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Afterwards, we give a geometrical interpretation of the order given
by this lattice.
We extend these results to other types of tilings (calisson tilings,
tilings with bicolored Wang tiles).

Keywords: Tiling, Height function,Lattice

Résumé

Mots-clés: pavage, fonction de hauteur, treillis

Lattices of tilings : an extension to figures with holes

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Abstract

We first prove that the set of domino tilings of a fixed finite figure is a distributive lattice, even in the case when the figure has holes. Afterwards, we give a geometrical interpretation of the order given by this lattice.

We extend these results to other types of tilings (calisson tilings, tilings with bicolored Wang tiles).

1   Introduction

During the last ten years, A lot of advancements have been done about the study of tiling. Especially, W. P. Thurston [15], using works of J. H. Conway and J. F. Lagarias [4], introduced the notion of height functions, which encode domino tilings and calisson tilings of a polygon P.

The notion of height function appears as a very powerful tool in tiling study. It has later been extended by different authors [6] [11], to study tilings algorithms for other sets of prototiles.

For domino tilings, height functions induce a lattice structure on the set of tilings of a fixed polygon (see [13]). From this structure, some important results are obtained : linear time tiling algorithm, rapidly mixing Markov chains for random sampling a tiling, computation of the number of necessary flips (local transformations involving two dominoes) to pass from a fixed tiling to another fixed tiling.

Notice that dominoes have a particular importance for theoretical physicists : for them, a domino is seen has a dimer, a diatomic molecule (as the molecule of hydrogen), and each tiling is seen as a possible state of a solid, or a fluid.

The present paper tries to generalize previous results to figure which are not polygons, i. e. figures with holes. This is done by the introduction of an *equilibrium function* on edges of cells of the figure. With this tool, we prove that the set of tilings of any finite figure has a structure of distributive lattice. Afterwards, we give a geometrical interpretation of this structure : to do it, we use classical local flips, but, moreover, we also have to introduce two new tools : *flips around holes*, and *reduction* of figures by *critical cycles*.

We finish proving that those ideas can directly be adapted for other types of tilings : calisson tilings and tilings with bicolored Wang tiles.

2 Tilings and induced height functions

2.1 The square lattice

Let Λ be the planar grid of the Euclidean plane R^2. A *vertex* of Λ is a point with both integer coordinates. Let $v = (x, y)$ be a vertex of Λ.

A *cell* of Λ is a (closed) unit square whose corners are vertices. Two cells are *4-neighbors* (respectively *8-neighbors*) if they share an edge (respectively (at least) a vertex).

Two vertices of Λ are *neighbors* if they both are ends of a same edge of a cell of Λ. Hence, each vertex v has four neighbors which are canonically called the *East, West, North and South neighbors* of v. An ordered pair of neighbor vertices is called an *arc* or an *edge* of Λ.

We assume that cells of Λ are colored as a checkerboard. By this way, we have black cells and white cells, and two cells sharing an edge have different colors. For each arc (v, v') of Λ, we define the *spin* of (v, v') (denoted by $sp(v, v')$) by :

- $sp(v, v') = 1$ if (v, v') if an ant moving form v to v' has a white cell on its left side (and a black cell on its right side),

- $sp(v, v') = -1$ otherwise.

A *path* of Λ is a sequence $(v_0, v_1, \ldots, v_p)$ of vertices such that for each integer i such that $0 \leq i < p$, v_{i+1} is a neighbor of v_i. This path is a *cycle* if, moreover, $v_p = v_0$. The cycle is *elementary* if $v_i = v_j$ and $i \neq j$ imply that $\{i, j\} = \{0, p\}$.

2.2 Figures

A *figure* F of Λ is a finite union of cells of Λ. The set of edges of F (denoted by E_F) is the set of arcs (v, v') such that the line segment $[v, v']$ is a side of a cell of F.

We state $F = F_1 \cup F_2 \ldots \cup F_p$, where, for each integer i, F_i is a 4-connected component of F. For each connected component F_i, we fix a vertex w_i of its boundary (for example, w_i can be chosen as the leftmost vertex of the lowest vertices of F_i).

The only infinite connected (for the 8-connectivity) component of $\mathbb{R}^2 - F$ is denoted by H_∞. The other ones are called *holes* of F. A connected figure such that H_∞ is the only connected component of $\mathbb{R}^2 - F$ is called a *polygon* of Λ.

Because of problems due to both types of connectivity for cells, we replace (until the end of the paper) each vertex v of F such that each edge issued from v is in the boundary of F by two vertices v_1 and v_2, each of them being connected two exactly two neighbors of v (see figure 1). By this way, the contour of each hole is an elementary cycle of F.

2.3 Tilings

A domino is a figure formed from two cells sharing an edge, which is called the central edge of the domino. A tiling T of a figure F is a set of dominoes included in F, with pairwise
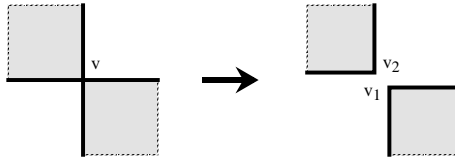
Figure 1: vertex duplication according to 4-connectivity of F and 8-connectivity of $\mathbb{R}^2 - F$.

disjoint interiors (i. e. there is no overlap), such that the union of tiles of T equals F (i. e. there is no gap).

Definition 1 *Let T be a tiling of a figure F and (v, v') be an arc of E_F. The T-value (denoted by val_T) is the function from E_F to $\mathbb{Z}$ is defined by :*

- *$val_T(v, v') = -3sp(v, v')$ and there exists a domino of T whose a symmetry axis is $[v, v']$,*

- *$val_T(v, v') = sp(v, v')$ otherwise.*

The function val_T is a tool to encode the tilings : for each pair (T, T') of tilings of F, if $val_T = val_{T'}$, then we have $T = T'$.

Notice that for each arc (v, v') such that $[v, v']$ is on the boundary of F, we necessarily have $val_T(v, v') = sp(v, v')$, thus $val_T(v, v')$ does not depend on T.

2.4 Equilibrium values

Informally, we can say that we want to work as if F had no hole. To to it, the informal idea is to introduce values on edges which make disappear holes.

Precisely, this is done by the use of *equilibrium functions* defined below :

Definition 2 *An equilibrium function (denoted by eq) is a function from E_F to $\mathbb{R}$ such that*

- *$eq(v, v') = -eq(v', v)$,*

- *for each elementary cycle $C = (v_0, v_1, v_2, v_3, v_4(= v_0))$ around a cell of F, we have : $\sum_{i=0}^{3} eq(v_i, v_{i+1}) = 0$*

- *for each hole H_i let $(v_{i,0}, v_{i,1}, \ldots, v_{i,p_i} = v_{i,0})$ be a cycle which clockwise follows the boundary of H_i. We have : $\sum_{j=0}^{p_i-1} eq(v_{i,j}, v_{i,j+1}) = -\sum_{j=0}^{p_i-1} sp(v_{i,j}, v_{i,j+1})$.*

An equilibrium function can be exhibited using cut lines (also see [14]) as follows : (see figure 2)

for each hole H_i of F, we (arbitrarily) fix a vertical line segment $L_i = [p_i, p_i']$ (which is called a *cut line* issued from H_i) of $\mathbb{R}^2$ such that p_i is the central point of a highest cell of H_i, there exists a positive integer n_i such that $p_i' = p_i + (0, n_i)$, the vertex p_i' is not in F, and, for each integer such that $0 < n_i' < n_i$, the point $p_i + (0, n_i')$ is the central point of a cell of F. Hence, the point p_i' is the central point of a cell of another 8 connected component H_j of $\mathbb{R}^2 - F$, with $j \neq i$ (and, possibly $j = \infty$).
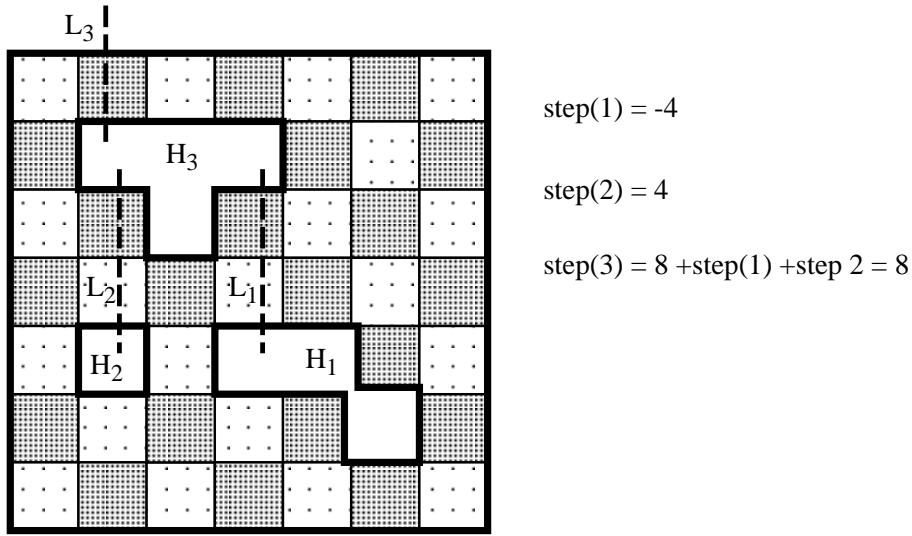
Figure 2: Computation of an equilibrium function by step values.

We say that H_j is the (immediate) predecessor of H_i. By this way, we construct a directed tree whose vertices are 8-connected components of $\mathbb{R}^2 - F$. This tree is deep-rooted in H_∞. We inductively define the *step value* of a hole H_i (denoted by $step(i)$) as follows : let $C_i = (v_{i,0}, v_{i,1}, \ldots, v_{i,p_i}(= v_{i,0}))$ be a cycle which (clockwise) follows the boundary of H_i. We have :

$$step(i) = -\sum_{j=0}^{p_i-1} sp(v_{i,j}, v_{i,j+1}) + \sum_{j \text{ s.t. } H_i \text{ is the predecessor of } H_j} step(j)$$

We now can define the equilibrium value of an arc (v, v') by :

- $eq(v, v') = step(i)$ if v' is the East neighbor of v and the line segment $[v, v']$ crosses the cut line L_i,

- $eq(v, v') = -step(i)$ if v' is the West neighbor of v and the line segment $[v, v']$ crosses the cut line L_i,

- $eq(v, v') = 0$ if the line segment $[v, v']$ crosses no cut line.

This function obviously satisfies the conditions of the definition.

In the following of the paper, we assume that an equilibrium function in F is fixed.

2.5 Height functions

The height difference of an arc (v, v') of E_F (denoted by $hd_T(v, v')$) is defined by :

$$hd_T(v, v') = val_T(v, v') + eq(v, v')$$

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Let $(v_0, v_1, \ldots, v_p)$ be a path of the figure F (i. e. for each integer i such that $0 \leq i < p$, (v_i, v_{i+1}) is in E_F) and T be a tiling of F. The height difference of this path for T is the sum : $\sum_{i=0}^{p-1} hd_T(v_i, v_{i+1})$.

Proposition 1 *Let T be a tiling of a figure F. The height difference of any cycle of F for T is null.*

This proposition is a generalization of a theorem from J. H. Conway [4] about tilings of polygons.

Proof. (sketch) It suffices to prove it for elementary cycles since the height difference of each cycle is the sum of the height differences of the elementary cycles which compose it. This is done by induction on the number of cells of Λ enclosed by the cycle.

We first treat the case when the cycle follows the boundary of a hole H_i. This case is easily treated, from the definition of equilibrium functions. We also verify that the proposition holds for elementary cycles of length 4 around a cell.

Now, we can apply the induction argument. If we are not in the cases treated above, then the area enclosed by the cycle can be cut by a path of F, which induces two new cycles, each of them enclosing less cells of Λ than the original cycle. Thus, by induction hypothesis, the height difference of both induced cycles is null, from which it is easily deduced that the height difference of the original cycle is null. □

This proposition guarantees the correctness of the definition below.

Definition 3 *Let F be a figure with a fixed equilibrium function, with $F = F_1 \cup F_2 \ldots \cup F_p$, where, for each integer i, F_i is a 4-connected component of F, with a fixed vertex w_i on the boundary.*

For each tiling T, the height function induced by T (denoted by h_T) is the function from the set V_F of vertices of cells of F (once necessary duplications have been done) to the set $\mathbb{R}$ of real numbers, defined by $h_T(w_i) = 0$ and, for each arc (v, v') of E_F, $h_T(v') - h_T(v) = hd_T(v, v')$.

Proposition 2 *For any pair (T, T') of tilings and each vertex v of F, we have : $h_T(v) - h_{T'}(v) = 0[4]$.*

Proof. obvious by induction on the length of a shortest path from w_i to v. □

Proposition 3 *Let (T, T') be a pair of tilings of F. If, for each vertex v of F, $h_T(v) = h_{T'}(v)$, then $T = T'$.*

Informally, this proposition means that a height function is a way to encode a tiling.

Proof. Let (v, v') be an arc of F such that $sp(v, v') = 1$. We have two cases :

- the edge joining v to v' cuts no domino of T. Thus, $h_T(v') - h_T(v) = eq(v, v') + 1$,

- the edge joining v to v' is a symmetry axis of a domino of T. Thus, $h_T(v') - h_T(v) = eq(v, v') - 3$,

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Thus, the tiling T is formed from tiles whose central axis is a segment $[v, v']$ such that $|h_T(v') - h_T(v) - eq(v, v')| = 3$. The same argument can be used for T', which yields : $T = T'$. □

The above proposition permits to define the distance between tilings.

Definition 4 *Let (T, T') be a pair of tilings of F. The distance $\Delta(T, T')$ between T and T' is defined by*

$$\Delta(T, T') = \sum_{v \text{ is interior of } F} |h_T(v) - h_{T'}(v)| + \sum_{H_i \text{ is a hole of } F} |h_T(H_i) - h_{T'}(H_i)|$$

where $|h_T(H_i) - h_{T'}(H_i)|$ denotes the common value $|h_T(v) - h_{T'}(v)|$ for any vertex v of the boundary of H_i.

The proposition below gives a characterization of height functions

Proposition 4 *Let f be a function from the set of vertices of F to the set $\mathbb{Z}$ of integers such that :*

- *$f(w_i) = 0$,*

- *for each arc (v, v') of E_F such that $sp(v, v') = 1$, either $f(v') - f(v) = eq(v, v') + 1$ or $f(v') - f(v) = eq(v, v') - 3$,*

- *if, moreover, the arc (v, v') is on the boundary of F, then $f(v') - f(v) = eq(v, v') + 1$.*

There exists a tiling T such that $f = h_T(v)$.

Proof. Let $(v_0, v_1, v_2, v_3, v_4 = v_0)$ be a cycle around a white cell, counterclockwise. The second constraint of the proposition implies that we have three vertices v_i such that $f(v_{i+1}) - f(v_i) - eq(v_i, v_{i+1}) = 1$ and a unique vertex v_j such that $f(v_{j+1}) - f(v_j) - eq(v_j, v_{j+1}) = -3$. One easily obtains a symmetric condition for black cells.

Thus, the set T of dominoes which are cut into both halves by an edge whose extremities, say v and v', are such $|f(v) - f(v') - eq(v, v')| = 3$, is a tiling of F. One obviously verifies (by induction on the distance from w_i to v) that, for each vertex v of F, $f(v) = h_T(v)$. □

2.6 Reduction

The height functions can sometimes be used to get a decomposition of the figure. It is the reason of the definition below.

Definition 5 *A clockwise elementary cycle $C = (v_0, v_1, \ldots, v_p = v_0)$ of F is direct critical (respectively indirect critical) if*

$$\sum_{i=0}^{p-1} sp(v_i, v_{i+1}) + eq(v_i, v_{i+1}) = 0$$

and, for each vertex v_i of C, at least one of the following condition holds :

- $sp(v_i, v_{i+1}) = 1$ *(respectively $sp(v_i, v_{i+1}) = -1$),*

- *the cell at the left side of the arc (v_i, v_{i+1}) is not in F.*

An insecable *figure is a figure with no critical cycle (except the boundary of H_∞).*

Proposition 5 *Let T be a tiling of a figure F and $C = (v_0, v_1, \ldots, v_p = v_0)$ be a critical cycle of F. The cycle C cuts no tile of T.*

Proof. This is a consequence of Proposition 1. Notice that, for each vertex v_i, $hd_T(v_i, v_{i+1}) \leq sp(v_i, v_{i+1}) + eq(v_i, v_{i+1})$ and $0 = \sum_{i=0}^{p-1} hd_T(v_i, v_{i+1}) = \sum_{i=0}^{p-1} sp(v_i, v_{i+1}) + eq(v_i, v_{i+1})$. This yields that, for each vertex v_i, $hd_T(v_i, v_{i+1}) = sp(v_i, v_{i+1}) + eq(v_i, v_{i+1})$.

Thus , when $sp(v_i, v_{i+1}) = 1$, we get $hd_T(v_i, v_{i+1}) = 1 + eq(v_i, v_{i+1})$, which means that the edge $[v_i, v_{i+1}]$ cuts no tile of T. $\square$

¿From this proposition, one can assume that the figure is insecable. Otherwise, the figure is cut along critical cycles and a new figure, with more connected components, is created. Both figures have isomorphic spaces of tilings.

We also have the corollary below, which will be used later.

Corollary 1 *If F has a critical cycle $C = (v_0, v_1, \ldots, v_p = v_0)$ such that for each integer i such that $0 \leq i < p$, v_i is an interior vertex of F, then there exists no tiling of F.*

Proof. Let $v_j = (x_j, y_j)$ be the vertex of this cycle with $x_j + y_j$ maximal, and, moreover, x_j minimal with the previous constraint. We necessarily have $v_{j-1} = v_j + (-1, 0)$, $v_{j-2} = v_{j-1} + (0, -1)$ and (on the other hand) $v_{j+1} = v_j + (-1, 0)$. Now, follow the cycle until a vertex v_{j+2k} such that $v_{j+2k} \neq v_j + (k, -k)$.
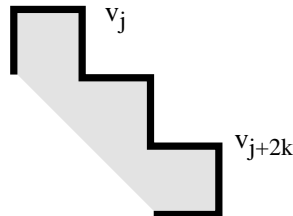

Figure 3: Proof of Corollary 1

Let T be a tiling of F. Necessarily, at least one tile of T is cut by an edge of the path formed from the part of the cycle from v_j to v_{j+2k}. But this is impossible, from Proposition 5. Thus there is no tiling. $\square$

3 The lattice of tilings

Height functions canonically induce an order on the set Γ_F of tilings of the figure F. Given a pair (T, T') of tilings of F, we say that $T \leq_{height} T'$ if and only if, for each vertex v of F, $h_T(v) \leq h_{T'}(v)$.

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Proposition 6 *Let (T, T') be a pair of tilings of F. The functions $f = inf(h_T, h_{T'})$ and $f' = sup(h_T, h_{T'})$ are height functions of tilings.*

Proof. We prove this proposition for f (the proof for f' is similar) using proposition 4. The first and last constraints are obviously satisfied, since $h_T(v') - h_T(v) = h_{T'}(v') - h_{T'}(v)$ for each arc (v, v') such that $[v, v']$ is included in the boundary of F.

Let (v, v') be an arc of E_F such that $sp(v, v') = 1$. Assume that $h_T(v) < h_{T'}(v)$. Thus, from proposition 2, we have : $h_T(v) \leq h_{T'}(v) - 4$. On the other hand, either $h_T(v') = h_T(v) + eq(v, v') + 1$ or $h_T(v') = h_T(v) + eq(v, v') - 3$. Thus :

$$h_T(v') \leq h_T(v) + eq(v, v') + 1 \leq (h_{T'}(v) - 4) + eq(v, v') + 1 = h_{T'}(v) + eq(v, v') - 3$$

Moreover, either $h_{T'}(v) = h_{T'}(v') + eq(v', v) - 1$ or $h_{T'}(v) = h_{T'}(v') + eq(v', v) + 3$, thus

$$h_{T'}(v) + eq(v, v') - 3 \leq (h_{T'}(v') + eq(v', v) + 3) + eq(v, v') - 3 = h_{T'}(v')$$

We have proven that if $h_T(v) < h_{T'}(v)$ then $h_T(v') \leq h_{T'}(v')$. Consequently, $f(v') - f(v) = h_T(v') - h_T(v)$, which guarantees the second constraint of proposition 4.

The cases when $h_T(v) > h_{T'}(v)$ and $h_T(v) = h_{T'}(v)$ are treated with the same kind of argument. $\qquad\square$

In the language of order theory (see [3] or [5]), the above proposition can be said as follows.

Corollary 2 *The order $\leq_{height}$, defined at the beginning of this section, induces a structure of distributive lattice on the set Γ_F of tilings of F.*

Proof. Obvious. $\qquad\square$

For the following, for each pair (T, T') of tilings of F, the tiling whose height function is $inf(h_T, h_{T'})$ (respectively $sup(h_T, h_{T'})$) is denoted by $inf(T, T')$ (respectively $sup(T, T')$).

3.1   Local Flips

We will now give a geometric interpretation of the lattice. To do it, we first have to introduce classical local transformations on tilings usually called local flips.

Definition 6 *Let T be a tiling of F. A* local maximum *(respectively minimum) of T is an interior vertex v of F such that, for each neighbor v' of v, $h_T(v') - h_T(v) < eq(v, v')$ (respectively $h_T(v') - h_T(v) > eq(v, v')$).*

Proposition 7 *An interior vertex v of F is a local extremum of T if and only if v is the center of a 2×2 square S which is exactly covered by two dominoes of T, with a common large side.*

Proof. If v is a local minimum, let v' and $''$ be the neighbors of v such that $sp(v',v) = sp(v'',v) = 1$. Notice that v is the middle of the line segment $[v',v'']$. Since one cannot have : $h_T(v') = h_T(v) + eq(v,v') - 1$, the equality $h_T(v') = h_T(v) + eq(v,v') + 3$ necessarily holds, thus the domino whose a symmetry axis is $[v,v']$ is a domino of T. The same argument holds with v'', which yields that S is exactly covered by dominoes of T. A similar proof can be done for a local maximum of T.

Conversely, assume that S is exactly covered by dominoes of T. One easily sees, applying rules which define a height function, that v is a local extremum of T. □

Definition 7 *With the notations above, a* local flip *(see figure 4) is the replacement in T of the pair of dominoes which cover S by the other pair which can cover S. After the flip is done, a new tiling T_v of F is obtained.*
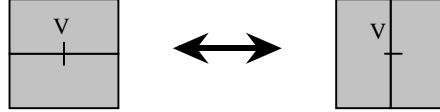

Figure 4: A local flip.

What are the consequences of a flip on the height function ? For each vertex v' such that $v' \neq v$, there exists a path of F from w_i (the origin vertex of the connected component containing v') to v' which does not pass through v. This yields that $h_T(v') = h_{T_v}(v')$.

For v, let v'' be a neighbor of v such that $sp(v'',v) = 1$. The segment $[v,v'']$ is a symmetry axis of exactly one domino of $T \cup T_v$, thus : $\{h_T(v), h_{T_v}(v)\} = \{h_T(v'') + eq(v'',v) + 1, h_{T_v}(v'') + eq(v'',v) - 3\}$. This yields that $|h_T(v) - h_{T_v}(v)| = 4$.

If $h_{T_v}(v) = h_T(v) + 4$, then we say that the flip is going up (remark that the local minimum v is transformed into a local maximum), and, if $h_{T_v}(v) = h_T(v) - 4$, then we say that the flip is going down (the local maximum v is transformed into a local minimum).

Local flips canonically induce an order on the set Γ_F of tilings of the irreducible figure F. Given a pair (T,T') of tilings of F, we say that $T \leq_{flip} T'$ if and only if there exists a sequence $(T_0, T_1, \ldots, T_p)$ of tilings of F such that $T_0 = T$, $T_p = T'$ and, for each integer i such that $0 \leq i < p-1$, T_{i+1} is deduced from T_i by an upwards (local or around a hole) flip. If $T \leq_{flip} T'$, then, one obviously has $T \leq_{height} T'$. We will prove a partial converse of this implication.

Definition 8 *Let (T,T') be a pair of tilings of F. We say that T and T' are* boundary equivalent *(denoted by $T = T'[\delta F]$) if, for each vertex v of the boundary of F, $h_T(v) = h_{T'}(v)$.*

Remark that if $T = T'[\delta F]$, then $T = inf(T,T')[\delta F]$ and $T = sup(T,T')[\delta F]$

Proposition 8 *Given a pair (T,T') of tilings of F, we have $T \leq_{flip} T'$ if and only if $T \leq_{height} T'$ and $T = T'[\delta F]$.*

Moreover, in such a case, the minimal number of necessary flips to pass from T to T' is $\Delta(T,T')/4$

Proof. The direct part of the first part of the proposition is obvious, after the study of flips above. The converse part of the proposition is proven by induction in the quantity $\Delta(T, T')$. The result is obvious if $\Delta(T, T') = 0$ (i. e. $T = T'$ from Proposition 3).

Now, assume that $\Delta(T, T') \neq 0$, $T \leq_{height} T'$ and $T = T'[\delta F]$. We will prove that there exists a vertex w such that $h_T(w) < h_{T'}(w)$ (which yields that $h_T(w) \leq h_{T'}(w) - 4$ from Proposition 2), and an upwards flip can be done from T around w.

Take a vertex v_0 such that $h_T(v_0) < h_{T'}(v_0)$. If for each neighbor v' of v such that $sp(v_0, v') = -1$, we have $h_T(v') = h_T(v_0) + eq(v_0, v') + 3$, then we are done. Otherwise, there exists a neighbor v_1 of v_0 such that $sp(v_0, v_1) = -1$ and $h_T(v_1) = h_T(v_0) + eq(v_0, v_1) - 1$. Thus

$$h_T(v_1) = h_T(v_0) + eq(v_0, v_1) - 1 \leq h_{T'}(v_0) - 4 + eq(v_0, v_1) - 1 = h_{T'}(v_0) + eq(v_0, v_1) - 5$$

On the other hand, either $h_{T'}(v_1) = h_{T'}(v_0) + eq(v_0, v_1) - 1$ or $h_{T'}(v_1) = h_{T'}(v_0) + eq(v_0, v_1) - 3$. Thus $h_T(v_1) < h_{T'}(v_1)$. Thus, either an upwards flip can be done around v_1, or the same argument can be repeated from v_1 to obtain a vertex v_2 such that $sp(v_1, v_2) = -1$, $h_T(v_2) = h_T(v_1) + eq(v_1, v_2) - 1$, and $h_T(v_2) < h_{T'}(v_2)$, which permits to repeat the process.

We have two alternatives : either a vertex v_i, around which an upwards flip can be done, is obtained, or an infinite sequence $(v_i)_{i \in \mathbb{N}}$ is obtained. But, since F is finite, the second alternative implies that there exists a critical cycle which does not meet the boundary of the figure, which contradicts corollary 1. Thus, this second alternative cannot holds.

We now have proven the existence of a vertex w such that an upwards flip can be done from T around w to obtain a tiling T_w. Notice that $\Delta(T_w, T') = \Delta(T, T') - 4$, which proves the first part of the proposition by induction, and, moreover, guarantees that there exists a sequence of $\Delta(T, T')/4$ flips to pass from T to T'.

To finish the proof, remark that, for each sequence $(T = T_0, T_1, \ldots, T_p = T')$ such that, for each integer i such that $0 \leq i < p$, T_{i+1} is deduced from T_i by a flip, we have $\Delta(T, T') \leq \sum_{i=0}^{p-1} \Delta(T_i, T_{i+1}) = 4p$. $\qquad\square$

Corollary 3 *Let (T, T') be a pair of tilings of F. We have $T = T'[\delta F]$ if and only if there exists a sequence $(T_0, T_1, \ldots, T_p)$ of tilings of F such that $T_0 = T$, $T_p = T'$ and, for each integer i such that $0 \leq i < p - 1$, T_{i+1} is deduced from T_i by a (downwards or upwards) flip.*

Moreover, in such a case, the minimal possible number p of necessary flips is equal to $\Delta(T, T')/4$.

Proof. The the first part of the proposition is easily proven using $inf(T, T')$. The equality is easy once it has been noticed that $\Delta(T, T') = \Delta(T, inf(T, T')) + \Delta(inf(T, T'), T')$. $\qquad\square$

Corollary 4 *Each equivalence class for the boundary equality relation has a structure of distributive lattice.*

Proof. Obvious. $\qquad\square$

3.2 Extension : flips around holes

In order to express the whole distributive lattice of tilings of the figure F in terms of flips, we have to introduce a new kind of flips, which make the height function change on the boundary of holes of F. We have to limit ourselves to the case when F is insecable (if F is not insecable, then F has first to be cut along critical cycles, according to the process of reduction until a figure insecable figure F' is obtained).

Let H be a hole of an insecable figure F. Notice that H has neither peak (i. e. cell of F with three of its 4-neighbors in F) nor isthmus (i. e. cell of F whose intersection with the boundary of H is not connected (for the classical topology)), since otherwise, a critical cycle would appear. Thus, the figure F_H formed from cells whose a vertex is on the boundary of H is a 4-connected elementary cycle of cells and, consequently, admits exactly two tilings.

Definition 9 *Let T be a tiling of F. A hole H is a* hole maximum *(respectively minimum) of T if, for each vertex v of the boundary of H, and for for each neighbor v' of v such that $[v, v']$ is not on the boundary of H, $h_T(v') - h_T(v) < eq(v, v')$ (respectively $h_T(v') - h_T(v) > eq(v, v')$).*
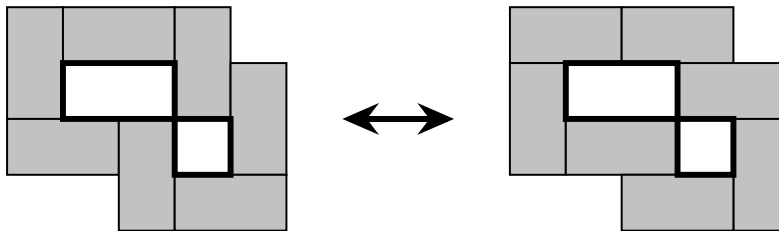

Figure 5: A hole flip.

Proposition 9 *A hole H of F is a hole extremum of T if and only if T contains one of the two tilings of F_H.*

Proof. If H is a hole minimum, let v and v' be neighbors such that v is on the boundary of H, $[v, v']$ is not on the boundary of H and $sp(v', v) = 1$. One cannot have : $h_T(v') = h_T(v) + eq(v, v') - 1$, the equality $h_T(v') = h_T(v) + eq(v, v') + 3$ necessarily holds, thus the domino whose a symmetry axis $[v, v']$ is a domino of T. The set of all dominoes obtained as above form a tiling of F_H. A similar proof can be done for a local maximum of T.

Conversely, assume that F_H is exactly covered by dominoes of T. One easily sees, applying rules which define a height function, that H is a hole extremum of T. □

Definition 10 *With the notations above, a* hole flip *(see figure 5) is the replacement in T of the et of dominoes which cover F_H by the other set which can cover F_H. After the flip is done, a new tiling T_H of F is obtained.*

As for local flips, one easily sees that $h_T = h_{T_H}$ except on the boundary of H and $h_{T_H} - h_T$ is constant on the boundary of H. The constant is in $\{-4, 4\}$ and we say that the flips is going upwards if the constant is 4.

Proposition 10 *Given a pair (T, T') of tilings of an insecable figure F, we have $T \leq_{height} T'$ if one can pass from T to T' by a sequence of (local or around hole) upwards flips.*

Moreover, in such a case, the minimal number of necessary flips to pass from T to T' is $\Delta(T, T')/4$

Proof. The direct part of the first part of the proposition is obvious, after the study of flips above. The converse part of the proposition is proven by induction in the quantity $\Delta(T, T')$. The result is obvious if $\Delta(T, T') = 0$ (i. e. $T = T'$ from Proposition 3).

Take a vertex v_0 such that $h_T(v_0) < h_{T'}(v_0)$. If v_0 either is a local minimum for T or is on the boundary or a hole which is a hole minimum for T, then we are done.

Otherwise, either there exists a neighbor v_1 of v_0 such that $sp(v_0, v_1) = -1$ and $h_T(v_1) = h_T(v_0) + eq(v_0, v_1) - 1$, or v_0 is on the boundary of a hole H such that there exists an arc (v_0', v_1) with v_0' on the boundary of H, $[v_0', v_1]$ is not on the boundary of H, $sp(v_0', v_1) = -1$ and $h_T(v_1) = h_T(v_0') + eq(v_0', v_1) - 1$. In both alternatives, $h_T(v_1) < h_{T'}(v_1)$. Thus, either an upwards flip can be done around v_1, or the same argument can be repeated from v_1 to obtain a vertex v_2 such $h_T(v_2) < h_{T'}(v_2)$, which permits to repeat the process.
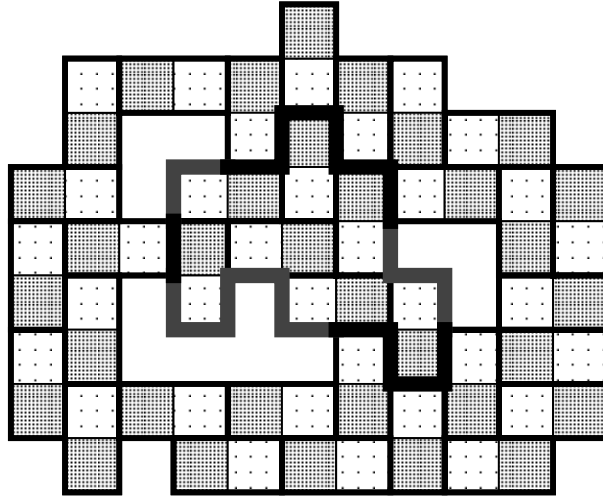

Figure 6: construction of a critical cycle

We have two alternatives : either a vertex v_i, around which an upwards flip can be done, is obtained, or an infinite sequence $(v_i)_{i \in \mathbb{N}}$ is obtained. But, since F is finite, the second alternative implies that a cycle can be constructed as follows : first construct paths joining holes using a period of sequence $(v_i)_{i \in \mathbb{N}}$. This defines a subfigure F' of F (formed from cell "inside" the path). The cycle is the boundary of F' (notice the cycle contains all paths drawn above and parts of contours of holes joined). This cycle is critical, which contradicts the fact that f is insecable. Thus, this second alternative cannot hold.

We now have proven that an upwards flip can be done from T to obtain a tiling T_1 such that $T_1 \leq_{height} T'$. Notice that $\Delta(T_1, T') = \Delta(T, T') - 4$, which proves the first part of the proposition by induction, and, moreover, guarantees that there exists a sequence of $\Delta(T, T')/4$ flips to pass from T to T'.

To finish the proof, remark that, for each sequence $(T = T_0, T_1, \ldots, T_p = T')$ such that, for each integer i such that $0 \le i < p$, T_{i+1} is deduced from T_i by an upwards flip, we have $\Delta(T, T') \le \sum_{i=0}^{p-1} \Delta(T_i, T_{i+1}) = 4p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Notice that the process of reduction can change the structure of the lattice of tilings, as it can be seen in the example below (figure 7). At the left side, the 2×2 surrounded square form the interior part of the critical cycle of the figure. After the cut is done, the figure at the right side is obtained, formed from a cycle of cells and a 2×2 square.

After the cut, on the right, the tiling given by the highest tiling one of the cycle and the lowest of the square and the tiling given by the lowest tiling of the cycle and the highest one of the square, are not comparable for $\le_{height}$.

At the opposite, before the cut, on the left, the first tiling is higher (for $\le_{height}$) than the second one, even though there is no direct flip the one.

The tiling lattices are not isomorphic. A precise study gives that, on the left, the order $\le_{height}$ is a chain of four elements, but on the right, the order $\le_{height}$ is product of two chains, each one with two elements.
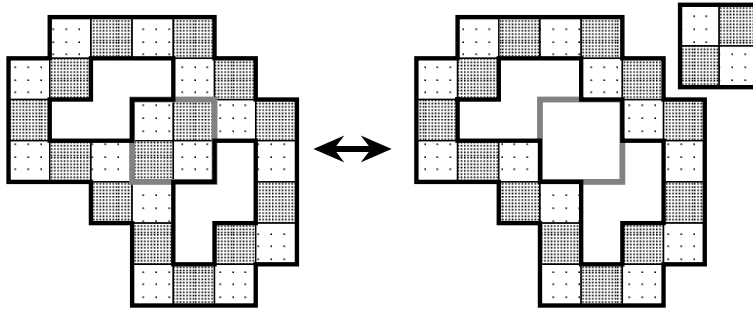

Figure 7: A figure and the figure obtained after the cut along the critical cycle.

4 Extension to other types of tilings

4.1 Calisson

The same study can easily be done for calisson (i. e. tiles formed from two neighboring cells of the triangular lattice) tilings to get similar results. In this case, local flips are induced by both tilings of hexagons of side of unit length, and flips around holes by both tilings of the neighboring of a hole. We only have two small differences, detailed below :

- We have two types of connectivity for triangular cells (3-connectivity for cells which share an edge, and 12-connectivity for cells which share a vertex) . Thus, some vertices have to be duplicated or triplicated (see figure 8).

- For the proof of corollary 1, we have two consider a part of the critical cycle with vertices $v = (x, y)$ such that y is maximal.
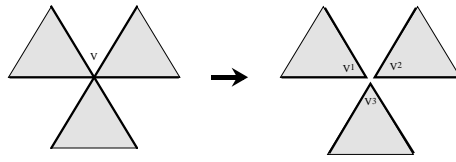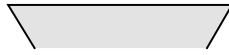
Figure 8: Example of "triplication".

Figure 9: A part of a critical cycle which implies that there is no tiling

4.2 Bicolored Wang tiles

The cases of dominoes is a particular case of tilings with Wang tiles (i. e. 1×1 squares with colored edges, see [8] for details). We have a tiling if colors of edges or neighbors squares are compatible.

An instance of the problem of tiling is given by a finite figure and a coloration of edges which are in its boundary. Hence, a domino tiling is a tiling by Wang tiles with one red edge and three blue edges, all the edges of the boundary of the figure being blue.

4.2.1 Eulerian orientations

The same study can be done for tiling with "balanced Wang tiles" such that each square has two blue edges and two red edges. In such a case, we have $val_T(v, v') = sp(v, v')$ if the corresponding edge is blue and $val_T(v, v') = -sp(v, v')$ otherwise. This is recognizable as the height function for Eulerian orientations of the dual lattice, called the *six-vertex ice model* by physicists [1]. This is also equivalent to the height function for three-colorings of the square lattice, and to alternating-sign matrices [10]. We obtain similarly result as for dominoes.

4.2.2 Examples with finite height functions

For the case of "odd tiles" (i. e. tiles with exactly three edges of the same color (blue or red) see [8]), we have to take a height function in $\mathbb{Z}/8\mathbb{Z}$ such that $val_T(v, v') = sp(v, v')$ if the corresponding edge is blue and $val_T(v, v') = -3sp(v, v')$. By our technique of equilibrium value, we can easily prove that the set of tilings of a fixed figure has a structure of boolean lattice (or hypercube), even if the figure has some holes.

The case of "even tiles" (i. e. tiles with an even number of blue edges and an even number of red edges) very similar, with $val_T(v, v') = (sp(v, v'), 0)$ if the corresponding edge is blue, and $val_T(v, v') = (sp(v, v'), sp(v, v'))$ otherwise, these values are taken in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

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