessarily ordered) structures.

Hence apart from all known, thus relatively consistent, infinite separation results such as NP$\neq$ co-NP, the following are interesting possible computational class combinations : LogSpace $\subseteq$ P and NLogSpace $\subseteq$ P. The third condition is a known result for (N)LogSpace and P in finite model theory (see [2][7.6.22]).

In the following, we compare, as indicated previously, the infinite versions of the logics behind usual complexity classes that verify the above ($\star$) conditions and therefore we do not talk anymore about the complexity classes themselves.

2. $\mu(L)$, $\nu(L)$ and strong limit cardinals

In the remainder of this paper, as we compare logics, we actually compare them on infinite *partially ordered* structures.

Our goal is to show that "Monotone $-$ FO[Operator^{②}] is not equal to a certain class $\mathcal{C}$ on infinite structures" is consistent with ZFC. To begin with, we are going to show that if Monotone[Operator^{②}] and $\mathcal{C}$ correspond on κ-structures (structures of cardinality κ), then κ is a strong inaccessible cardinal. Monotone on a structure $\mathcal{A}$ is the class of functions from 2^{A^k} to 2^{A^k} which are monotone (not necessarily definable).

Let $\mathcal{L} = (L, \leq, \ldots)$ be a partially ordered structure. We take the structure $\mathcal{A} = (A, \preccurlyeq, \ldots)$ to be the power set of $\mathcal{L}$, with $\preccurlyeq$ being defined as a suitable combination of $\leq$ (pointwise) and $\subseteq$ (we will precisely define $\preccurlyeq$ when we come to ν' later on). We can then use $\mathcal{A}$ to shift the comparison of the logics to the comparison of the functions used in the logics' fixed point operators.

We show that if there is a structure L (by abuse of language, L also denotes the structure $\mathcal{L}$) where the inclusion of our two logics is not proper, then the cardinality of that structure is an inaccessible cardinal (strong limit regular cardinal).

Such a structure, where every *monotone* function with an OPERATOR② operator is equivalent to a formula in $\mathcal{C}$, will be called a *nice* structure.

The approach consists in studying the properties of some cardinals $\mu(L)$ and $\nu(L)$, where L is a partially ordered set with certain properties, which are defined below.

We now define some of the notations that we hereafter use.

2.1. Definition. Let $(L, \leq)$ be a partial order.

(1) We say that a set $A \subseteq L$ is "co-well-ordered" iff $(A, \geq)$ is a well-ordered set.
(2) We call a set $A \subseteq L$ "uniform" iff A is either an antichain, or a well-ordered chain, or a co-well-ordered chain.

First, let us now recall some common set theoretical definitions (see [10]) to be used later on :

2.2. Definition. For any cardinals κ, λ,

- κ^+ is the least cardinal $> \kappa$. κ is a *successor* cardinal iff $\kappa = \lambda^+$ for some λ. κ is a *limit* cardinal iff $\kappa > \omega$ and is not a successor cardinal.
- κ is a *strong limit* cardinal iff $\forall \lambda < \kappa$, $2^\lambda < \kappa$.
- if $f : \kappa \to \lambda$, f maps κ *cofinally* iff $\operatorname{ran}(f)$ is unbounded in λ $(\forall \xi < \lambda, \exists \gamma \in \operatorname{ran}(f), \xi < \gamma)$.
- the *cofinality* of λ $(\operatorname{cf}(\lambda))$ is the least κ such that there is a map from κ cofinally into λ.
- κ is *regular* iff $\operatorname{cf}(\kappa) = \kappa$ (singular otherwise).
- κ is a *strong inaccessible* cardinal iff κ is a regular strong limit cardinal.

A strong inaccessible (which is sometimes called inaccessible) cardinal is called a large cardinal (see [12]). Its existence is independent of the axioms of set theory.

A part of the study of μ and ν is from [7], it is adapted and modified for our purpose. Let $(L, \leq)$ be a partial order. We try to get some information on the structure of L by considering certain "cardinal characteristics" $\mu(L)$ and $\nu(L)$, which are defined as follows:

2.3. Definition. (1) $\mu(L)$ is the smallest cardinal μ such that there is no uniform set $A \subseteq L$ of cardinality μ. Hence, $\kappa < \mu(L)$ iff there is a uniform subset $A \subseteq L$ of size κ.
(2) $\mu_n(L) = \mu(L^n)$ for $n > 0$.
(3) $\nu(L)$ is the smallest cardinal ν such that there is no family $(f_i : i < \nu)$ of ν many pairwise incomparable (pointwise) monotone functions from L to L.
(4) $\nu_n(L)$ is the smallest cardinal ν such that there are no pairwise incomparable monotone functions $(f_i : i < \nu)$ from L^n to L.
(5) $\nu(L_1, L_2)$ is the the smallest cardinal ν such that there are no pairwise incomparable monotone functions $(f_i : i < \nu)$ from L_1 to L_2.
(6) $\mu_\infty = \sup\{\mu_n : n \in \omega\}$, $\nu_\infty = \sup\{\nu_n : n \in \omega\}$.

From the previous definitions, we trivially have that for all $n \in \omega$, $\mu_n \leq \mu_{n+1}$ and $\nu_n \leq \nu_{n+1}$.

2.4. Fact. *Let L be infinite. Then $\mu_n(L) \leq |L|^+$ and $\nu(L) \leq (2^{|L|})^+$.*

Our goal is to see the link with large cardinals, in particular strong inaccessible cardinals, and how those relations can be used to get a hint about $\mu(L)$ and $\nu(L)$. The partition relations (see the proof of the following proposition) help us in understanding μ and we will see later how this gives us a better understanding of ν.

2.5. Proposition. *Let $(L, \leq)$ be a partial order.*

(a) *If κ is an infinite cardinal, $|L| > 2^\kappa$, then $\mu(L) > \kappa$. (In fact, $\mu(L) > \kappa^+$.)*
(a') $|L| \leq 2^{\mu(L)}$.
(b) *If L is infinite, then $\mu(L) > \aleph_0$.*
(c) *If κ is a strong limit cardinal, then $\kappa \leq \mu(L)$ iff $\kappa \leq |L|$.*
(d) *If κ is a strong limit cardinal, then $|L| > \kappa$ implies $\mu(L) > \kappa$,*
(e) *If κ is a strong limit cardinal, then $\mu(L) = \kappa$ implies $|L| = \kappa$.*

Proof. (a) Write ρ for $(2^\kappa)^+$. Let $(a_i : i < \rho)$ be distinct elements of L, and define $F : [\rho]^2 \to \{<, >, =, \|\}$ by requiring $F(i,j) =$ "comparability[6] of a_i and a_j" whenever $i < j$.

To go ahead, we use *partition relations*[7]. There are some well-known partition relations such as Ramsey's theorem and most of them are from Erdős (see [3] for a detailed exposition from which the following partition relations are taken) :

(1) **(Ramsey)** For any natural number k, $\aleph_0 \to (\aleph_0)^2_k$.
(2) **(Erdős, Rado)** For any infinite κ, $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.
(3) **(Erdős, Rado)**
  (a) If κ is an infinite cardinal, k finite, then $(2^{<\kappa})^+ \to (\kappa)^2_k$.
  (b) If κ is a strong limit cardinal, then $\kappa^+ \to (\kappa)^2_4$.

The Erdős-Rado partition relation (2) promises us an F-homogeneous set $\{i_\zeta : \zeta < \kappa^+\}$ of size κ^+, which naturally induces a uniform set $\{a_{i_\zeta} : \zeta < \kappa^+\}$ of the same cardinality.

(a') follows from (a).

The proofs of (b) and (d) are similar, using (1) and (3), respectively, instead of (2).

(c) easily follows from fact 2.4. (e) follows from (c) and (d).

□

And the following facts are well-known results, proved using an independence lemma of Shelah and Goldstern. It gives an evaluation (lower estimates) of ν for simple (uniform) sets that we use in order to evaluate μ, as soon as we have some strong relation between ν and μ.

2.6. Fact. (a) *If A is uniform, $|A| > 2$, then $\nu(A) > 2$.*
(b) *If A is uniform, $|A| = \kappa \geq \aleph_0$, then $\nu(A) > 2^\kappa$, i.e., there are 2^κ many pairwise incomparable monotone functions from A to A.*
(c) *If A is an antichain, $|A| = \kappa \geq \aleph_0$, then $2^\kappa < \nu(A, \{0,1\})$, i.e., there are 2^κ many incomparable (necessarily monotone) functions from A into the two-element set $\{0,1\}$.*

3. Relations between μ and ν

Recall that our final goal is to compare $\mathcal{C}$ and $\mathcal{C}' =$ Monotone $-$ FO[Operator^{②}] that verify the $(\star)$ conditions. We first want to show that if $\mathcal{C}$ and Monotone[Operator^{②}] correspond on structures of cardinality κ, then κ is a large cardinal.

We finally investigate the relation between μ and ν. It turns out to be slightly simpler if we look at μ_∞ and ν_∞ first.

First we show in proposition 3.1 that the existence of many incomparable monotone functions from L^n to L $(\kappa < \nu_n(L))$ implies the existence of a large antichain in some L^m $(\kappa < \mu_m(L))$, assuming that L is nice. (This is actually the only place in the whole proof where we talk about nice structures rather than general partial orders.)

Then we show in lemma 4.1 that a large (anti)chain in L^m $(\kappa < \mu_m(L))$ implies the existence of *very* many incomparable monotone functions from L^m to L $(2^\kappa < \nu_m(L))$.

Finally, in theorem 4.2, we combine proposition 3.1 and lemma 4.1 to show that $\mu = \mu_\infty$ must be a strong limit cardinal.

First, we need to precise the underlying partial order in $\mathcal{A} = \langle 2^L, \preccurlyeq \rangle$. It is defined such that any monotone$^\preccurlyeq$ function can effectively be used with the Operator^{②} operator of $\mathcal{C}'$.

$$a \preccurlyeq b \quad \text{iff} \quad a \subseteq b \text{ and } a \leq b \setminus a$$

and a new incomparability notion (completely independent of the incomparability due to $\preccurlyeq$) :

$$a \wr\wr b \quad \text{iff} \quad \forall c \in a\, \forall d \in b \;\; c \,\|\, d$$

[6] "comparability of a and b" is $<$, $>$, $=$ or $\|$ whenever respectively $a < b$, $a > b$, $a = b$ or $a \,\|\, b$.

[7] Let κ, λ and c be cardinals. The "partition symbol" $\lambda \to (\kappa)^2_c$ means: whenever $F : [L]^2 \to C$, where $[L]^2$ is the set of unordered pairs from L, $|L| = \lambda$, $|C| = c$, then there is an *F-homogeneous set* $K \subseteq L$ of cardinality κ, i.e., a set K such that F restricted to $[K]^2$ is constant.

We now introduce $\nu'(L)$: it is the smallest cardinal ν' such that there is no family $(f_i : i < \nu')$ of pairwise incomparable$^{\wr}$ monotone$^{\preccurlyeq}$ functions from $A = 2^L$ to A. In the following, when considering elements of A, *incomparable* stands for incomparable$^{\wr}$. Trivially, we have $\nu(L) \leq \nu'(L)$. We define also $\nu'_n(L) = \nu'(L^n)$ as we did for μ_n and ν_n.

We could as well introduce $\nu''(L)$ as the smallest cardinal such that there is no family of pairwise incomparable monotone functions *definable in first order logic* but we would then get $\nu'' < \nu'$ which would not be of any help. So we decide to stick to monotone functions and we will end up with definable functions later on.

3.1. Proposition. *Let $(L, \leq)$ be a nice structure, κ a cardinal of cofinality $> cf(2^{\aleph_0})$. If $\kappa < \nu'_n(L)$, then $\kappa < \mu_\infty(L)$.*

Proof. Let us assume $\kappa < \nu'_n(L)$.

So, let $(f_i : i < \kappa)$ be a family of pairwise incomparable monotone functions from 2^{L^n} to 2^L. Since $\mathcal{L}$ is a nice structure, each of these functions with an OPERATOR^{②} operator can be written as t_i in $\mathcal{C}$. Thus, for each i there is some natural number k_i and a definable function $g_i(\overline{x}, y_1, \ldots, y_{k_i})$ and a k_i-tuple $\bar{b}^i = (b^i_1, \ldots, b^i_{k_i})$ such that

$$t_i = [\text{OPERATOR}^{①} g_i](\bar{b}^i)$$

Since there are only $\leq 2^{\aleph_0}$ many pairs (t_i, k_i) and we have assumed $cf(\kappa) > \mathrm{cf}(2^{\aleph_0})$, we may assume that they are all equal, say to (t^*, k^*). But then $(\bar{b}^i : i < \kappa)$ must be pairwise incomparable in $2^{L^{k^*}}$, because, with our assumptions on $\mathcal{C}$ and $\mathcal{C}'$, if $\bar{b}^i$ and $\bar{b}^j$ were comparable then f_i and f_j would be comparable. Hence we have found an antichain of size κ in $2^{L^{k^*}}$. And by definition of $\wr$, this implies that we also have an antichain of size κ in L^{k^*}.

□

4. Main result

To show the relative consistency of the infinite versions of $\mathcal{C} \neq$ MONOTONE[OPERATOR^{②}], we first show (using all the previous lemmata) that the cardinality of L is necessarily a strong limit cardinal and then using a lemma of Goldstern and Shelah, that it is also regular.

In order to show the next important lemma, it is necessary to have a bounded (with a smallest and a greatest element) structure, which we have because of the implication of the $(\star)$ conditions on nice structures.

4.1. Lemma. *Let $(L, \leq, 0, 1)$ be a bounded partially ordered structure, κ an infinite cardinal. If $\kappa < \mu_n(L)$, then $2^\kappa < \nu_n(L)$. In particular, $\kappa < \mu_\infty$ implies $2^\kappa < \nu_\infty$.*

Proof. Let $A \subseteq L^n$ be uniform of size κ.

Case 1: A is a chain.

If $A \subseteq L^n$ is well-ordered of order type κ, then there is $A' \subseteq L$, also well-ordered of order type κ[8]. So, without loss of generality, $n = 1$.

By fact 2.6, $\nu(A) > 2^\kappa$.

For every complete partial order $\bar{A} \subseteq L$ such that $A \subseteq \bar{A}$, every monotone map $f : A \to A$ can be extended to a monotone map $\hat{f} : L \to \bar{A}$[9]. If f, g are incomparable, then so are $\hat{f}$, $\hat{g}$.

Let $\bar{A} = A \cup \{0, 1\}$. Then since $\bar{A}$ is a complete partial order, we get $\nu(L) \geq \nu(A)$. Hence $\nu(L) > 2^\kappa$.

Case 2: A is an antichain. Use fact 2.6(c).

□

4.2. Theorem. *If L is infinite and nice, then*

(a) *$\mu_\infty(L)$ must be a strong limit cardinal,*
(b) *$\mu(L) = \mu_\infty(L)$*
(c) *$|L| = \mu(L)$.*

[8]Let $\bar{a}^i = (a^i(1), \ldots, a^i(n))$ for $i < \kappa$, and $i < j \Rightarrow \bar{a}^i < \bar{a}^j$. For each $k \in \{1, \ldots, n\}$ the sequence $(a^i(k) : i < \kappa)$ is weakly increasing. If the sequence $(a^i(k) : i < \kappa)$ does not contain a strictly increasing sequence of length κ, then it must be eventually constant. However, this cannot happen for every $k \in \{1, \ldots, n\}$.

[9]$\hat{f}(x) = \sup_{\bar{A}}\{f(y) : y \in \mathrm{dom}(f), y \leq x\}$

Proof. (a) If $\kappa < \mu_\infty(L)$, then $2^\kappa < \nu_\infty(L)$ by lemma 4.1. So, $2^{2^\kappa} < \nu'_\infty(L)$. Now 2^{2^κ} always has cofinality greater than $2^{\aleph_0} \geq \mathrm{cf}(2^{\aleph_0})$, so we get $2^{2^\kappa} < \mu_\infty(L)$ by proposition 3.1. And hence $2^\kappa < \mu_\infty(L)$.

(b) Assume that $\mu(L) < \mu_\infty(L)$. Let $\lambda = 2^{2^{\mu(L)}} < \mu_\infty(L)$. By proposition 2.5, $|L| \leq 2^{\mu(L)} < \lambda$, so $\mu_n(L) \leq |L|^+ \leq \lambda$ for all $n \in \omega$, hence $\mu_\infty(L) \leq \lambda$, a contradiction.

(c) Use proposition 2.5(e): Let $\kappa = \mu(L)$. From proposition 2.5(c) we conclude $\kappa \leq |L|$, and from proposition 2.5(d), we conclude $\kappa \geq |L|$.

□

We have already shown that for a nice structure L the cardinal characteristic $\mu(L)$ must be a strong limit cardinal. Now we need to show that $\mu(L)$ must be regular.

Letting $\kappa := \mu(L)$, a lemma of Goldstern and Shelah [7, 4.1] shows that the singularity of κ would imply the existence of $\gg \kappa$ many incomparable monotone functions, and we show from there that this would imply $\mu(L) > \kappa$.

4.3. Lemma. *Let $(L, \leq, 0, 1)$ be a bounded partially ordered structure, and let κ be a singular strong limit cardinal, $\kappa \leq |L|$.*

Then $\nu(L) > \kappa$.

If moreover $cf(\kappa) = \aleph_0$, then we even get $\nu(L) > 2^\kappa$.

Proof. See [7, 4.1]. □

4.4. Theorem. *If $(L, \leq)$ is a nice structure, then $\mu(L) = |L|$ is an inaccessible cardinal.*

Proof. Let $\kappa = \mu(L)$. From theorem 4.2, we know that κ is a strong limit, and that $|L| = \kappa$. Assume that κ is singular.

First, let us assume that $cf(\kappa)$ is uncountable. The previous lemma 4.3 tells us that $\nu(L) > \kappa$, so $\nu'(L) > 2^\kappa$. Now, we know that κ is a strong limit cardinal and so because of its singularity, $2^{\mathrm{cf}(\kappa)} < \kappa$. Moreover, $\mathrm{cf}(\kappa) > \aleph_0$, so $2^{\mathrm{cf}(\kappa)} \geq 2^{\aleph_0} \geq \mathrm{cf}(2^{\aleph_0})$ and by König's theorem, $\mathrm{cf}(2^\kappa) > \kappa > \mathrm{cf}(2^{\aleph_0})$. We can then apply our proposition 3.1 : $\mu_\infty(L) > 2^\kappa$, a contradiction.

Now we consider the second case: $cf(\kappa) = \aleph_0$. Here lemma 4.3 tells us $\nu(L) > 2^\kappa$ and so $\nu'(L) > 2^{2^\kappa}$. Since 2^{2^κ} has cofinality $> 2^{\aleph_0} \geq \mathrm{cf}(2^{\aleph_0})$, we can again apply proposition 3.1 and again get $\mu_\infty(L) > 2^{2^\kappa} > \kappa$, a contradiction.

We then know that L has a strong inaccessible cardinality because $|L| = \mu(L)$ when L is infinite and nice. □

4.5. *Remark.* Note that the cardinality of a nice structure cannot be a weakly compact cardinal.

4.6. *Remark.* Note also that this result means that the cardinality of our structure $\mathcal{A}$ will be the cardinality of the powerset of a large cardinal, which, of course, cannot be anything else than a large cardinal itself.

Theorem 4.4 tells us that "Monotone[Operator^{②}] is not equal to $\mathcal{C}$ on any infinite structures" is consistent relative to ZFC. Recall that our goal is to compare $\mathcal{C}$ to Monotone$-$FO[Operator^{②}]. We are able to come to our ends through the following modified forcing theorem of Goldstern and Shelah [6]. It is easily shown that our partial order $\preccurlyeq$ (from $\mathcal{A}$) still verifies the forcing conditions of the theorem whenever we have an original partial order $\leq$ (from $\mathcal{L}$) that verifies them.

4.7. Theorem. *The statement*

> *"There is a partial order $(P, \preccurlyeq)$ such that all monotone$^{\preccurlyeq}$ functions $f : P \to P$ are definable in P"*

is consistent relative to ZFC*. Moreover, the statement holds in any model obtained by adding (iteratively) ω_1 Cohen reals to a model of CH.*

Hence we can reach our goal :

4.8. Theorem. *"Monotone$-$FO[Operator^{②}] is not equal to $\mathcal{C}$ on some infinite structure" is consistent relative to* ZFC.

Proof. "(Strong) inaccessible cardinals do not exist" is consistent with the continuum hypothesis and also with Cohen reals. Therefore it is consistent with "MONOTONE = MONOTONE−FO on a particular structure". And so we have, by the previous theorem and by theorem 4.4 that "on this particular structure, MONOTONE−FO[OPERATOR②] $\neq \mathcal{C}$" is consistent with ZFC. □

5. CONCLUDING REMARKS

The existence of structures, such as in the previous section, cannot be derived from the "usual" axioms of mathematics, as codified in the Zermelo-Fraenkel axioms for set theory. Moreover, also certain additional assumptions such as the (generalized) continuum hypothesis are not sufficient to prove the existence of such structure, or in other words, the theory ZFC + GCH + "there is no infinite such $\mathcal{L}$" is consistent. In fact, the well-known consistent theory ZFC + "there is no inaccessible cardinal", a natural extension of ZFC, proves that there is no such $\mathcal{L}$ and *a fortiori* that the involved logics separate on all infinite structures.

One natural question is then to know if those questions, in the infinite case, are independent of the axioms of set theory. This does not appear to be an easy task.

One can also try to give some interpretation of our results in terms of special "infinite" Turing machines. Infinite time Turing machines have been considered in the past in various ways. Hamkins and Lewis [9] considered *countable* tape machines that at limit ordinal stages of the computation make their cell values the lim sup of the cell values (0 or 1) before the limit and enter a special distinguished limit state with the head of each tape plucked from wherever it might have been racing towards and placed on top of the first cell of that tape. At successor ordinal stages, they behave as classical Turing machines. Hamkins and Lewis also obtain some recursive analogues of large cardinals. For example, the supremum of the writable (countable ordinals are somehow coded in reals) ordinals is recursively inaccessible : it is recursively Π^1_1-indescribable. So again, the large cardinals were predictable for special machines with *non-countable* tapes. With specific special infinite time and space Turing machines, one can give analogues of usual complexity class definitions using ordinal arithmetic (polynomial, logarithmic ...) and even obtain the same logical characterizations as in the finite Turing machine case. In this *infinite* Turing machine framework, our result then states that the separation of most complexity classes is relatively consistent with set theory for any infinite analogue of complexity class definitions, with the only requirement that our ($\star$) conditions on the classes (finite and infinite versions) are met. What do the ($\star$) conditions mean in this Turing context?

Another open question is whether there are some other complexity classes, apart from LOGSPACE, NLOGSPACE and P, that verify the ($\star$) conditions and whose separations are not trivial in the infinite case.

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