Contractions of octagonal tilings with rhombic tiles.

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Abstract

We prove that the space of rhombic tilings of a fixed octagon can be given a canonical order structure. We make a first study of this order, proving that it is a graded poset. As a consequence, we obtain the diameter of the space and a lower bound for the distance between tilings.

Keywords: Tilings, order theory

Résumé

Dans ce papier, nous prouvons que les pavages d'un octogone par des parallélogrammes peut être structuré de façon canonique. Une première étude de cette ordre est effectuée, et une preuve est faite de sa structure d'ordre gradué. Il en découle naturellement le diamètre de l'espace de pavages associé, ainsi qu'une borne inférieure pour la distance de flips entre deux pavages.

Mots-clés: pavages, theorie des ordres

Contractions of octagonal tilings with rhombic tiles

Frédéric Chavanon [*] Éric Rémila [†]

Résumé

We prove that the space of rhombic tilings of a fixed octagon can be given a canonical order structure. We make a first study of this order, proving that it is a graded poset. As a consequence, we obtain the diameter of the space and a lower bound for the distance between tilings.

1 Introduction

In this paper, we are concerned with tilings of octagons by parallelograms. These are a part of the class of rhombic tilings of polygons, which appear in physics, as a model for quasicrystals [BDMW02] and aperiodic structures [Sen95].

A *hexagonal tiling* is a tiling with rhombic tiles, each tile being a parallelogram constructed with two vectors chosen in a set of 3 fixed vectors (thus, we have 3 kinds of tiles). Given a tiling, a *flip* consists in replacing three tiles pairwise adjacent by three other adjacent tiles covering the same domain.

A hexagonal tiling of a hexagon can be interpreted as a set of cubes partially filling a box. In the box representation, a flip is the addition or the suppression of a cube. From this interpretation, a structure of distributive lattice is easily given on the space of tilings. W. Thurston [Thu90] has extended this result to the space of tilings of any simply connected domain of the plane. From the interpretation and the lattice structure, some important results are obtained : a linear time tiling algorithm [Thu90], rapidly mixing Markov chains for random sampling [MLS01] [Wil], computation of the number of necessary flips to pass from a fixed tiling to another fixed tiling, efficient exhaustive generation of tilings [Des] [SD03].

An *octagonal tiling* is a tiling with rhombic tiles, each tile being constructed with two vectors chosen in a set of 4 fixed vectors (thus, we have 6 kinds of tiles). For these tilings the situation is much more complex. To our knowledge, the main structural result is the connectivity of the space of tilings [Ken93]. In this paper, we use ideas issued from matroid theory [Zie92] [BVS+99] to get a decomposition method for tilings. By this way, each tiling can be encoded as a set of cubes as for the hexagonal case. Moreover, we prove that a set of cubes can be seen as a tiling if and only if two types of ”local” (i. e. involving a set of 2 or 4 cubes) properties are satisfied. From this interpretation, we have a representation of tilings, which allows to obtain some structural results on the space of octagonal tilings : even though there is no lattice structure, flips induce a structure of graded poset with minimal and maximal element.

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2 Preliminaries

All the following takes place in $\mathbb{R}^2$, its canonical basis will be noted (e_1, e_2). $\mathbb{R}^2$ is identified with the subspace $\mathbb{R}^2 \times \{0\}$ of $\mathbb{R}^3$. Let v and v' be two vectors of $\mathbb{R}^2$. Each of them can be represented by a complex number $re^{i\theta}$ with $r \geq 0$ and $0 \leq \theta < 2\pi$. For convenience, we assume in the following that $r = 1$ for all v. Let $v = e^{i\theta}$ and $v' = e^{i\theta'}$, we will say that $v < v'$ if $\theta < \theta'$.

We set $V = (v_1, v_2, v_3, v_4)$ be a sequence of vectors of $\mathbb{R}^2$ such that $v_i < v_{i+1}$ for all i and $\theta_i \leq \pi$. Let $M = (m_1, m_2, m_3, m_4)$ be a sequence of nonnegative integers ; m_i is associated to v_i and called the *multiplicity* of the vector v_i. The *octagon* associated to (V, M) is the region of the affine plane defined by : $\left\{\sum_{i=1}^{4} \lambda_i v_i, 0 \leq \lambda_i \leq m_i, m_i \in M, v_i \in V\right\}$ denoted $O_{(V,M)}$. $S = \sum_{i=1}^{4} m_i$ is the *size* of the octagon ; in the following such an octagon is called a S-*octagon*. See Figure 1 for an example.

The tiles we will use are parallelograms, constructed with vectors in V. The set of vectors used to define a tile t is the *type* of t. Hence, one counts $\binom{4}{2} = 6$ different types of tiles.

An *octagonal tiling* is then a covering of $O_{(V,M)}$ by translated tiles built with V, such that the intersection between two tiles is either empty, a point, or a segment. See Figure 1 for an illustration of a tiled octagon.

Let t be a tile of type $\{v_i, v_j\}$. Such a tile has four vertices. In the following, we call them the *points* of the tiling. Moreover t contains the four segments defining its perimeter, associated to v_i and v_j. They will be called the *edges* of the tiling.

Octagonal tilings present size one hexagons, tiled by three parallelograms. There exist two tilings of such an hexagon. Passing from one tiling to the other is called a *geometric flip* (see Figure 1).

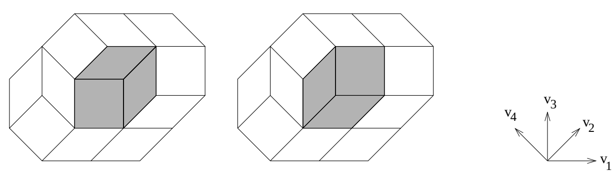

FIG. 1 – Two tilings of a 6-octagon, the grey tiles are the flipped ones.

One can then build the *space of tilings* of an octagon, which is the graph whose vertices are tilings, and there is an edge between two vertices if and only if the two corresponding tilings differ by a flip. See Figure 2 for examples of spaces of tilings. Those are spaces of tilings for a unitary octagon (i.e. $M = \{1, 1, 1, 1\}$) and for a hexagon with one repeated vector (i.e. $M = \{2, 1, 1, 0\}$). They are the fundamental examples used as the basis of the representation tool introduced later.

It is known that for hexagons (i.e. one component of M is null), the space of tilings can be structured as a distributive lattice. Our aim is to investigate the structure of larger spaces of tilings, begining with octagons, for which nothing but the flip-connectivity is known (see [Ken93, Eln97]).

Definition 1 (lifting) *Let $V = \{v_1, v_2, v_3, v_4\}$ be a set of vectors of $\mathbb{R}^2$, and v_0 a vector of $\mathbb{R}^3$ not in $\mathbb{R}^2$. Let p be the projection onto the plane orthogonal to v_0. Let $U = \{u_1, u_2, u_3, u_4\}$ be a set of vectors of $\mathbb{R}^3$, such that $\forall i, i = 1, ..., 4$ $p(u_i) = v_i$. Then U is called a* lifting *of V.*

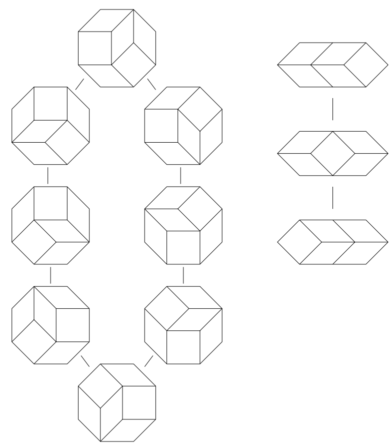

FIG. 2 – left : space of a unitary octagon tilings ; right : space of tilings of an hexagon with one repeated vector.

In the following, the mostly used lifting functions are the *principal lifting function*, defined by $u_i = v_i + e_3 \; \forall v_i \in V$, and the *k-located* function, $u_i = v_i + e_3$ if $i = k$ and $u_i = v_i$ if $i \neq k$.

Definition 2 (lifting of a tiling) *Let $V = \{v_1, v_2, ..., v_S\}$ be a set of vectors in $\mathbb{R}^2$, M a set of multiplicities, $O_{(V,M)}$ the associated octagon, and T_O a tiling of O. Let X and X' be two points of the tiling such that $\overrightarrow{XX'} = v_i$ and $[XX']$ is an edge of the tiling. A* lifting *of T_v is a function f such that $f(X') = f(X) + u_i$, where $u_i = \tilde{f}(v_i)$ is a lifting of v_i in $\mathbb{R}^3$, and $f(0) = 0$.*

The lifting of a tiling is well defined, since an octagon is simply connected.

Definition 3 (height function) *One defines a height function $h(v_i)$ associated to a lifted tiling, as the component upon e_3 of $\tilde{f}(v_i)$.*

For the principal lifting of a tiling, and given a unitary hexagon, one can build the associated heights. The point whose coordinates are $(0, 0, 0)$ has height 0, and the height of the other points are computed from this one. In the 2 possible tilings, the heights of the points that surround the hexagon are the same, whereas the height of the interior point varies. We will define the tiling with the lowest height as the *low tiling*, and the other as the *high tiling*. Notice that the compared interior points in the two tilings have different projections onto $\mathbb{R}^2$. See Figure 3.

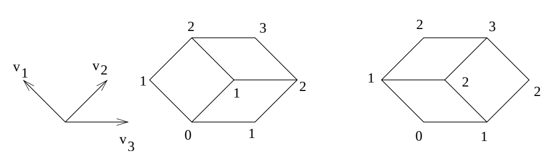

FIG. 3 – Two lifted tilings in $\mathbb{R}^3$, and the associated principal heights.

Notice that the i-located height function has the same value on the four points of a tile whose type does not contain v_i, whereas tiles containing v_i in their type have different height values on the extremities of an edge of type v_i.

Definition 4 (de Bruijn line, family) *Let T be a tiling of an octagon $O_{(V,M)}$, and h_i the i-located height function. The* de Bruijn family *associated with the vector v_i is the set of all tiles which have different height function on their opposite sides. The $j-th$ de Bruijn line of the family is the set of tiles which have $j-1$ for height function on one side (i.e. the two vertices at the extremities of the edge), and j on the opposite side. This line will be noted $\{v_i, j\}$. This was introduced in [dB81]. See Figure 4.*

Let $\{v_{i_0}, j_0\}$ be a de Bruijn line. We consider the vertices : $v_{i_0,j_0} = \sum_{i<i_0} v_i m_i + v_{i_0} j_0$, and $v'_{i_0,j_0} = \sum_{i>i_0} v_i m_i + v_{i_0} j_0$. The de Bruijn line is the sequence $(t_1, t_2, \ldots, t_p)$ of tiles of T such that the line segment $[v_{i_0,j_0}, v_{i_0,j_0} - v_{i_0}]$ is a side of t_1 , the line segment $[v'_{i_0,j_0}, v'_{i_0,j_0} - v_{i_0}]$ is a side of t_p, and for $1 \leq k < p$, $t_k \cap t_{k+1}$ is a line segment of the form $[v, v - v_{i_0}]$. One can thus set a total order on the tiles of the de Bruijn line, taking the indexes of the sequence. Given two tiles t_1 and t_2 of a de Bruijn line, we will say that t_1 is *below* t_2 if its index is smaller in the given order. This gives the *relative positions* of the tiles according to the de Bruijn line.

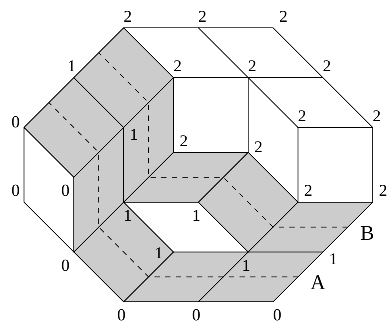

FIG. 4 – A tiled octagon, the 2-located height, and two de Bruijn lines ; A is the $1st$ line and B the $2nd$ line of the $2nd$ family.

Remarkable properties of these lines can be highlighted. Each tile belongs to two de Bruijn lines, and by definition, two de Bruijn lines $\{v_{i_0}, j_0\}$ and $\{v_{i_1}, j_1\}$ cross at a tile of type (v_{i_0}, v_{i_1}) for $i_0 \neq i_1$ (otherwise they do not cross). Hence every tiling of an octagon $O_{(V,M)}$ contains $\sum_{1 \leq i < j \leq 4} m_i m_j$ tiles.

Deleting the de Bruijn line associated to $\{v_i, j\}$ disconnects the tiling into two parts, say $T^+_{\{v_i,j\}}$ and $T^-_{\{v_i,j\}}$. The first one is composed of the tiles which vertices have i-located height functions larger than j, and the second corresponds to the tiles which vertices have i-located height functions less than j. Therefore we will say that $T^+_{\{v_i,j\}}$ is *higher* than the de Bruijn line, and $T^-_{\{v_i,j\}}$ is *lower*. Then, after the deletion, one can translate all the tiles of $T^+_{\{v_i,j\}}$ by the vector $-v_i$. The final configuration is a tiling of $O = (V, M')$, where M' is defined by $m'_k = m_k \ \forall\, k \neq i$ and $m'_i = m_i - 1$.

Let us now define a *contraction* function on octagonal tilings.

Definition 5 (contraction) *Let $O_{(V,M)}$ be a S-octagon, T one of its tilings, and v_i one of the vectors it is constructed with. Consider a lifting of T and the associated i-located height function. The $(S-1)$-*contracted *of O upon $\{v_i, j\}, j \leq m_i$, noted $C_{\{v_i,j\}}$, is the tiled octagon constructed by deleting all the tiles of the j-th de Bruijn line associated to v_i, translating all the tiles having height at least j by $-v_i$, thus obtaining a new octagon (or hexagon) with one less de Bruijn line. See Figure 5 for an example.*

Notice that this contraction function along a de Bruijn line preserves the T^+ and T^- parts according to other lines. Moreover, a contraction preserves also the relative positions on the tiles belonging to other de Bruin lines.

Proposition 1 *Assuming $S \geq 4$, every tiling is defined by the sequence of its $(S-1)$-contracted upon each vector $\{v_i, j\}$.*

Proof : Let T be a tiling of a S-octagon, and $\mathcal{C}$ the sequence of its $(S-1)$-contracted. We focus on one the Bruijn line, say $\{v_i, j\}$, which is the j-th de Bruijn line of the i-th family. The point is to find the relative positions of two tiles of this de Bruijn line, or more precisely the total order previously defined on the tiles of the de Bruijn line. If one can do this for every pair of tiles, the entire line can be constructed, and then the entire tiling. Consider two vectors v_k and v_l such that the tiles of type $\{v_i, v_k\}$ and $\{v_i, v_l\}$ appear in the line $\{v_i, j\}$. Since $S \geq 4$, there exists another de Bruijn line l' which contains none of those tiles. Consider the $S-1$-contracted along l'. This contracted gives the relative position of the two given tiles. Therefore, given two tiles, one can find their relative positions, and then construct the de Bruijn line. □

Let T be a tiling of a S-octagon. Following the definitions and assuming $v_i \neq v_k$, it is quite obvious that $C_{\{v_i,j\}}(C_{\{v_k,l\}}(T)) = C_{\{v_k,l\}}(C_{\{v_i,j\}}(T))$, because they both correspond to the deletion of the tiles of type containing i or j.

So we notice $C_{\{v_i,j\}\,\{v_k,l\}}(T)$ the $(S-2)$-contracted of T upon $\{v_i, j\}$ and $\{v_k, l\}$. See Figure 5. Similarly, we define the p-contracted, for any positive integer $p \leq S-2$. Iterating the proof for $(S-1)$-contracted, one obtains the following corollary for proposition 1 :

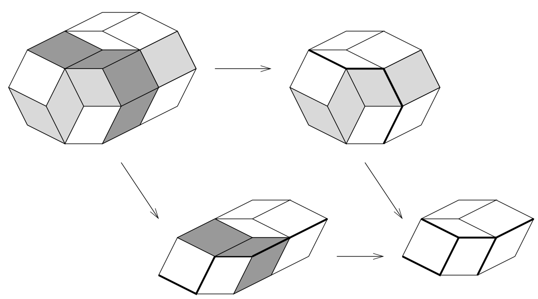

FIG. 5 – a tiling of an octagon and the two ways of contracting two different vectors (grey : the tiles affected by a contraction ; bold are the path corresponding to the tiles deleted formerly).

Corollary 1 *Assuming $S \geq 4$, every tiling T of octagon is defined by the sequence of its S'-contracted, for $3 \leq S' \leq S$. This is especially true for the 3-contracted.*

Informally, this means that a large octagonal tiling is obtained as a compound of its unitary tilings (i.e. its smallest non trivial contracted : tiled hexagons). If one sees such tilings as pilled up sets of cubes (which is a direct extension of the hexagonal case), the unitary hexagons represent the cubes that appear (up position 3-contracted) or miss (low position) in the stacking corresponding to the tiling. This gives a encoding of an octagonal tiling as a $\sum_{i=1,j>i,k>j}^{4} m_i m_j m_k$-long word, the 3-contracted being placed in lexicographic order, and the corresponding letter representing the position (up or down) of the cube.

2.1 Reconstruction

Continuing with the encoding idea, we have to show how the 3-contracted can be used to reconstruct the tiling. We are interested in the following problem : given, for each subset of 3 vectors of a size S set, a tiling of the associated unitary hexagon, is it possible to construct a tiling of the unitary S-octagon constructed with all the given vectors, such that the 3-contracted are precisely the given tiled hexagons ?
This can obviously be solved by constructing the 4-contracted, then the 5-contracted, and so on until the searched tiling is reached. If there is a contradiction, the reconstruction is impossible, otherwise the tiling is obtained. But this can be done faster.

Proposition 2 *The reconstruction of a tiling is possible if the 4-contracted can be correctly constructed.*

Proof : In the following, we call *possible* a contracted which has to be reconstructed, without regarding the fact that it can indeed be or not.
We do the proof by induction on the size S of the octagon. The case $S = 4$ is obvious.
Let $S > 4$. By assumption, the set of hexagons not containing tiles of the de Bruijn line $\{v_4, m_4\}$ is the set of 3-contracted of a $S-1$ octagon tiling T'. For each tile t of T', there exists a hexagon containing t and two tiles of $\{v_4, m_4\}$ which has obviously not been used in the reconstruction process leading to T'. Hence t can be assigned a sign $+$ or $-$ depending on its position relatively to the positions of the tiles of $\{v_4, m_4\}$. Let $T^+_{\{v_4,m_4\}}$ be the part of T' formed by tiles marked $+$ and $T^-_{\{v_D,m_D\}}$ the part formed by tiles marked $-$.

Let us now consider a straight line l directed along v_4. We claim that if one follows l in the orientation of v_4, then one first meets tiles marked $-$, then tiles marked $+$. This means that T^+ and T^- are convex along l, i.e. that the new de Bruijn line can be inserted correctly in the tiling, leading to a new tiling T. See Figure 6 left. Two cases may occur :

- l meets no vertex of the tiling. Consider two tiles, say t_i and t_{i+1}, which share an edge of type v_k, and such that t_{i+1} is following t_i in the succession of tiles crossed by l in the direction of v_4. Hence there is a possible 4-contracted containing tiles of $\{v_4, m_4\}$ that also contains t_i and t_{i+1}. There are 4 possible sign assignment for t_i, t_{i+1}. The assignment $(t_i, +), (t_{i+1}, -)$ is impossible, otherwise the tile of type $\{v_4, v_k\}$ cannot be placed in the possible 4-contracted. See Figure 6 right.

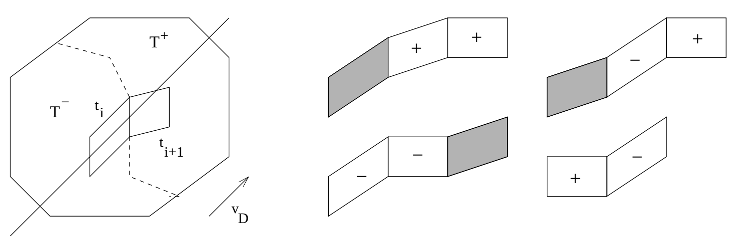

FIG. 6 – left : the tiling T', the crossing line l and the two tiles. right : the four sign assignments. Notice the lower right case which does not allow to add the grey tile.

- l meets a vertex v of the tiling. Then there are two tiles t_i and t_{i+1} playing the same role as in the previous case, but sharing only a single vertex v. There exists one line, close to l, which corresponds to the previous case. This line l' can be chosen as close as possible to l, and the continuity of this translation induces that the order on the tiles t_i and t_{i+1} is the same for l and for l'. Then the result on l' translates to the same result on l. See Figure 7.

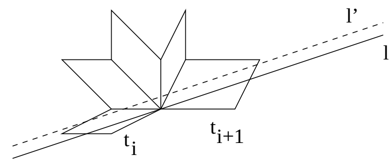

FIG. 7 – l crossing a vertex, and its derived line l'

Hence the $T^+_{\{v_4,m_4\}}$ and $T^-_{\{v_4,m_4\}}$ sets are coherent according to $\{v_4, m_4\}$. Therefore one may translate the part $T^+_{\{v_4,m_4\}}$ of T' by v_4 in order to insert there the de Bruijn line $\{v_4, m_4\}$. The tiling T obtained by this way is the one searched, which ends the proof. □

3 Associated order

We will now define two orders that can be associated to these tilings, and see in which way they correspond.

Definition 6 (set order) *Consider an hexagon built upon 3 vectors, and its two possible tilings T_1 and T_2. We will say that $T_1 < T_2$ if T_1 is in low position whereas T_2 is in high position. Let now O be an octagon, and T, T' two of its tilings. We say that $T \leq_{set} T'$ if on every 3-contracted $C(T) < C(T')$ or $C(T) = C(T')$.*

Thus,two tilings T and T' of a same octagon and differing on only one 3-contracted are said to differ by a *set flip*.

Proposition 3 *The set flip and the geometric flips are equivalent.*

Proof : Let T be a tiling of an octagon. Set flipping one of its 3-contracted is exchanging the positions of 3 tiles, say t_1, t_2 and t_3. Consider the the Bruijn line on which are t_1 and t_2. The flip on this de Bruijn line has just exchanged the positions of t_1 and t_2, which means that their relative positions with all the other tiles of the line are the same. Hence they have to be adjacent. Repeating this argument with the other couples of tiles, one can see that the 3 tiles have to be pairwise adjacent. In this case, a geometric flip can be done on these tiles, and gives the same result. The converse is obvious. □

Definition 7 (flip order) *A flip is said* ascending *if the height of the interior point is increased during the flip, otherwise it is said to be* descending. *This allows to define an order relation on tilings via geometric flips . Let O be an octagon, T and T' two of its tilings. $T \leq_{flip} T'$ if T' can be obtained from T by a sequence of ascending flips.*

Remark : The set of tilings of an octagon O ordered by $\leq_{set}$ has unique smallest and largest elements. One can see that the largest element is the one which has all 3-contracted in high position, and the smallest has all them in low position.

The flip orders on the fundamental tilings, which are the unitary octagon and the hexagon with one repetition, are depicted in Figure 2. In both cases, the $\leq_{set}$ order is the same.

3.1 Representation and characterization

3.1.1 Representation

We will now construct a representation tool for octagon tilings based upon the structure of the contracted set. First, we introduce a few notations. As seen before, the main idea is that 3-contracted, or unitary hexagons, define the entire tiling. Therefore we will use them as the basis of the representation tool. Each hexagon is represented by a vertex, called a *point*, which is black if the associated 3-contracted is in high position, and white if it is in low position, according to the principal height function.

This representation tool relies on the two fundamental examples shown formerly. As seen in proposition 2, the correctness of the tiling is encoded in unitary octagonal tilings structures. These structures are of two types, namely the hexagon with one repeated vector, and the unitary octagon. So we present only these two structures on the diagram.

An octagon is represented by a line containing four points. Each of them represents a 3-contracted, hence a hexagon, and has 3 coordinates taken in the 4 of the octagon. A point is called ijk if constructed upon vectors v_i, v_j, v_k. Presenting the four points 123, 124, 134 and 234 on a line in this order, one can check that all the tilings correspond to coloured half-lines. There is a *midpoint* which has only black vertices on one of its sides and only white vertices on the other side. The white line and the black line are particular cases of tilings. Hence a necessary condition for a line to represent a unitary octagonal tiling is that it has half-lines of only black and only white points.

For the hexagon with one repeated vector, there are 3 tilings in total order, and two 3-contracted. One of the contracted (say 123) can be in high position only if the other ($1'23$, $1'$ being the repetition of the vector 1) is already, so we set an arrow which has 123 as origin and $1'23$ as endpoint. This induces that no arrow in a tiling diagram may have a black endpoint and a white origin. This corresponds to the interpretation that the cube $12'3$ is pilled up on the cube 123.

Taking these remarks together, one may build a diagram for a given tiling. See Figure 8 for an example.

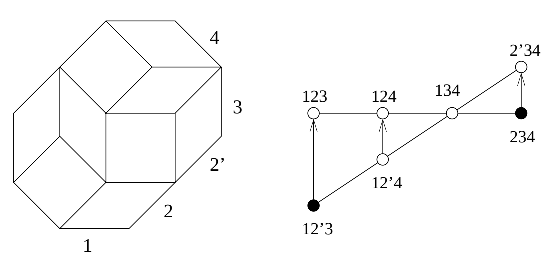

FIG. 8 – One tiling and its associated diagram. Notice the sense of the arrows, according to the inversion property.

Important remark (Inversion property) Let l_1 and l_2 be two lines of a tiling crossing on a point ijk. Then the arrows between points on the left of ijk (i.e. points before ijk in the lexicographic order) have their origin vertex on the same line, and the arrows between points on the right of ijk (i.e. points after ijk in the lexicographic order) have their origin vertex on the same line. Moreover, the arrows between points on the left of ijk start in l_1 if and only if the points on the right of ijk start in l_2. One easily checks the correctness of this property by a case by case analysis. See Figure 8 for an illustration of this property.

3.2 Structure of $\leq_{flip}$

It is obvious that for each pair (T, T') of tilings $T \leq_{flip} T'$ implies that $T \leq_{set} T'$. The converse is not clear. We prove it below.

For convenience, we introduce now a few vocabulary. Let T and T' be two tilings of a same octagon, D and D' their associated diagrams. The points black in both D and D' are said *wholly black*, those white in both D and D' are *wholly white*. The points black only in T are *positive*, and those black only in T' are *negative*.

The arrows of the diagram are the domination, or covering relations. A positive point cannot be covered by a negative or a black point. A positive point is *critical* if it is not covered by any black point. A negative point is critical if it covers no negative point (and obviously no white or positive one). A positive point P is *removable* if, for every line, its transformation into white does not contradict the half-line property, i.e. T with P removed is still a tiling. Symmetrically, a negative point P' is *"add-able"* if its transformation into black preserves the tiling property.

The *cluster* $F(P)$ generated by a point P is the set of all lines passing trough P. Notice that the covering relation induces a total order on the points having same type in a cluster. Moreover, a cluster can be seen as an octagon, and the trace of T and T' over $F(P)$ are tilings. One can then consider critical points removable or addable over $F(P)$.

In order to study the structure of the set of tilings, one sees now that the behavior of the points of one line is clear. The crucial point is to study the interferences between points of different lines. To do this, our investigation process is the following one : we define a graph based on lines of a diagram, and study its behavior, then transpose this to the set of tilings.

Definition 8 (Flip graph) *Let D be the diagram associated to the tilings of a octagon O. The* flip graph *associated to D is $G = (V, E)$, where :*

- *V is the set of vertices of G. $v \in V$ is a line l of D together with the colorations of the points of l.*
- *E is the set of edges of G. $e \in E$ is a couple $(v_1, v_2), v_1 \in V, v_2 \in V$, such that v_1 and v_2 belong to the cluster generated by $v = v_1 \cap v_2$, and the colorings on v_2 prevent a flip on v. v being numbered i on v_1 and v_2, we label e by i^+ if v_1 is before v_2 in the trigonometrical orientation, and i^- otherwise. See Figure 9.*

Notice the following fact, which enables us to define one graph for an entire diagram.

Proposition 4 *Let P be a point removable in $F(P)$, and P' a positive point which covers P. Then P' is removable in $F(P')$.*

Proof : Consider L a line passing through P, and L' a line passing through P' which meets L (there must be, since P' covers P, they differ on one coordinate only). Then, it is clear that

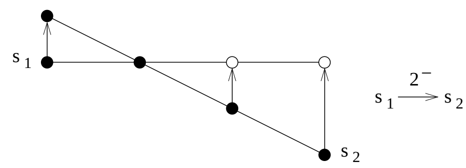

FIG. 9 – example of an edge of an octagonal tiling flip graph

if P is removable on L, then P' is removable on L', because all colorations of L allowing to remove P induce a coloration on L' which enables the removal of P'. See Figure 10 for an example. One can check the other possible configurations lead to the same result. $\square$

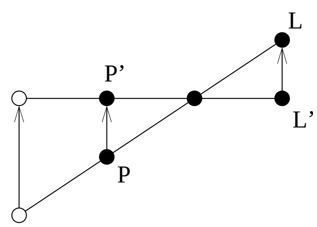

FIG. 10 – The configuration for P and P'

Proposition 5 *For the case of dimension octagonal tilings with $D = 4$, $\leq_{flip} = \leq_{set}$.*

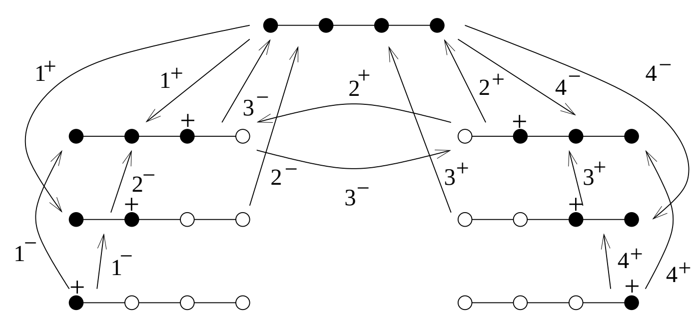

FIG. 11 – flip graph of a octagon

Proof : It is obvious that $\leq_{flip} \Rightarrow \leq_{set}$, because $T \leq_{flip} T'$ is equivalent to the fact that there exists a sequence of ascending flips (or additions of points) leading from T to T'. We do the converse by iterating on the number of differing points. Let T and T' be two tilings such that $T \leq_{set} T'$. We have to show that every positive point of T' can be removed.

To do this, we use the flip graph shown on picture 11. Consider there exists a positive point P in T' that cannot be removed. Then there must be a cycle in the graph containing P and such that the $+$ and $-$ are equilibrated for every index. This is equivalent to the fact that there is a sequence of cubes in the tiling preventing each other to be flipped.

Consider now there is an equilibrated cycle on the flip graph. This cannot contain any edge labelled 1^- or 4^+, because there is no edge leading to their origin vertices. Then, as the cycle has to be equilibrated, there can be no edge labelled 1^+ or 4^+. Removing all these edges, and repeating the preceding arguments for 3^+ and 2^-, one sees there cannot be any equilibrated cycle in the flip graph.

Hence each positive point can be removed, which ends the proof. □

Corollary 2 *$\leq_{flip}$ is a graded poset, its rank function is $rank(T) = B$, where B is the number of 3-contracted in high-position in the tiling T.*

Remark : $\leq_{flip}$ is not a lattice in the general case.

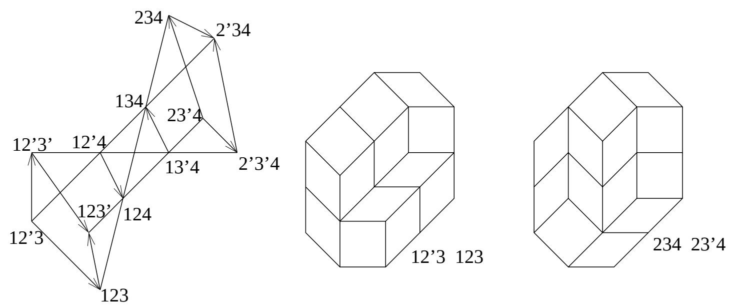

FIG. 12 – left : the diagram for $D = 4$ and 2 repeated vectors, 2 and 3, as 2' and 3' ; right : two tilings and their associated sets of black points (up position 3-contracted).

Consider the example of Figure 12. The octagon has two vectors repeated. We notice $v_1, v_2, v_2', v_3, v_3', v_4$, where the primes stand for the repeated vectors. Then one can build the two tilings having every point white except 123 and $12'3$ for the first tiling and 234 and $23'4$ for the second one. Those tilings are the right part of the figure. There exist two tilings containing both of them (which is the case of every tiling higher in the $\leq_{set}$ or $\leq_{flip}$ order) : in T_1, 12'4, 134, 124, 13'4 and 12'3' become black points, and T_2 is like T_1, but 12'3' is replaced by 2'3'4 as black point. These two tilings are incomparable.

4 Perspectives

We have seen in this paper a decomposition method applied to tilings of octagons. This method also applies to hexagons, and gives an interesting approach for encoding tilings. The associated representative diagrams induces nontrivial results on the spaces of tilings, which are not well known.

The tilings of hexagons and octagons are indeed a special case of a wide class of tilings, called *zonotopal* tilings, defined for larger numbers of vectors, and higher dimensions. The encoding and representation methods described in the core of the paper seem really promissing, and we expect new results for a larger class of zonotopes, and for tilings of more general figures by parallelograms.

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