The theory of Liouville functions

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Abstract

A Liouville function is an analytic function $H : \mathbb{C} \to \mathbb{C}$ with a Taylor series $\sum_{n=1}^{\infty} x^n/a_n$ such the a_n's form a "very fast growing" sequence of integers. In this paper we exhibit the complete first-order theory of the complex field expanded with H.

Keywords: model theory, amalgamation, complex variable, analytic functions

Résumé

Une fonction de Liouville est une fonction analytique $H : \mathbb{C} \to \mathbb{C}$ avec une série de Taylor $\sum_{n=1}^{\infty} x^n/a_n$ telle que les a_n forment une suite d'entiers à croissance "très rapide". Dans cet article nous exhibons la thorie du premier ordre de la structure obtenue en enrichissant le corps des nombres complexes d'une telle fonction.

Mots-clés: théorie des modèles, amalgames, variable complexe, fonctions analytiques

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A Liouville function is an analytic function $H : \mathbb{C} \to \mathbb{C}$ with a Taylor series $\sum_{n=1}^{\infty} x^n/a_n$ such the a_n's form a "very fast growing" sequence of integers. In this paper we exhibit the complete first-order theory of the complex field expanded with H.

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1 Introduction

In [8] Wilkie calls "Liouville function" a function $H : \mathbb{C} \to \mathbb{C}$ with a Taylor series of the form

$$H(x) = \sum_{i=1}^{\infty} x^i/a_i$$

where the a_i are non-zero integers satisfying the condition:

$$\text{for every } l \geq 1,\ |a_{i+1}| > |a_i|^{i^l} \text{ for all sufficiently large } i. \tag{1}$$

A fragment of the first-order theory of the complex field expanded with H is described in [8]. In this paper we exhibit the complete first-order theory. It turns out that this theory is the "limit theory of generic polynomials" recently studied in [4] (this answers a question of Zilber [9]). We recall the axiomatization of the theory in section 2, where we also present another equivalent axiomatization which is closer in spirit to [8]. In section 3 we give a short overview of the proof of our main result and present two of the main tools: continuity of the roots of polynomial systems and effective quantifier elimination. The last two sections are devoted to the proof of the main result.

A model of our theory can be constructed by a Hrushovski-style amalgamation method [4, 10]. It is therefore natural to ask whether analytic models exist for other theories constructed by this method. The limit theory of generic curves [3] and the theory of generic functions with derivatives [10] are two natural candidates. Additional examples and further discussion can be found in the surveys [6, 9].

2 Axiomatization

In this section only we work within an arbitrary algebraically closed field K of characteristic 0. In the remainder of the paper we set $K = \mathbb{C}$. We do not work in the language of "curved fields" (the language of fields expanded with a binary predicate) as in [4] but in the language of fields expanded with a unary function symbol H. We call this language $\mathcal{L}$. Given a tuple $\overline{x}$ of elements of K, $H(\overline{x})$ denotes the tuple obtained by applying H componentwise. This notation will be used freely throughout the paper, for H as well as for other unary functions.

2.1 The Limit Theory of Generic Polynomials

A generic polynomial of degree d is of the form $g_d(x) = \sum_{i=1}^{d} \alpha_i x^i$ where the coefficients α_i are algebraically independent over $\mathbb{Q}$. Let F be a sentence of $\mathcal{L}$. We have shown in [4] that F is either true for all generic polynomials of sufficiently high degree, or false for all generic polynomials of sufficiently high degree. The set T of sentences which are ultimately true therefore forms a complete theory. We recall that this theory is defined by the following axioms.

1. The axioms of algebraically closed fields of characteristic 0.
2. $H(0) = 0$.
3. The universal axioms. Let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a conjunction of polynomial equations with coefficients in $\mathbb{Q}$. If the subset of K^{2n} defined by ϕ is of dimension $< n$, we add the axiom
$$\forall x_1, \ldots, x_n \bigwedge_i x_i \neq 0 \wedge \bigwedge_{i \neq j} x_i \neq x_j \to \neg\phi(\overline{x}, H(\overline{x})). \tag{2}$$
4. The inductive axioms. Let $\phi(x_1, y_1, \ldots, x_n, y_n, \overline{z})$ be a conjunction of polynomial equations with rational coefficients. For any fixed value of the parameter $\overline{z}$, ϕ defines an algebraic subset $V_{\overline{z}}$ of K^{2n}. Let $\xi(\overline{z})$ be a formula of the language of fields which states that $V_{\overline{z}}$ is irreducible, has dimension n and is not contained in a subspace of the form $x_i = x_j$ for some $i \neq j$, or of the form $x_i = c$ for some element c in the model.

   Let ϵ be a function which chooses one variable $u_i^\epsilon \in \{x_i, y_i\}$ for every $i \in \{1, \ldots, n\}$. For each value of the parameter $\overline{z}$, the formula $\exists u_1^\epsilon, \ldots, u_n^\epsilon \phi(\overline{x}, \overline{y}, \overline{z})$ defines a constructible set $C_{\overline{z}}^\epsilon \subseteq K^n$. As pointed out in [3], there is a formula $\psi_\epsilon(\overline{z})$ of the language of fields which states that $C_{\overline{z}}^\epsilon$ is dense in K^n. Let $\psi(\overline{z})$ be the disjunction of the 2^n formulas $\psi_\epsilon(\overline{z})$. Let θ be the conjunction of ξ and ψ. We add the following axiom:
$$\forall \overline{z}\ \exists x_1, \ldots, x_n\ \theta(\overline{z}) \to \phi(\overline{x}, H(\overline{x}), \overline{z}). \tag{3}$$

In fact these inductive axioms are slightly different from those of [4]. Indeed, there was not requirement that $\dim V_{\overline{z}} = n$ in that paper. An inspection of the completeness proof in [4] reveals that the inductive axioms are used only in the case $\dim V_{\overline{z}} = n$. The corresponding theories are therefore identical.

We have shown in [4] that a universal axiom is satisfied by a generic polynomial g_d of degree d as soon as $d \geq n$ and that an inductive axiom is satisfied as soon as $d \geq n(nr+n+r+1)$. More precisely, we have obtained the following result.

Theorem 1 *Let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_n, \overline{z})$ be a conjunction of polynomial equations with coefficients in $\mathbb{Q}$. Given a tuple $\overline{z}$ of parameters we denote by $V_{\overline{z}}$ the algebraic subset of $\mathbb{C}^{2n}$ defined by ϕ.*

Fix a tuple $\overline{z}$ which satisfies the associated condition $\theta(\overline{z})$ from (3). Let r be the transcendence degree of $\mathbb{Q}(\overline{z})$ over $\mathbb{Q}$. For any $k \geq 0$, if $d \geq n(nr+n+r+1)$ there exists $\overline{x} \in \mathbb{C}^n$ such that $(\overline{x}, g_d(\overline{x}))$ is a generic point of $V_{\overline{z}}$ (i.e., a point of transcendence degree n over $\mathbb{Q}(\overline{z})$).

2.2 Axiomatization à la Wilkie

Given $I \subseteq \{1, \ldots, n\}$, say $I = \{i_1, \ldots, i_k\}$ (in increasing order), and a n-tuple $x = (x_1, \ldots, x_n)$ of elements or variables, we denote by $\overline{x}_I$ the k-tuple $(x_{i_1}, \ldots, x_{i_k})$.

Consider n polynomials $f_i(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_r)$ with integer coefficients. For each value of the parameter $\overline{z}$ we obtain a polynomial map $F_{\overline{z}} : K^n \times K^n \to K^n$. Recall that a zero $(\overline{a}, \overline{b})$ of $F_{\overline{z}}$ is said to be regular if the Jacobian matrix of $F_{\overline{z}}$ at $(\overline{a}, \overline{b})$ has rank n, or in other words if

$$\det\left(\frac{\partial F_{\overline{z}}}{\partial(\overline{x}_I, \overline{y}_J)}(\overline{a}, \overline{b})\right) \neq 0 \tag{4}$$

for some $I, J \subseteq \{1, \ldots, n\}$ such that $|I|+|J| = n$. Let $V(F_{\overline{z}})$ be the set of zeros of $F_{\overline{z}}$. A regular zero lies on a unique irreducible component of $V(F_{\overline{z}})$ and this component is of dimension n. Following Wilkie, we say that $(\overline{a}, \overline{b})$ is a balanced zero[1] of $F_{\overline{z}}$ if $a_1, \ldots, a_n$ are all non-zero and pairwise distinct, and if one can choose I and J so that $I \cup J = \{1, \ldots, n\}$ (or equivalently, so that $I \cap J = \emptyset$).

In a geometric language, condition (4) means that the tangent space to $V(F_{\overline{z}})$ at $(\overline{a}, \overline{b})$ has dimension n. If this tangent space is not included in a subspace of the form $x_i = c$ for some $i \in \{, \ldots, n\}$ and some constant $c \in K$ and if additionally $(\overline{a}, \overline{b})$ is a balanced zero of $F_{\overline{z}}$, we say that $(\overline{a}, \overline{b})$ is a *well balanced zero.*

In this axiomatization we keep the axioms 1,2 and 3 from section 2.1 but we replace the inductive axioms by:

4'. Let $\theta'(\overline{z})$ be a formula of the language of fields which expresses that $F_{\overline{z}}$ has a well balanced zero. We add the following axiom.

$$\forall z_1, \ldots, z_r \; \exists x_1, \ldots, x_n \; \theta'(\overline{z}) \to F(\overline{x}, H(\overline{x}), \overline{z}) = 0. \tag{5}$$

The following lemma is standard.

[1] Wilkie uses the terminology "balanced, non-singular zero". We just write "balanced zero" for short.

Lemma 1 *Let $(\overline{a}, \overline{b})$ be a regular zero of $F_{\overline{z}}$. For any neighbourhood V of $(\overline{a}, \overline{b})$ in $\mathbb{C}^{2n}$ there exists a neighbourhood U of $\overline{z}$ such that for any $\overline{\zeta} \in U$, $F_{\overline{\zeta}}$ has a regular zero in V.*

Proof. Let I and J be such that condition (4) is satisfied. There exist n affine functions $l_1, \ldots, l_n$ such that $(\overline{a}, \overline{b})$ is an isolated solution of the system $f_1 = 0, \ldots, f_n = 0, l_1 = 0, \ldots, l_n = 0$. Since we now have as many equations as unknowns (namely, $2n$) we can apply Proposition 3: there exists a neighbourhood U of $\overline{z}$ such that for any $\overline{\zeta} \in U$, the system $F_{\overline{\zeta}} = 0, l_1 = 0, \ldots, l_n = 0$ has a zero $(\overline{\alpha}, \overline{\beta}) \in V$. Since $\det\left(\frac{\partial F_{\overline{\zeta}}}{\partial(\overline{x}_I, \overline{y}_J)}(\overline{\alpha}, \overline{\beta})\right)$ is a continuous function of $\overline{\zeta}$, $\overline{\alpha}$ and $\overline{\beta}$ we can choose U so small that $\det\left(\frac{\partial F_{\overline{\zeta}}}{\partial(\overline{x}_I, \overline{y}_J)}(\overline{\alpha}, \overline{\beta})\right) \neq 0$. $\square$

The point of working with this second axiomatization is that we have the following proposition.

Proposition 1 *The set of parameters $\overline{z}$ such that $F_{\overline{z}}$ has a balanced zero is an open subset of $\mathbb{C}^r$. The same is true of the set of parameters $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero.*

Proof. Note that in the proof of Lemma 1, the subsets $I, J \subseteq \{1, \ldots, n\}$ which witness the fact that F_ζ has a regular zero are the same for all $\zeta \in U$. This implies immediately the first part of the proposition. The second part follows from a similar continuity argument. $\square$

2.3 Equivalence of these axiomatizations

We first show that any model of the limit theory of generic curves is also a model of the theory defined in section 2.2.

Let (K, H) be a model of the limit theory of generic polynomials. Let $F : K^n \times K^n \times K^r \to K^n$ be a polynomial map with integer coefficients. Fix a tuple $\overline{z}$ such that $F_{\overline{z}}$ has a well-balanced zero $(\overline{x}, \overline{y})$. This well balanced zero lies on an irreducible component V of $V(F_{\overline{z}})$ defined by a conjunction $\phi(\overline{x}, \overline{y}, \overline{\zeta})$ of polynomial equations where the parameters $\overline{\zeta}$ lie in the algebraic closure of $\mathbb{Q}(\overline{z})$. We claim that $\overline{\zeta}$ satisfies the associated condition $\theta(\overline{\zeta})$ from the inductive axioms. Indeed, it follows from the implicit function theorem that $K \models \psi(\overline{\zeta})$. Moreover, V is not included in a subspace of the form $x_i = x_j$ for some $i \neq j$ since the components of $\overline{x}$ are pairwise distinct. Finally, V is not included in a subspace of the form $x_i = c$ for some constant c due to the condition on the tangent space at $(\overline{x}, \overline{y})$. We can therefore apply the inductive axioms: there exists $\overline{x} \in \mathbb{C}^n$ such that $\phi(\overline{x}, H(\overline{x}), \overline{z})$. In particular we have $F(\overline{x}, H(\overline{x}), \overline{z}) = 0$.

Next we show that any model of the theory defined in section 2.2 is a model of the limit theory of generic polynomials. Consider therefore a model (K, H) of the theory defined in section 2.2. We assume without loss of generality that K is of infinite transcendence degree over $\mathbb{Q}$.

Lemma 2 *Let $S \subseteq K^m$ be a constructible set defined by a boolean combination of polynomial equations with coefficients in a subfield k of K. Assume that the projection of S on the first d variables is dense in K^d.*

There exists a polynomial map $F = (f_1, \ldots, f_{m-d}) : K^m \to K^{m-d}$ which satisfies the following properties:

(i) f_i depends only on the first $d + i$ variables and its coefficients are in k (i.e., $f_i \in k[X_1, \ldots, X_{d+i}]$).

(ii) The algebraic set $V(F) = \{x \in K^m;\ F(x) = 0\}$ has a dense projection on the first d variables.

(iii) There exists a nonzero polynomial $P \in k[X_1, \ldots, X_d]$ such that for any $x \in V(F)$, $P(x_1, \ldots, x_d) \neq 0$ implies that $x \in S$ and that x is a regular zero of F (more precisely, the matrix of partial derivatives of f_i, $i = 1, \ldots, m - d$ with respect to X_j, $j = d + 1, \ldots, m$ has rank $m - d$ at x).

Note that property (iii) implies (ii) by the implicit function theorem. We will use this lemma only in the case where S is an algebraic set.

Proof of Lemma 2. We assume without loss of generality that k is of finite transcendence degree over $\mathbb{Q}$. Since the projection of S on the first d variables is dense, there exists a point $(\alpha_1, \ldots, \alpha_d)$ of transcendence d over k which is in the projection. In fact, there exists $\alpha \in S$ such that $\alpha_1, \ldots, \alpha_d$ is a transcendence basis of α over k. Let f_i be the minimal polynomial of α_{d+i} over $k(\alpha_1, \ldots, \alpha_{d+i-1})$. Condition (i) is satisfied by definition, and condition (ii) is also satisfied since $\alpha \in V(F)$.

Let $(x_{d+1}, \ldots, x_m)$ be such that $(\alpha_1, \ldots, \alpha_d, x_{d+1}, \ldots, x_m) \in V(F)$ and let $\phi(X_1, \ldots, X_m)$ be a a boolean combination of polynomial equations with coefficients in k which is satisfied by $\alpha_1, \ldots, \alpha_m$. This formula is also satisfied by $(\alpha_1, \ldots, \alpha_d, x_{d+1}, \ldots, x_m)$ since the fields $k(\alpha_1, \ldots, \alpha_d, x_{d+1}, \ldots, x_m)$ and $k(\alpha_1, \ldots, \alpha_m)$ are isomorphic. In other words, $(\alpha_1, \ldots, \alpha_d)$ satisfies formula $\Phi'(x_1, \ldots, x_d)$ below:

$$\forall x_{d+1}, \ldots, x_m F(x_1, \ldots, x_m) = 0 \Rightarrow \Phi(x_1, \ldots, x_m).$$

Since $\alpha_1, \ldots, \alpha_d$ are algebraically independent over k, Φ' defines a Zariski dense subset of K^d. This is exactly the first part of condition (iii), if we take for Φ the formula defining S. To obtain the second part of this condition, we apply the same observation to a different Φ. Namely, we apply it to the formula $\Phi(x_1, \ldots, x_m) \equiv \bigwedge_{i=1}^{m-d} \dfrac{\partial f_i}{\partial X_{d+i}}(x_1, \ldots, x_{d+i}) \neq 0$. This formula is satisfied by α since f_i is the minimal polynomial of α_{d+i} over $k(\alpha_1, \ldots, \alpha_{d+i-1})$. Our observation now implies that exists a nonzero polynomial $R \in k[X_1, \ldots, X_d]$ such that for any point $x \in V(F)$ with $R(x_1, \ldots, x_d) \neq 0$, the $m - d$ partial derivatives $\frac{\partial f_i}{\partial X_{d+i}}$ do not vanish at x. Since the jacobian matrix of F contains a triangular matrix with these partial derivatives on the diagonal, it has maximum rank $m - d$ at x and this point is by definition a regular zero of F. $\square$

There is of course nothing special about the first d variables in this lemma: we may project on any tuple of variables as long as the projection is dense. This is just what we shall do now.

Let $\phi(x_1, y_1, \ldots, x_n, y_n, \overline{z})$ be a conjunction of polynomial equations with rational coefficients. Fix $\overline{z}$ such that the associated formula θ in (3) is satisfied. Since $K \models \theta(\overline{z})$ there exists $I, J \subseteq \{1, \ldots, n\}$ such that $|I| + |J| = n$, $I \cap J = \emptyset$ and the projection of $V_{\overline{z}}$ on the variables $\overline{x}_I \,\hat{}\, \overline{y}_J$ is dense in K^n. Let us apply Lemma 2 to $S = V_{\overline{z}}$. We obtain a polynomial map $F : K^n \times K^n \to K^n$ and a polynomial P in n variables with coefficients in $k = \mathbb{Q}(\overline{z})$ such that for any point $\overline{x} \,\hat{}\, \overline{y} \in V(F)$, $P(\overline{x}_I, \overline{y}_J) \neq 0$ implies that $\overline{x} \,\hat{}\, \overline{y} \in V_{\overline{z}}$ and that $\overline{x} \,\hat{}\, \overline{y}$ is a regular zero of F. Since $\dim V_{\overline{z}} = n$ and $V_{\overline{z}}$ is irreducible, this variety is an irreducible component of $V(F)$. Now we consider the polynomial map $G : K^{2(n+1)} \to K^{n+1}$ which sends $(x_1, \ldots, x_n, x_{n+1}, y_1, \ldots, y_n, y_{n+1})$ to $F(x_1, \ldots, x_n, y_1, \ldots, y_n) \,\hat{}\, f_{n+1}(x_1, \ldots, x_n, x_{n+1}, y_1, \ldots, y_n, y_{n+1})$. Here x_{n+1} and y_{n+1} are two additional variables, and $f_{n+1} = P(\overline{x}_I, \overline{y}_J) y_{n+1} - 1$. We claim that G has a well balanced zero. To obtain such a zero, pick a generic point $\overline{a} \,\hat{}\, \overline{b}$ of $V_{\overline{z}}$. Since $K \models \theta(\overline{z})$, the components $a_1, \ldots, a_n$ are all nonzero and distinct from each other. Moreover we can set $b_{n+1} = 1/P(\overline{a}_I, \overline{b}_J)$ since $\overline{a}_I \,\hat{}\, \overline{b}_J$ has transcendence degree n over k. Pick an arbitrary a_{n+1} different from 0 and from $a_1, \ldots, a_n$. The matrix of partial derivatives of $f_1, \ldots, f_{n+1}$ at $(a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1})$ with respect to the variables x_i $(i \in J)$ and y_i $(i \in I \cup \{n+1\})$ has the block form

$$B = \begin{pmatrix} A & 0 \\ 0 & P(\overline{a}_I, \overline{b}_J) \end{pmatrix}$$

where A is the matrix of partial derivatives of $f_1, \ldots, f_n$ with respect to the variables x_i $(i \in J)$ and y_i $(i \in I)$. We know from Lemma 2 that A has rank n. Hence B has rank $n+1$ and $(a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1})$ is a balanced zero of G. In order to show that this zero is well balanced, we have to check the condition on the tangent space. This condition is indeed satisfied due to Lemma 3 below and to the fact x_{n+1} does not appear in $f_1, \ldots, f_{n+1}$.

Lemma 3 *The tangent space to $V(F)$ at $\overline{a} \,\hat{}\, \overline{b}$ is not included in a subspace of the form $x_i = c$ for some constant $c \in K$.*

Proof. Assume the opposite. We have seen that $V_{\overline{z}}$ is an irreducible component of $V(F)$. Since $\overline{a} \,\hat{}\, \overline{b}$ is a generic point of $V_{\overline{z}}$ this variety would be included in the subspace $x_i = c$. This is in contradiction with the hypothesis $K \models \theta(\overline{z})$. □

We can therefore apply the modified inductive axioms (5) to G: there exists $x_1, \ldots, x_{n+1}$ such that $F(x_1, \ldots, x_n, H(x_1), \ldots, H(x_n), \overline{z}) = 0$ and $P(\overline{x}_I, \overline{y}_J) y_{n+1} - 1 = 0$. Since $P(\overline{x}_I, \overline{y}_J) \neq 0$, $(x_1, \ldots, x_n, H(x_1), \ldots, H(x_n)) \in V_{\overline{z}}$ by property (iii): we have proved that the inductive axioms (3) are satisfied.

3 Structure of the Proof

Wilkie has shown in [8] that Liouville functions satisfy the universal axioms. In order to show that the theory of Liouville functions is the limit theory of generic polynomials, it remains to show that the inductive axioms are also satisfied. Using the axiomatization of section 2.2, we need the following result.

Theorem 2 (Main Theorem) *Let $G : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If G has a well balanced zero, there exists $\overline{a} \in \mathbb{C}^n$ such that $G(\overline{a}, H(\overline{a})) = 0$.*

The proof will be given at the end of section 5. A special case of this theorem (which we will not use here) was obtained in [8]: let $G : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with integer coefficients. If G has a balanced zero then there exists $\overline{a} \in \mathbb{C}^n$ such that $G(\overline{a}, H(\overline{a})) = 0$. Wilkie's proof of this result relies in particular on Newton's method.[2] By contrast the proof of Theorem 2 is based on a method which is reminiscent of homotopy methods for solving systems of polynomial equations. Let $H_d(x)$ be the partial sum $\sum_{i=1}^{d} x^i/a_i$. In section 4 we show that the system $G(\overline{x}, H_d(\overline{x})) = 0$ has an isolated solution $\overline{x}_d \in \mathbb{C}^n$ for all sufficiently large d. Then we track $\overline{x}_d$ as d goes to infinity. It turns out that these roots remain in a compact subset of $\mathbb{C}^n$. Theorem 2 then follows immediately from a standard uniform convergence argument.

As in [8] effective quantifier elimination also plays an important role. Here there is an additional complication due to the presence of arbitrary complex parameters in G. Our solution to this problem is to work not for a single value of the parameters (i.e. for a single G), but simultaneously for all parameters in a compact set. This is made possible in particular by Proposition 1. We will use the following version of quantifier elimination (note that we need to work over the real numbers).

Proposition 2 *Let $\Phi(x_1, \ldots, x_n)$ be a formula of the language of ordered rings. The k polynomials occuring in Φ have integer coefficients. Let $h \geq 2$ be an upper bound on their absolute values. Let $d \geq 2$ be an upper bound on the degrees of these polynomials, and let m be the number of occurrences of quantifiers in F.*

In the theory of real-closed fields, ϕ is equivalent to a quantifier-free formula $\Psi(x_1, \ldots, x_n)$ in which all polynomials are of degree at most d^c, and have integer coefficients of absolute value bounded by h^{d^c}. The constant c depends only on n, m and k.

Almost any reasonable quantifier elimination method will yield the above result. Much more precise bounds are known, see for instance [1, 7]. We will not need them here since the parameters n, m and k can be treated as constants for our purposes.

In the remainder of this section we present another important tool: continuity of the roots of polynomial systems. For $\overline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $||\overline{z}|| = \sum_{i=1}^{n} |z_i|^2$ (this is just the Euclidean norm on $\mathbb{R}^{2n}$).

[2] In fact he even shows that the map $\overline{x} \mapsto G(\overline{x}, H(\overline{x}))$ has a non-singular zero with pairwise distinct, nonzero coordinates.

Proposition 3 (Continuity of roots) *The following property holds for any polynomial map $F : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ and any $\overline{z} \in \mathbb{C}^r$.*

Let $\overline{x}$ be an isolated root of the map $F_{\overline{z}} : \overline{x} \mapsto F(\overline{x}, \overline{z})$. For any sufficiently small neighbourhood U of $\overline{x}$ there exists a neighbourhood V of $\overline{z}$ such that for all $\overline{\zeta} \in V$, the number of roots of $F_{\overline{\zeta}}$ in U is positive and finite.

This follows for instance from the "extended geometric version" of Bézout's theorem [2].

Corollary 1 *Let $F : \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ be a polynomial map. The map*

$$\begin{aligned} N_F \quad : \quad & \mathbb{C} \to \mathbb{R} \cup \{+\infty\} \\ & z \mapsto \min\{||\overline{x}||;\ \overline{x} \text{ is an isolated root of } F(\overline{x}, \overline{z}) = 0\} \end{aligned}$$

is upper semi-continuous (we set $N_F(\overline{z}) = +\infty$ if the system $F(\overline{x}, \overline{z}) = 0$ has no isolated roots).

Proof. Fix $\overline{z} \in \mathbb{C}^r$ such that $N_F(\overline{z}) < +\infty$. We have to show that for every $\epsilon > 0$ there is a neighbourhood U of $\overline{z}$ such that $N_F(\overline{z}) \geq N_F(\overline{\zeta}) - \epsilon$ for every $\overline{\zeta} \in U$. let $\overline{x}$ be an isolated root of the system $F(\overline{x}, \overline{z}) = 0$ such that $N_F(\overline{z}) = ||\overline{x}||$. By Proposition 3.(i) there is a neighbourhood U of $\overline{z}$ such that for every $\overline{\zeta} \in U$ the system $F(\overline{x}, \overline{z}) = 0$ has an isolated root in the ball $B(x, \epsilon)$. Hence $N_F(\overline{\zeta}) \leq ||\overline{x}|| + \epsilon = N_F(\overline{z}) + \epsilon$. $\Box$

One can show that if $\overline{z}$ is such that $F_{\overline{z}}$ has finitely many roots then N_F is continuous in $\overline{z}$. The example of the polynomial $F(x, z) = zx^2 - 2x + 1$ shows that no such continuity property holds for the map

$$z \mapsto \max\{||\overline{x}||;\ \overline{x} \text{ is an isolated root of } F(\overline{x}, \overline{z}) = 0\}.$$

Indeed, F has a single root for $z = 0$; for $z \not\in \{0, 1\}$ it has a second root which goes to infinity as z goes to 0.

4 A Starting Point for the Homotopy

We denote $H_{d,\epsilon}(x) = \sum_{i=1}^{d} x^i/a_i + \epsilon x^{d+1}$ and $H_d = H_{d,0}$. Let g_d be a generic polynomial of degree d. The *modified partial sum* $\mu_{k,d}$ is the polynomial function $x \mapsto H_k(x) + x^k g_d(x)$.

4.1 Finiteness for modified partial sums

We temporarily revert to the language of curved fields to cite a simple combinatorial result (Lemma 7 from [5]).

Lemma 4 *Let k be a p-sufficient substructure of a curved field (K, C) and $\overline{z} = (z_1, \ldots, z_r)$ a tuple of r elements of $K \backslash k$. Set $q = \lfloor \frac{p-r}{r+1} \rfloor$. Let j be the smallest integer such that there exists an extension ℓ of $k(\overline{z})$ satisfying* $\mathrm{tr.deg}(\ell/k) \leq p - q(j+1)$ *and $\delta(\ell : k) \leq j$ (note that j always exists and is upper bounded by r). Then ℓ is q-sufficient.*

Some explanations are in order. In this lemma K is an arbitrary field and the "curve" C is an arbitrary subset of K^2. The symbol $\delta(\ell : k)$ is defined by the formula

$$\delta(\ell : k) = \text{tr.deg}(\ell : k) - \text{Card}(C \cap \ell^2 - C \cap k^2).$$

A subfield k of K is said to be p-sufficient if $\delta(\ell : k) \geq 0$ for any subfield ℓ of K which contains k, and is of transcendence degree at most p over k. For instance, we have seen is section 2.1 that the universal axiom (2) is satisfied by a generic polynomial of degree d as soon as $d \geq n$. This implies that $\mathbb{Q}$ is d-sufficient if we interpret C by the graph of a generic polynomial $g : \mathbb{C} \to \mathbb{C}$ of degree d.

Theorem 3 *Let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_r)$ be a boolean combination of polynomial equations with coefficients in $\mathbb{Q}$. Fix a tuple $\overline{z}$ such that the constructible subset $D_{\overline{z}}$ of $\mathbb{C}^{2n}$ defined by $\phi(.,.,\overline{z})$ has dimension at most n. Let C_d be the graph of a generic polynomial of degree d. If $d \geq n(r+1)+r$ the system*

$$\bigwedge_i x_i \neq 0 \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i C_d(x_i, y_i) \wedge \phi(\overline{x}, \overline{y}, \overline{z}) \tag{6}$$

has at most finitely many solutions in $\mathbb{C}^{2n}$.

Proof. Since $\mathbb{Q}$ is d-sufficient, by choice of d and Lemma 4 there exists a n-sufficient extension ℓ of $\mathbb{Q}(\overline{z})$ of transcendence degree at most $d-n$ over $\mathbb{Q}$. There are at most $d-n$ nonzero points on C_d with both coordinates in ℓ^2. Moreover, outside ℓ^{2n} any solution of the system

$$\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i C_d(x_i, y_i)$$

must be of transcendence degree at least n over ℓ (by choice of ℓ). We conclude that up to a finite set, all solutions of (6) are of transcendence degree at least n over $\mathbb{Q}(\overline{z})$.

This implies that the subset S of $D_{\overline{z}}$ defined by (6) is finite. Indeed, if S is infinite this (constructible) subset of $D_{\overline{z}}$ must contain infinitely many non-generic points of $D_{\overline{z}}$ (i.e., points of transcendence degree over $\mathbb{Q}(\overline{z})$ smaller than $\dim D_{\overline{z}}$). $\square$

We have the same property for modified partial sums.

Corollary 2 *Let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_r)$ be a boolean combination of polynomial equations with coefficients in $\mathbb{Q}$. Fix a tuple $\overline{z}$ such that the algebraic subset $D_{\overline{z}}$ of $\mathbb{C}^{2n}$ defined by $\phi(.,.,\overline{z})$ has dimension at most n. If $d \geq n(r+1)+r$ the system*

$$\bigwedge_i x_i \neq 0 \wedge \bigwedge_{i \neq j} x_i \neq x_j \wedge \phi(\overline{x}, \mu_{k,d}(\overline{x}), \overline{z})$$

has at most finitely many solutions in $\mathbb{C}^{2n}$.

Proof. Let $\overline{x}$ be a solution of the system. Note that $(\overline{x}, g_d(\overline{x}))$ lies in the constructible subset of $\mathbb{C}^{2n}$ (call it $C_{\overline{z}}$):

$$\bigwedge_i x_i \neq 0 \wedge \phi(x_1, \ldots, x_n, H_k(x_1) + x_1^k y_1, \ldots, H_k(x_n) + x_n^k y_n).$$

Let $P : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the polynomial map

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n, H_k(x_1) + x_1^k y_1, H_k(x_n) + x_n^k y_n).$$

Since $P(C_{\overline{z}}) \subseteq D_{\overline{z}}$ and every point in $D_{\overline{z}}$ has finitely many preimages in $C_{\overline{z}}$ we have $\dim C_{\overline{z}} \leq \dim D_{\overline{z}} \leq n$. By Theorem 3 (applied to $C_{\overline{z}}$) we conclude that our system has finitely many solutions. □

4.2 Existence for Modified Partial Sums

The only property of the coefficients of Liouville functions that will be used in the next proposition is that they are rational numbers.

Proposition 4 *Let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_n, \overline{z})$ be a conjunction of polynomial equations with coefficients in $\mathbb{Q}$. Given a tuple $\overline{z}$ of parameters we denote by $V_{\overline{z}}$ the algebraic subset of $\mathbb{C}^{2n}$ defined by ϕ.*

Fix a tuple $\overline{z}$ which satisfies the associated condition $\theta(\overline{z})$ from (3). Let r be the transcendence degree of $\mathbb{Q}(\overline{z})$ over $\mathbb{Q}$. For any $k \geq 0$, if $d \geq n(nr+n+r+1)$ there exists $\overline{x} \in \mathbb{C}^n$ such that $(\overline{x}, \mu_{k,d}(\overline{x}))$ is a generic point over $\mathbb{Q}(\overline{z})$ of $V_{\overline{z}}$ (note that this genericity condition implies in particular that the components of $\overline{x}$ are nonzero and pairwise distinct).

Proof. We argue as in Corollary 2. One would like to find a point $(\overline{x}, g_d(\overline{x}))$ on the algebraic subset $W_{\overline{z}}$ of $\mathbb{C}^{2n}$ defined by the formula

$$\phi(x_1, \ldots, x_n, H_k(x_1) + x_1^k y_1, \ldots, H_k(x_n) + x_n^k y_n).$$

Pick a generic point $(\overline{\alpha}, \overline{\beta})$ of $V_{\overline{z}}$. Then $(\overline{\alpha}, \overline{\gamma}) \in W_{\overline{z}}$ where $\gamma_i = (\beta_i - H_k(\alpha_i))/\alpha_i^k$. Note that $\alpha_1, \ldots, \alpha_n$ are pairwise distinct and do not belong to the algebraic closure K of $\mathbb{Q}(\overline{z})$. Moreover there exists $I, J \subseteq \{1, \ldots, n\}$ such that $|I| + |J| = n$, $I \cap J = \emptyset$ and $\overline{\alpha}_I \,\hat{}\, \overline{\beta}_J$ is of transcendence degree n over K. The tuple $\overline{\alpha}_I \,\hat{}\, \overline{\gamma}_J$ is also of transcendence degree n over K since $K(\overline{\alpha}_I, \overline{\beta}_J) = K(\overline{\alpha}_I, \overline{\gamma}_J)$. We can therefore apply the inductive axioms to the irreducible component of $W_{\overline{z}}$ which contains $(\overline{\alpha}, \overline{\gamma})$. More precisely, by Theorem 1 there exists a point $(\overline{x}, g_d(\overline{x})) \in W_{\overline{z}}$ of transcendence degree n over K. We conclude that $(\overline{x}, \mu_{k,d}(\overline{x}))$ is a point of $V_{\overline{z}}$ of transcendence degree n over K. □

4.3 Isolated Solutions

The results of sections 4.1 and 4.2 can be summarized as follows.

Theorem 4 *Let $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ be a polynomial map with integer coefficients. For any $k \geq 0$, any $d \geq n(nr+n+r+1)$ and any $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero, the system $F(\overline{x}, \mu_{k,d}(\overline{x}), \overline{z}) = 0$ has an isolated solution.*

Proof. Fix $k \geq 0$, $d \geq n(nr+n+r+1)$ and $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero. This well balanced zero lies on an irreducible component V of $V(F_{\overline{z}})$ of dimension n and V satisfies the inductive axioms. More precisely, we have seen at the beginning of section 2.3 that V is defined by a conjunction $\phi(\overline{x},\overline{y},\overline{\zeta})$ of polynomial equations where the parameters $\overline{\zeta}$ lie in the algebraic closure of $\mathbb{Q}(\overline{z})$ and satisfy the associated condition $\theta(\overline{\zeta})$. By Proposition 4 there exists $\overline{x} \in \mathbb{C}^n$ with nonzero, pairwise distinct coordinates such that $(\overline{x}, \mu_{k,d}(\overline{x}))$ is in V. Since $\dim V = n$, by Corollary 3 there are only finitely many such $\overline{x}$. To make sure that $\overline{x}$ is an isolated solution of the system $F(\overline{x}, \mu_{k,d}(\overline{x}), \overline{z}) = 0$ it is therefore sufficient to satisfy the following requirement: $(\overline{x}, \mu_{k,d}(\overline{x}))$ should lie on no other irreducible component of $V(F_{\overline{z}})$ but V (indeed, other components might be of dimension $> n$). This is possible thanks to the genericity condition in the conclusion of Proposition 4. □

Any inductive axiom is eventually satisfied by H_d if d is sufficiently large. More precisely:

Theorem 5 *Let $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ be a polynomial map. If d is sufficiently large and $|\epsilon| \leq 1/|a_{d+1}|$ the following property holds: for any $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero, the system $F(\overline{x}, H_{d,\epsilon}(\overline{x}), \overline{z}) = 0$ has an isolated solution.*

Note that we do not rule out the the possibility that non-isolated solutions might also exist.

Proof of Theorem 5. Set $d_0 = n(nr+n+r+1)$. For any $d \geq 1$ and $\overline{\alpha} \in \mathbb{R}^{d_0}$, let

$$\nu_{k,\overline{\alpha}}(x) = H_k(x) + x^k \sum_{j=1}^{d_0} \alpha_j x^j.$$

One can easily write down a formula $\phi(\overline{\alpha})$ of the language of ordered fields which expresses the fact for any $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero, the system $F(\overline{x}, \nu_{k,\overline{\alpha}}(\overline{x}), \overline{z}) = 0$ has an isolated solution in $\mathbb{C}^n$ (of course this involves the separation of the real and imaginary parts of variables such as $x_1, \ldots, x_n$, which range over the complex numbers). It follows from Theorem 4 that $\mathbb{R} \models \phi(\overline{\alpha})$ if $\alpha_1, \ldots, \alpha_{d_0}$ are algebraically independent. By Proposition 2, $\phi(\overline{\alpha})$ is equivalent to a quantifier-free formula $\psi(\overline{\alpha})$ involving polynomials of degree $k^{O(1)}$ with integer coefficients of absolute value $|a_k|^{k^{O(1)}}$ (the implied constants may depend on F but not on k). Any $\overline{\alpha}$ which is not a root of any of these polynomials will satisfy ϕ. If k is sufficiently large and $|\epsilon| \leq 1/|a_{k+d_0}|$, it is indeed the case that $(1/a_{k+1}, \ldots, 1/a_{k+d_0-1}, \epsilon)$ is not a root of any of these polynomials. This follows from the growth rate condition (1). We conclude that for any $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero, the system $F(\overline{x}, H_{k+d_0-1,\epsilon}(\overline{x}), \overline{z}) = 0$ has an isolated solution. □

Instead of $\phi(\overline{\alpha})$, one could (less easily) write down a formula $\phi'(\overline{\alpha})$ of the language of fields such that $\mathbb{C} \models \phi'(\overline{\alpha})$ iff the system $F(\overline{x}, \nu_{k,\overline{\alpha}}(\overline{x}), \overline{z}) = 0$ has an isolated solution for any $\overline{z}$ such that $F_{\overline{z}}$ has a well balanced zero. Allowing the order relation just makes it easier to express the fact that there is an isolated solution. By contrast order plays an essential role in the proof of the next result.

Corollary 3 *Let $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ be a polynomial map and let K be a rational ball of $\mathbb{C}^r$ such that $F_{\overline{z}}$ has a well balanced zero for all $\overline{z} \in K$. There is a constant $c > 0$ such that the following property holds if d is sufficiently large: for any $\overline{z} \in K$, the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated solution which satisfies $||\overline{x}|| \leq |a_d|^{d^c} - 1/d$.*

By *rational ball* of $\mathbb{C}^r$ we mean a closed ball $B(\overline{z}, R) = \{\zeta \in \mathbb{C}^r;\ ||\zeta - z|| \leq R\}$ such that the radius R is a rational number and the real and imaginary parts of $\zeta_1, \ldots, \zeta_r$ are also rational.

Proof of Corollary 3. By Theorem 5, if d is sufficiently large (say, $d \geq d_0$) the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has isolated solutions for all $\overline{z} \in K$. Pick any $d \geq d_0$ and consider the map $N_F : K \to \mathbb{C}$ which sends x to $N_F(x) = \inf\{||\overline{x}||;\ F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0\}$. By Corollary 1 this function is upper semi-continuous on K. Since K is compact, N_F reaches its (finite) supremum $R(d)$ on K. Translating the definition of $R(d)$ in first-order logic immediately yields a formula $\phi(u)$ in the language of ordered rings such that $\mathbb{R} \models \forall u\ (\phi(u) \leftrightarrow u = R(d))$. Note that ϕ has only rational parameters since K is a rational ball. By elimination of quantifiers from ϕ we conclude that $R(d)$ is a root of a polynomial of degree $d^{O(1)}$ with integer coefficients bounded in absolute value by $|a_d|^{d^{O(1)}}$ (the implied constants may depend on K and F but not on d). We conclude that $R(d) \leq |a_d|^{d^\alpha}$ for some constant α, so that $R(d) \leq |a_d|^{d^{\alpha+1}} - 1/d$ if d is sufficiently large. $\Box$

This property is not only valid for K a rational ball: one could generalize to arbitrary compact sets (see Corollary 4). One may wonder why we insist on a bound of the form $|a_d|^{d^c} - 1/d$ in Corollary 3 instead of e.g. $|a_d|^{d^c}$. The reason will become apparent in the next section.

5 The Path Following Method

In the preceding section we have proved the existence of isolated roots for systems of the form $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$. In this section we show that some of the roots stay inside a fixed compact ball as d goes to infinity. The main theorem then follows easily.

Lemma 5 *Let $G : \mathbb{C}^n \times \mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^n$ be a polynomial map. We denote by $G_{\overline{z},\epsilon}$ the map $\overline{x} \mapsto G(\overline{x}, \overline{z}, \epsilon)$. Let K be a compact subset of $\mathbb{C}^r$ and C a compact subset of $\mathbb{C}^n$ such that for any $\overline{z} \in K$, $G_{\overline{z},0}$ has isolated roots in the interior of C. There exists $\delta > 0$ such that $G_{\overline{z},\epsilon}$ has isolated roots in C if $|\epsilon| \leq \delta$ and $\overline{z} \in K$.*

Proof. By Proposition 3, for any $\overline{\zeta} \in K$ there exists $\delta(\overline{\zeta}) > 0$ such that $G_{\overline{z},\epsilon}$ has isolated roots in C if $|\epsilon| \leq \delta(\overline{\zeta})$ and $\overline{z}$ belongs to the open ball $B(\overline{\zeta}, \delta(\overline{\zeta}))$. Since K is covered by the open balls $B(\overline{\zeta}, \delta(\overline{\zeta}))$, by compactness there exists a finite cover of the form $B(\overline{\zeta_1}, \delta(\overline{\zeta_1})), \ldots, B(\overline{\zeta_k}, \delta(\overline{\zeta_k}))$. Set $\delta = \min(\delta(\overline{\zeta_1}), \ldots, \delta(\overline{\zeta_k}))$. Now fix any $\overline{z} \in K$ and ϵ such that $|\epsilon| \leq \delta$. Since $\overline{z} \in B(\overline{\zeta_i}, \delta)$ for some i, $G_{\overline{z},\epsilon}$ has isolated roots in C by choice of δ. $\Box$

Proposition 5 *Fix a polynomial map $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ with integer coefficients and a rational ball $K \subseteq \mathbb{C}^r$ such that $F_{\overline{z}}$ has a well-balanced zero for all $\overline{z} \in K$. For any $c > 0$ the following property holds if d is sufficiently large.*

Suppose that $R \leq |a_d|^{d^c}$ is an integer such that for all $\overline{z} \in K$ the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated solution in the closed ball $\overline{B}(0, R - 1/d)$. If $|\epsilon| \leq 1/|a_{d+1}|$, for any $\overline{z} \in K$ the system $F(\overline{x}, H_{d,\epsilon}(\overline{x}), \overline{z}) = 0$ has an isolated solution in the closed ball $\overline{B}(0, R - 1/(d+1))$.

Proof. By Theorem 5 if d is sufficiently large (say, $d \geq d_0$) for any $\overline{z} \in K$ the system $F(\overline{x}, H_{d,\epsilon}(\overline{x}), \overline{z}) = 0$ has isolated solutions. Pick any $d \geq d_0$ and let $R \leq |a_d|^{d^c}$ be an integer such that for all $\overline{z} \in K$ the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated solution in the closed ball $\overline{B}(0, R - 1/d)$.

Let $C = \overline{B}(0, R - 1/(d+1))$. We can apply Lemma 5 to $G(\overline{x}, \overline{z}, \epsilon) = F(\overline{x}, H_{d,\epsilon}(\overline{x}), \overline{z})$: there exists $\delta > 0$ such that for any $\overline{z} \in K$, the system $F(\overline{x}, H_{d,\epsilon}(\overline{x}), \overline{z}) = 0$ has an isolated root $\overline{x}$ in C if $|\epsilon| \leq \delta$. The same quantifier elimination argument as in Corollary 3 shows that one may take $1/\delta = |a_d|^{d^{O(1)}}$ (the implied constant depends only on K, F and c). One may therefore take $\delta = 1/|a_{d+1}|$ if d is sufficiently large. □

Theorem 6 *Fix a polynomial map $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ with integer coefficients and a rational ball $K \subseteq \mathbb{C}^r$ such that $F_{\overline{z}}$ has a well-balanced zero for all $\overline{z} \in K$. There exists $R > 0$ such that the following property holds for all sufficiently large d: for any $\overline{z} \in K$ the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated root in the closed ball $\overline{B}(0, R)$.*

Proof. By Corollary 3 there is a constant $c > 0$ such that if d is sufficiently large (say, $d \geq d_0$) then for any $\overline{z} \in K$, the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated solution which satisfies $||\overline{x}|| \leq |a_d|^{d^c} - 1/d$. Let us choose d_0 so large that Proposition 5 also applies for $d \geq d_0$. The proof will be complete if we can show that the following claim is true: for $d \geq d_0$ and any $\overline{z} \in K$, the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated solution in the closed ball $\overline{B}(0, R - 1/d)$ where $R = |a_{d_0}|^{d_0^c}$. The proof of this claim is a straightforward induction on d. Indeed, the claim is true for d_0 by choice of d_0, and one can go from d to $d+1$ by Proposition 5. □

Although this is not really needed for the proof of our main result we note that the same property does not hold only for rational balls, but for all compact subsets of $\mathbb{C}^r$.

Corollary 4 *Fix a polynomial map $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ with integer coefficients and a compact $K \subseteq \mathbb{C}^r$ such that $F_{\overline{z}}$ has a well-balanced zero for all $\overline{z} \in K$. There exists $R > 0$ such that the following property holds if d is sufficiently large: for any $\overline{z} \in K$ the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated root in the closed ball $\overline{B}(0, R)$.*

Proof. By compactness of K and Proposition 1, K can be covered by a finite set $\{K_1, \ldots, K_p\}$ of rational balls such that $F_{\overline{\zeta}}$ has a well balanced zero for any $i \in \{1, \ldots, p\}$ and any $\overline{\zeta} \in K_i$. By Theorem 6 there exist $R_1, \ldots, R_p$ such that the following property holds for all sufficiently large d: for any $i \in \{1, \dots, p\}$ and any $\overline{z} \in K_i$, the system $F(\overline{x}, H_d(\overline{x}), \overline{z}) = 0$ has an isolated root in the closed ball $\overline{B}(0, R_i)$. Now set $R = \max(R_1, \dots, R_p)$. $\Box$

Proof of the main theorem. We can write $G = F_{\overline{z}}$ where $F : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^r \to \mathbb{C}^n$ is a polynomial map with integer coefficients and $\overline{z} \in \mathbb{C}^r$ is a tuple of parameters. By Proposition 1 there exists a rational ball $K \subseteq \mathbb{C}^r$ containing $\overline{z}$ such that $F_{\overline{\zeta}}$ has a well balanced zero for all $\overline{\zeta}$ in K. By Theorem 6 there exists an increasing sequence $(d_i)_{i \geq 0}$ of integers and a sequence $(\overline{x}_i)_{i \geq 0}$ of points of $\mathbb{C}^n$ such that $F(\overline{x}_i, H_{d_i}(\overline{x}_i), \overline{z}) = 0$. Moreover the sequence $(\overline{x}_i)_{i \geq 0}$ remains inside a fixed compact ball $\overline{B}(0, R)$. Extracting a subsequence if necessary, we may therefore assume that $\overline{x}_i$ converges to a limit point $\overline{a} \in \mathbb{C}^n$ as i goes to infinity. We conclude that $F(\overline{a}, H(\overline{a}), \overline{z}) = 0$ since $\lim_{d \to +\infty} F(\overline{x}, H_d(\overline{x}), \overline{z}) = F(\overline{x}, H(\overline{x}), \overline{z})$ uniformly with respect to $\overline{x} \in \overline{B}(0, R)$. $\Box$

Remark 1 *We have only defined the notion of well balanced zero for a polynomial map $G : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, but this notion clearly makes sense if G is an arbitrary analytic function. The main theorem is no longer true in this more general context. Indeed, set $G(x, y) = y - H(x) + 1$. This function has well balanced zeros, but there does not exist a such that $G(a, H(a)) = 0$.*

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