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Abstract

In this paper, we are interested in signals, form whereby the data can be transmitted in a cellular automaton. We study generation of some signals. In this aim, we investigate a notion of constructibility of increasing functions related to the production of words on the initial cell (in the sense of Fischer for the prime numbers). We establish some closure properties on this class of functions. We also exhibit some impossible moves of data.

Keywords: Cellular automata, computability, moves of information

Résumé

Nous nous intéressons à la notion de signal sur une ligne d'automates. Par là, nous modélisons le mouvement d'une information élémentaire. Cette notion est étroitement reliée à la construction en temps réel de fonctions croissantes an sens de Fisher. Nous donnons des propriétés de clôture des fonctions ainsi calculables. En outre nous exhibons des mouvements d'information impossibles.

Mots-clés: Automates cellulaires, calculabilité, mouvement de l'information

Signals in one dimensional cellular automata*

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1 Introduction

One of the greatest interest of Cellular automata (in short CA) is the modelization of massively parallel computation. In particular, for one dimensional CA, the interest focuses on these following topics:

- synchronization problems such that French Flag and Firing Squad ([9], [1] and [6]),
- real time production of words on the first cell ([4] and [8]),
- real time recognition of languages ([2], [3] and [5]).

It seems that signals are intrinsic objects of massively parallel computation. Indeed the signals are not only a natural tool to collect and dispatch the information through the network but more deeply this notion appears to be a strength way to encode and combine the information.

Thus signals seem to be objects interesting to be studied in themselves. In this paper, we investigate what kind of set of sites or, in other words, what kind of path can draw a signal in CA.

In section 2, we propose a formal definition of CA and we introduce a notion of Fisher's constructible functions connected to the production of words on the initial cell (in the sense of Fisher ([4]) for prime numbers).

In section 3, we list some examples of signals.

In section 4, we exhibit some impossible fast moves of the data.

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In section 5, we show that the set of Fisher's constructible functions is stable by some operations: addition,some subtractions, recurrent construction, composition, minimum, maximum and multiplication.

In section 6, we point out the links between Fischer's constructible functions and other notions like rightward signals of a given ratio, real time unary languages and real time constructibility.

2 Definitions

Definition 1 *A one dimensional cellular automaton $\mathcal{A}$ is a 4-tuple $(Q, \sharp, L, \delta)$ with :*

- *Q is a finite set (*states*),*
- *$\sharp$ is a special state not in Q (the* border state*),*
- *$\delta : Q \cup \{\sharp\} \times Q \times Q \longrightarrow Q$ is the* state transition function*,*
- *L is a another special state such that $\delta(L, L, L) = \delta(\sharp, L, L) = L$ (the* quiescent state*).*

We consider an half line of identical finite automata (cells) indexed by $\mathcal{N}$.Each cell communicates with its two neighbors. All cells evolve synchronously inducing a discrete time. The state (in Q) of the k-th cell at time t is denoted by $< k, t >$.

At each step, every cell enters a new state according to the state transition function, its own state and states of its two neighbors. For $t > 0$ and $k > 0$, the state $< k, t >$ is defined by:

$$< k, t >= \delta(< k-1, t-1 >, < k, t-1 >, < k+1, t-1 >).$$

The first cell having no left neighbor, we use the border state:

$$\forall t \in \mathcal{N}^* \quad < 0, t >= \delta(\sharp, < 0, t-1 >, < 1, t-1 >).$$

We depict the evolution of a CA on $\mathcal{N} \times \mathcal{N}$ elementary squares ; on the square of coordinates k and t, we mark state $< k, t >$ (by a number, a letter or a pattern). Such a picture is called the *space time diagram* of $\mathcal{A}$.

When we want to emphasize not the states but the communication between cells, the previous elementary squares are reduced to points, called sites. The lines between sites (k, t) and $(k + \epsilon, t + 1)$ ($\epsilon \in \{-1, 0, 1\}$) are marked in such a way that they depict the data sent by cell k at time t to itself and its two neighbors. Such a representation is called a *communication space time diagram.*

In order to study how an information can be moved through the network, we start with a special initial line. All cells are in the quiescent state except the leftmost one (cell 0). This fact will allow us to study the possible moves of the data regardless of the input words. This leads us to the following definition.

Definition 2 *A one dimensional impulse cellular automaton $\mathcal{A}$ (in short ICA) is a 5-tuple $(Q, \sharp, G, L, \delta)$ where $(Q, \sharp, L, \delta)$ is a cellular automaton, with a distinguished state G of Q such that, at initial time, all cells are in the quiescent state L except the cell 0 which is in state G.*

The case where the input word is considered constant can be easily reduced to definition 2: it is sufficient to define a new half line whose cells are obtained by grouping the n significant cells in one cell.

We will study the sites distinguished by the initial impulse when they appear as a line in the space time diagram. In this case, at each time, only one cell is distinguished. This remark induces the following definition of a signal.

Definition 3

1. *A signal S is a set of sites $\{(c(t), t)\ ;\ t \in \mathcal{N}\}$ where c is a mapping from $\mathcal{N}^*$ on $\mathcal{N}$ such that $(c(t+1), t+1)$ is $(c(t)-1, t+1)$ or $(c(t), t+1)$ or $(c(t)+1, t+1)$.*
   A signal is called rightward *(resp.* leftward*) if $(c(t+1), t+1)$ belongs to $\{(c(t), t+1), (c(t)+1, t+1)\}$ (resp. $\{(c(t)-1, t+1), (c(t), t+1)\}$).*

2. *A signal S is constructed by an ICA if there exists a subset Q_0 of Q such that $< k, t >\in Q_0$ if and only if $(k, t) \in S$. Such a signal is called CA constructible.*

3. *A signal is* basic *if the sequence of its elementary moves $\{c(t+1)-c(t)\}_{t\in\mathcal{N}}$ (whose values are in to $\{-1, 0, 1\}$) is ultimately periodical.*

Fact 1 *Basic signals are CA constructible. If an impulse generates a signal S such that all sites, not in S, are in the quiescent state (i.e. $(k,t) \notin S \Longleftrightarrow < k, t >= L$), then S is finite or basic.*

Proof The CA which sets up a basic signal S of period T from t_0, has t_0 states which define S for $t < t_0$ (including the impulse state G), and T states for the periodic part. The figure 1 illustrates this trick on an example.

For $t \in \mathcal{N}$, we denote the state $< c(t), t >$ by q_t. If S is infinite, then, for all time t, q_t is not quiescent (if $q_{t_0} = L$, then the signal S does not exist for time greater than t_0). The infinite sequence $\{q_t\ ;\ t \geq 0\}$ of states of $Q \setminus L$ becomes periodical: q_{t+1} is obtained from q_t by one of the transitions $\delta(q_t, L, L)$, $\delta(L, q_t, L)$ or $\delta(L, L, q_t)$ (the choice between these three possibilities only depends on the value of q_t). Thus the signal S is basic. □

Let S be a basic signal of period T from time t_0 and U be the sum of all elementary moves of a period: $U = c(t+T) - c(t)$ for any $t > t_0$. The rational number $\frac{T}{U}$ is called the *slope* of S. Clearly $\mid \frac{T}{U} \mid$ is greater or equal to 1.

To visualize signals in a more convenient way, we represent a signal of slope ν by a straight line of slope ν. Thus any signal can be depicted by straight lines. Such a representation is called a *geometric diagram*.

Definition 4 *Let ρ be an increasing function from $\mathcal{N}$ into $\mathcal{N}$. ρ is the ratio of a rightward signal S if S reaches the cell n at time $\rho(n)$. More precisely, $(n, \rho(n)) \in S$ but $(n, \rho(n) - 1) \notin S$.*

A rightward signal S is of speed $\sigma(n)$ *if its ratio is $\frac{n}{\sigma(n)}$.*

We note that $\forall n \in \mathcal{N}$, $\rho(n) \geq n$ and that the maximal speed is 1.

In [4], Fischer shows how the binary sequence representing the set of prime numbers can be generated by an initial impulse on the first cell. In this point of view, we will develop the notions of words Fischer's produced and of functions Fischer's constructible.

Definition 5 *Let $\omega = \omega_0, \ldots, \omega_i, \ldots$ be an infinite word on an alphabet A, ω is Fischer produced if there exists an ICA $(Q, \sharp, \omega_0, L, \delta)$ such that $\forall i \in \mathcal{N}$; $<0, i>= \omega_i$ with $A \subseteq Q$ (the i^{th} letter of ω appears on cell 0 at time i).*

This allows us to define a new notion of computation for increasing function.

Definition 6 *An increasing function f is* Fischer's constructible *(or constructible) by ICA if there exist a subset of states D of Q and a word Fischer's produced $\omega = \omega_0, \ldots, \omega_i, \ldots$ such that $\omega_i \in D \Longleftrightarrow \exists n \in \mathcal{N} \;\; i = f(n)$. It means that the sites $(0, f(n))$ can be distinguished by D.*

3 Some examples of signals

3.1 Signals of exponential ratio

1. C. Choffrut and Čulik II [2] have given a typical example of signal: their cellular automaton marks the cell 0 at every time h^m (the function $f : m \longrightarrow h^m$ is Fischer's constructible where h is an fixed integer and $m \in \mathcal{N}^*$). Figure 2 illustrates this construction on a geometric diagram when $h = 2$ and $h = 3$. We first consider a basic signal Σ_h of slope $\frac{h+1}{h-1}$: this signal appears on diagrams as a line which starts from site $(0,0)$ and reaches cells $\alpha\frac{h(h-1)}{2}$ at time $\alpha\frac{h(h+1)}{2}$. Another signal S remains on the cell 0 until time h, then it goes rightward at maximal speed until it reaches the signal Σ_h and then it comes back, at maximal speed, to cell 0. Reaching cell 0, it repeats this process and thus it zigzags between cell 0 and Σ_h.

   If the signal S leaves the cell 0 at time h^m, it reaches Σ_h on the cell $\frac{h^m(h-1)}{2}$ at time $h^m + \frac{h^m(h-1)}{2}$ (this site is on Σ_h, taking $\alpha = h^{m-1}$). Then, coming back, it reaches cell 0 at time $h^m + h^m(h-1)$ which is h^{m+1}.

2. We can transform this Fischer's construction in a signal of ratio h^m. Figure 3 illustrates this transformation on a geometric diagram when $h = 2$ and $h = 3$. In point 1, an undefined signal always remains on cell 0. The feature to obtain a signal of ratio h^n is to move this signal one cell to the right at each h^n units of time. Clearly, the signals S and Σ_h must also be shifted to the right. The shifted signals are denoted by S^* (instead of S) and Σ_h^* (instead of Σ_h). We note S_{exp} the signal of ratio h^m.

   The signal Σ_h^* is basic but with a non periodic part : it goes h cells to the right during $h+1$ units of time and then it becomes periodic with a slope $\frac{h+1}{h-1}$ until it meets again signal S^*.
   When a signal S^* reaches the signal S_{exp}, it remains one unit of time on the same cell and then it goes rightward at maximal speed until the signal Σ_h^*. Then it immediately comes back to the left at maximal speed.
   The signal S_{exp}, when it is reached by a signal S^*, remains one unit of time on the same cell, goes one cell to the right and then it remains on this new cell until it is reached again by signal S^*.

   The previous process is initialized as follows. Signals S_{exp} and S^* are created on cell 0 at time $h-2$ (using a finite signal S_{init1}). The signal Σ_h^* is created on the cell $\frac{h(h-1)}{2}$ at time $\frac{h^2+h-2}{2}$ (using a finite signal S_{init2}).

   We prove the correctness of the process by induction on m. Let the induction hypothesis be:
   $[H_m]$ The signal S^* reaches the signal S_{exp} on cell $m-1$ at time h^m-2 and then it reaches Σ_h^* at time $\frac{h^{m+1}+h^m-2}{2}$ on cell $m-1+\frac{h^m(h-1)}{2}$.
   $[H_1]$ is obvious by our initialization choice.
   We assume $[H_m]$ and we prove $[H_{m+1}]$. After its meeting with Σ_h^*, S^* goes leftward at maximal speed and reaches the signal S_{exp} on the cell m at time $\frac{h^{m+1}+h^m-2}{2}-1+\frac{h^m(h-1)}{2}=h^{m+1}-2$. At time $h^{m+1}-1$, signals S_{exp} and S^* remain on cell m. Then at time h^{m+1}, signal S_{exp} goes on cell $m+1$ (and then stay on it); and signal S^* runs rightward at maximal speed. Thus signal S^* visits sites $(m+1+\alpha, h^{m+1}+\alpha)$; $\alpha \in \mathcal{N}$. Taking $\alpha = -1+\frac{h^{m+1}(h-1)}{2}$, we see that signal S^* is on cell $m+\frac{h^{m+1}(h-1)}{2}$ at time $\frac{h^{m+2}+h^{m+1}-2}{2}$. Now, signal Σ_h^* moves right for h cells during $h+1$ units of time and runs rightward with a sloper $\frac{h+1}{h-1}$. Thus it visits sites $(m-1+\frac{h^m(h-1)}{2}+h+\alpha(h-1), \frac{h^{m+1}+h^m-2}{2}+h+1\alpha(h+1))$; $\alpha \in \mathcal{N}$. Taking $\alpha = \frac{h^{m+1}+h^m-2}{2}$, we obtain that signal Σ_h^* is on cell $m+\frac{h^{m+1}(h-1)}{2}$ at time $\frac{h^{m+2}+h^{m+1}-2}{2}$.

3. Figure 4 illustrates these signals on a communication space time diagram when $h = 2$ and $h = 3$: a signal of ratio $\frac{\alpha}{\beta}$ (with $\frac{\alpha}{\beta} \geq 1$) is set up with β right moves and $\alpha - \beta$ stays.

3.2 Signals of ratio n^k with $k \in \mathcal{N}^*$

Figure 5 illustrates these signals on a geometric diagram

1. The first example of a quadratic signal can be found in [4]. A signal of ratio n^2 is easily obtained using the formula: $(n+1)^2 = n^2+2n+1$. From the site (n, n^2), we obtain the site $(n+1,(n+1)^2)$ waiting $2n$ units of time on cell n and moving in one unit of time of one cell to the right. To wait $2n$ units of time is easy: it is the delay needed for a signal, created on site (n, n^2) to go to cell 0 and to come back on cell n.

2. A signal of ratio n^3 is constructed in a similar way using quadratic signals. From the site (n, n^3), we obtain the site $(n+1,(n+1)^3)$ waiting $3n^2+3n$ units of time on cell n and then moving in one step of one cell to the right. The delay of $3n$ is the delay needed for a signal, born on site (n, n^3) to go to cell 0, to come back to cell n^3 and to go, once time more, to cell 0. The delay of $3n^2$ is the delay needed to a quadratic signal, born on site $(0, n^3+3n)$ to go to cell n, to come back to cell 0 and then to go again to cell n.

3. Clearly, it is easy to set up signals of any ration n^k.

3.3 Signals of ratio involving roots

We can construct signals of ratio $rn + (\lfloor\sqrt{n}\rfloor)$ for $r \in \mathcal{N}$ and $r > 1$. We do not know if a signal of ratio $n + \lfloor\sqrt{n}\rfloor$ exists. Figure 6 illustrates the case of $r = 2$.

Let S_{root} be the signal which starts from the site $(0,0)$, it remains one step on the cell 0 and then it runs rightward with a slope r. A signal T starts from the site $(0,0)$ and moves one cell on the right in one unit of time and then it runs rightward to the right with a slope r. A signal Z starts from the cell 1 at time 1, it remains on cell 1. At the intersection of the signals S_{root} and Z, Z runs at maximal speed to the right and S_{root} remains one unit of time on its current cell and moves again to the right with a slope r. At the intersection of Z and T, Z and T move one cell to the right in one unit of time, then, Z remains on the same cell and T runs to the right with the slope r. S_{root} characterizes the sites $(n, rn + \lfloor\sqrt{n}\rfloor)$.

3.4 Signals of ratio involving logarithms

We can construct signals of ratio $n + \lfloor\log_q n\rfloor$. Figure 7 illustrates the case of $r = 2$.

Let n be written in basis q. Note that to add 1 to n can be made by a finite automaton with no delay, i.e. the i-th digit of $n+1$ is defined after the reading of the i-th digit of n. So, if the n-th vertical sends n, precisely if each site $(n, n+i)$ sends the i-th digit of n to the site $(n+1, n+1+i)$, then the site

$(n+1, n+1+i)$ can send the i-th digit of $n+1$. The signal which delimits the non quiescent area, distinguishes the sites $(n, n+\lfloor \log_\alpha(n) \rfloor)$.

3.5 Fischer's construction of a factorial

As an example of a Fischer's constructible function which grows faster than an exponential one, there is the function $n \longrightarrow 2(n!)$. Let us describe this construction depicted on figure 8.

From the site $(0, 2(n!))$, we obtain the site $(0, 2(n+1)!)$ by waiting n times $2n!$ units of time. The delay of $2n!$ units of time is the delay needed to achieve a zigzag, at maximal speed, from the cell 0 to the cell $n!$. So, we have to characterize the cell $n!$. For that, a signal S of slope 3 is created on the site $(0,0)$ and a signal T of slope 1 starts from the site $(0, 2(n!))$. They intersect on the site $(n!, 3(n!))$. From this site, a vertical signal V which characterizes the cell $n!$ is created.

Now, to count n zigzags, i.e. n times $2(n!)$, we have to characterize the cell n. Indeed, if at the beginning of the computation of $2(n+1)!$, a signal C starts from the cell n and at each zigzag it moves one cell to the left, then it will reach with the last zigzag the cell 0 at time $2((n+1)!)$. To characterize the cell n, we use a signal M which starts on the site $(0,0)$, it moves vertically except at its meeting with a signal T, on which it moves one cell to the right. At the beginning of the computation of $2(n+1)!$, M has met n signals T, thus it runs on the cell n.

Clearly, we can construct the function $n \longrightarrow (n!)$, grouping cells two by two. As we shall see, this induces the existence of a signal of speed $n!$.

4 Periodicity on diagonals

Proposition 1 *For any rightward signal of ratio $\rho(n)$, $\rho(n) - n$ becomes constant or there exists an integer α such that $\rho(n) \geq n + \log_\alpha(n)$ ($\forall n \in \mathcal{N}$).*

Proof We consider the time space diagram of some cellular automaton $\mathcal{A}$. Let us consider the words $\omega(i,n) = \omega_0(i), \ldots, \omega_n(i)$ where $\omega_k(i) =< i, i+k >$, it is to say $\omega(i,n)$ is the sequence of the n first non quiescent states of the i^{th} vertical.
Let q be the cardinal of Q (the set of states). Let n_0 be such that $\rho(n_0) < n_0 + \log_q(n_0)$. On Q, there only exist n_0 words of length $\log_q(n_0)$. Then there exist two integers i and T such that, for $i \geq 0$ and $T > 0$, $i + T \leq n_0$ and $\omega(i, \log_q(n_0)) = \omega(i+T, \log_q(n_0))$. And thus, $\forall j \geq i$, $\forall k > 0$, $\omega(j, \log_q(n_0)) = \omega(j+kT, \log_q(n_0))$.

But as $\rho(n_0) - n_0 < \log_q(n_0)$ and the site $(n_0, \rho(n_0))$ belonging to the signal S of ratio $\rho(n)$ is in a special state, we have that all sites $(n_0+kT, \rho(n_0)+kT)$ are in the same state and then belong to the signal S. Thus, $\rho(n_0+kT) = \rho(n_0)+kT$.

Since ρ is an increasing function, we get $\ell(n_0 + k) = \ell(n_0) + k$. In other words, $\forall n \geq n_0$, $\rho(n) - n = \rho(n_0) - n_0$, i.e. $\rho(n) - n$ becomes constant. □

Remark 1

The proposition 1 shows that there exists a gap in the ratios of signals. We can define a new notion of computation by: an increasing function f is constructible if there exists a signal of ratio f.

5 Properties of stability

Now, we come back to the notion of Fischer's constructibility. We proof some properties of stability on the set of Fischer's constructible functions. In this section, we denote by f and g two constructible functions. The two ICA which set up them are viewed as a black box which distinguishes the sites $< 0, f(i) >$ and $< 0, g(i) >$. To obtain new ICA computing new functions, we consider new impulses generated on sites $< 0, f(i) >$ and $< 0, g(i) >$, we describe behavior of these impulses in such a way that they distinguish some new site of the first cell.

5.1 Stability by multiplication with a rational

Proposition 2 *The set of Fischer's constructible functions is stable by multiplication by a rational.*

Proof

Construction of pf with $p \in \mathcal{N}^*$.
The figure 9 illustrates this proof. On the site $(0, 0)$, a basic signal T of slope $\frac{p+1}{p-1}$ is created. From each site $(0, f(n))$, a basic signal F of slope 1 is sent. This signal F reaches the signal T on the site $(\frac{(p-1)f(n)}{2}, \frac{(p+1)f(n)}{2})$. A signal R of slope -1 starts from this site. It reaches the cell 0 at time $pf(n)$. Thus the sites $(0, pf(n))$ $(\forall n \in \mathcal{N})$ are distinguished.

Construction of $\lfloor \frac{f}{p} \rfloor$ with $p \in \mathcal{N}^*$ We consider the ICA $\mathcal{A}'$ such that the cell (i, j) represents the cells $\{(pi+u, pj+v)\ ;\ 0 \leq u, v < p\}$ of $\mathcal{A}$: it is sufficient to group the cells $p \times p$ in space and time. By this way, the states of $\mathcal{A}'$ are a $p \times p$ matrix of states of $\mathcal{A}$. A state of $\mathcal{A}'$ is distinguished if and only if a state (of $\mathcal{A}$) of the first column of the matrix is distinguished. And thus, the sites $(0, \lfloor \frac{f(n)}{p} \rfloor)$ are distinguished by $\mathcal{A}'$.

□

5.2 Stability involving addition

Proposition 3 *Fischer's constructible functions are stable by addition.*

Proof The figure 10 illustrates this proof. From the site $(0, 0)$, a signal T of slope 3 is created. From each site $(0, f(n))$ (resp. $(0, g(n))$) a signal F (resp. G) of slope 1 is sent.
When $f(n) \leq g(n)$, we construct $f(n) + g(n) = 2f(n) + (g(n) - f(n))$ in the following way: the signal F which starts from the site $(0, f(n))$, meets the signal T on the site $(\frac{f(n)}{2}, \frac{3f(n)}{2})$. From this site, a signal F' is sent; this signal always remains on the same cell. This signal F' meets the signal G on the site $(\frac{f(n)}{2}, \frac{f(n)}{2} + g(n))$. At the intersection of F' and G, a signal R of slope -1 is created. This signal reaches the initial cell at time $f(n) + g(n)$.
When $g(n) \leq f(n)$, the signal G, created on the site $(0, g(n))$, meets the signal T before the signal F. In this case, the roles played by F and G are inverted and we construct $2g(n) + (f(n) - g(n)) = f(n) + g(n)$.
We observe that the choice between the two previous cases is not ambiguous: signals F (resp. G) are suppressed when they meet signals G' (resp. F'). □

Corollary 1 *Fischer's constructible functions are stable by linear combinations with rational coefficients.*

Proof According to proposition 2, af and bg are constructible. And thus is $af + bg$ by proposition 3. The figure 11 shows a direct construction of $af + bg$; we do not detail this construction. □

Corollary 2 *Fischer's constructible functions are stable by iterated addition.*

Proof Let $F(n)$ be $\sum_{i=0}^{i=n} f(i)$ where f is Fischer's constructible. Replacing the site $(0, g(n))$ by the site $(0, F(n))$, the proof of proposition 3 shows that F is Fischer's constructible. □

Proposition 4 *Fischer's constructible functions are stable by recurrent addition with k steps.*

Proof Let $a_1, a_2 \ldots a_k$ be positive integers, we prove that the function defined by the data $f(0), f(1), \ldots, f(k-1)$ and $f(n) = \sum_{i=1}^{k} a_i f(n-i)$ is Fischer's constructible.
The figure 12 illustrates this proof in the case of $k = 3$ and $a_1 = a_2 = 1$.
We have: $f(n) = b_k f(n-k) + b_{k-1}(f(n-k+1) - f(n-k)) + \ldots + b_i(f(n-i) - f(n-i+1)) + \ldots + b_1(f(n-1) - f(n-2))$ with $b_i = \sum_{s=1}^{i} a_s$
We define the evolution of the ICA computing f, in the following way: from the site $(0, 0)$, a signal T_k of slope $\frac{b_k+1}{b_k-1}$ is sent. From each site $(0, f(n))$, a signal H of slope 1 is sent.
When a signal H meets the signal T_k:

- Signal H dies and signal T_k pursues its move,
- a signal T_{k-1} of slope $\frac{b_{k-1}+1}{b_{k-1}-1}$ is created.

When a signal H meets a signal T_i with $i \in \{2, \ldots, k-1\}$:

- Signal T_i dies and signal H pursues its move,
- a signal T_{i-1} of slope $\frac{b_{i-1}+1}{b_{i-1}-1}$ is initialized.

At the intersection of a signal H and a signal T_1:

- T_1 dies, H pursues its moves,
- a signal R of slope -1 is created.

Now we show that the signals R reach the cell 0 at times $f(n)$.

- The signal H, which follows the diagonal of equation $y = x + f(n)$, reaches the signal T_k on the site $(\frac{(b_k-1)f(n)}{2}, \frac{(b_k+1)f(n)}{2})$. Between two consecutive signals H which follow diagonals $y = x + f(n)$ and $y = x + f(n+1)$, every signal T_i $(i \in \{2, \ldots, k-1\})$ moves of $\frac{(b_i-1)(f(n+1)-f(n))}{2}$ cells on the right in $\frac{(b_i+1)(f(n+1)-f(n))}{2}$ units of time.
- Then the signal T_{k-1}, emitted from the site $(\frac{(b_k-1)f(n)}{2}, \frac{(b_k+1)f(n)}{2})$, reaches the next signal H on the site $(\frac{(b_k-1)f(n)}{2} + \frac{(b_{k-1}-1)(f(n+1-f(n))}{2}, \frac{(b_k+1)f(n)}{2}) + \frac{(b_{k-1}+1)(f(n+1-f(n))}{2}$. From this last site, a signal T_{k-2} runs to the next signal H, and so on.
- Finally, the signal T_1 reaches a signal H on the site: $(\frac{(b_k-1)f(n)}{2} + \frac{(b_i-1)(f(n+k-i)-f(n+k-i-1))}{2}, \frac{(b_k+1)f(n)}{2}) + \frac{(b_i+1)(f(n+k-i)-f(n+k-i-1))}{2}$. On this last site, a signal R of slope -1 is created and it reaches the cell 0 at time $b_k f(n) + \sum_{i=1}^{k-1} a_i b_i (f(n+k-i) - f(n+k-i-1))$ which is $f(n+k)$.

□

5.3 Stability involving subtraction with extra conditions

Lemma 1 *If f and g are Fischer's constructible functions, $f \geq g$ (i.e. $\forall n \in \mathcal{N}$, $f(n) \geq g(n)$) and $(b+1)f - bg$ $(b \in \mathcal{N}^*)$ is an increasing function, then the function $(b+1)f - bg$ is Fischer's constructible.*

Proof The figure 13 illustrates this proof. From the first cell, at each time $f(n)$, a signal F of slope 1 is sent and, at each time $g(n)$, a signal G of slope $\frac{b+2}{b}$ is sent.
Since $f \geq g$, the signal G meets the signal F on the site $(\frac{b(f(n)-g(n))}{2}, f(n) +$

$\frac{b(f(n)-g(n))}{2}$). On this site, both signals F and G die and a signal H of slope -1 is created. This signal H reaches the cell 0 at time $(b+1)f(n) - bg(n)$.
But, if we consider all signals F and G, the following fact can happen: if, for some n, we have $g(n) < g(n+1) \leq f(n)$, then the $n+1$-th signal G will meet the n-th signal F before the signal n-th G. So, we introduce a signal E, indicating the active signal G. This signal E is created on site $(0, g(1))$ and follows the first signal G. The process of the signal E is to follow a signal G until the meeting of this signal G and a signal F, then to run leftward with slope -1 until it reaches the next signal G and then to follow it. We observe that as $(b+1)f - bg)$ is increasing, this signal E reaches the n-th signal G before the meeting of the n-th signal G and n-th signal F. By this way, the signal H which the cell 0 marks the cell 0 at times $(b+1)f(n) - bg(n)$ is created by the simultaneous meeting of three signals G, F and E. □

Proposition 5 *Let a and b be two positive integers and f and g be two Fischer's constructible functions. If there exists a positive integer m such that $\frac{f}{g} \geq \frac{mb+1}{ma}$ and if $af - bg$ is increasing, then $af - bg$ is Fischer's constructible.*

Proof By the proposition 2, maf and $(mb+1)g$ are constructible. The condition of proposition 5 ensures us that $maf \geq (mb+1)g$. The function $(mb+1)maf - mb(mb+1)g$ can be written $m(mb+1)(af - bg)$ and, thus, is increasing. By the lemma 1 and the proposition 2, $af - bg$ is Fischer's constructible. □

Remark 2

The proposition 5, in fact, induces that f and $af - bg$ are of the same order. Let us consider $f(n) = n^3 + n$ and $g(n) = n^3$, we have $(f - g)(n) = n$. f and g do not satisfy the conditions of the proposition 5, indeed $f(n) - g(n) = n$ cannot be constructed from $f(n) = n^3 + n$ with a simple linear acceleration.

Below, we shall need the following corollary.

Corollary 3 *Let a and b be two positive integers and f and g be two Fischer's constructible functions. If f is $ah + bg$, if there exists a positive integer m such that $g \leq mh$ and if h is increasing then h is Fischer's constructible.*

Proof In this case, $\frac{f}{g} \geq \frac{mb+1}{m}$ and $f - bg = ah$ is increasing. So, according to the proposition 5, h is Fischer's constructible. □

5.4 Stability involving recurrent functions

Proposition 6 *Let $a_1, \ldots, a_k$ be k integers, if the function h defined by the data: $h(1)$, $h(2)$, ..., $h(k)$ and $h(n) = \sum_{i=1}^{k} a_i h(n-i)$ is increasing, then h is Fischer's constructible.*

Proof There exist positive integers b_i and c_i such that $h(n) = \sum_{i=1}^{k} b_i h(n - i) - \sum_{i=1}^{k} c_i h(n-i)$. We prove that, if h is increasing, then h is Fischer's constructible.
First, the following functions are Fischer's constructible by the proposition 4: $f(n) = \sum_{i=1}^{k} b_i h(n-i)$ and $g(n) = \sum_{i=1}^{k} c_i h(n-i)$.
Secondly, as h is increasing, we have: $\sum_{i=1}^{k} c_i h(n) \geq \sum_{i=1}^{k} c_i h(n-i) = g(n)$. It is to say $g \leq \sum_{i=1}^{k} c_i h$.
Finally, according to the corollary 3, h is Fischer's constructible. □

5.5 Stability by composition

Lemma 2 *If f and g are Fischer's constructible functions, then $f \circ g + 2g$ is Fischer's constructible.*

Proof The figure 14 illustrates this proof.

Characterization of the sites $(n, n + f(n))$.
From each $(0, f(i))$, a signal F of slope 1 is created. A signal T starts from the site $(0, 0)$, it moves of one cell to the right in one unit of time, and then remains on cell 1. When a signal F meets the signal T, the signal F dies and the signal T moves of one cell to the right in one unit of time, and then remains on the same cell. Meetings of signals F and T occur on the sites $(n, n + f(n))$.

Constructibility of the sites $f \circ g + 2g$.
A signal U of slope 2 is sent from the site $(0, 0)$. From each site $(0, g(i))$, a signal G of slope 1 is created. It reaches the signal U on the site $(g(i), 2g(i))$. On this site, a signal V, which remains on cell $g(i)$ is created. As f is increasing, we have $2g(i) \leq g(i) + f(g(i))$; and, thus, the signal V reaches the site $(g(i), g(i) + f(g(i))$ on which occurs the meeting of the previous signals F and T. Then, from this site, a signal R of slope -1 is sent and it reaches the cell 0 at the time $f(g(i)) + 2g(i)$.

□

Proposition 7 *If f and g are Fischer's constructible functions, then $f \circ g$ is Fischer's constructible.*

Proof By lemma 2, $f \circ g + 2g$ is Fischer's constructible. By hypothesis, g is Fischer's constructible. As f and g are increasing, $f \circ g$ is increasing and we have $f \circ g \geq g$. Thus by the corollary 3, $f \circ g$ is Fischer's constructible. □

5.6 Stability by minimum and maximum

Proposition 8 *If f and g are Fischer's constructible functions, then the functions $min(f, g)$ and $max(f, g)$ are Fischer's constructible.*

Proof We only give the proof for $min(f,g)$, the case of $max(f,g)$ is similar. The sum σ_s of the digits which reach the diagonal $y = x + s$ is the difference between the number of integers i such that $g(i) < s$ and the number of integers i such that $f(i) < s$. Thus: $\sigma_s = |\{i\ ;\ g(i) < s\}| - |\{i\ ;\ f(i) < s\}|$. So, at time s, a digit is taken in or out if s is equal or not to $f(n)$ or $g(n)$.
The transitions of states are indicated on the figure 15. We observe that, for a transition on a cell c ($c > 0$), if $|i| < 2$ and $|j| < 1$, then $|k| < 1$ and $|p| < 2$. For a transition on the cell 0, if $|i| < 1$ and $|j| < 1$, then $|k| < 1$ and $|p| < 2$. This shows that the number of signals is finite. The figure 16 illustrates this proof on an example. □

5.7 Stability by multiplication

Proposition 9 *If f and g are Fischer's constructible functions, then $f \times g$ is Fischer's constructible.*

Proof We may assume that $f \geq g$: if it is not the case, we replace f and g by $min(f,g)$ and $max(f,g)$ according to the proposition 8.
By the corollary 2 and the proposition 3, we have two ICA which construct $G(n) = \sum_{i=1}^{n} g(i)$ and $G(n-1) + f(n)$.
First, we characterize the sites $(0, 2f(n)g(n) + 2G(n-1) + G(n))$. The figure 17 illustrates this construction. On the site $(0,0)$, a signal T of slope 2 is initialized. At each time $G(n)$, a signal G of slope 1 is created on the first cell. When this signal G meets T on the site $(G(n), 2G(n))$, G dies and a new signal V which always remains on cell $G(n)$ is sent. In the same way, at each time $G(n-1) + f(n)$, a signal F of slope 1 is initialized on the first cell, dies at its meeting with T on the site $(G(n-1) + f(n), 2G(n-1) + 2f(n))$ and a new signal C, which remains on cell $G(n-1) + f(n)$, is created on this site.
The distance between two consecutive signals V is $g(n)$. Thus to achieve a zigzag at maximal speed between these two signals need exactly $2g(n)$ units of time. The distance between signals V and C is of $f(n)$ cells. we use the signal C as a counter: at each zigzag, it moves of one cell to the left in one unit of time.
More precisely, when the n-th signal G meets T, a signal R of slope 1 is created; on its meeting with the $n+1$-th signal V, it dies and creates a signal A of slope 1. This signal A dies on its meeting with the n-th signal V, creating a new signal R., and so on. During this process, when a signal R passes through the signal C, C moves of one cell to the left. This process ends when signals C and R simultaneously reach the signal V. At this time, the signals R have achieved $f(n)$ moves and the signals A have achieved $f(n)-1$ moves. Thus, C reaches V at time $2G(n)+(2f(n)-1)g(n)$ which is $2f(n)g(n)+G(n-1)+G(n)$. On the site $(G(n-1), 2f(n)g(n)+G(n-1)+G(n))$ of this meeting, a signal K of slope -1 is created. This last signal reaches the cell 0 at time $2f(n)g(n)+2G(n-1)+G(n)$.
We observe that $\forall n \in \mathcal{N}$, $f(n)g(n) \geq ng(n) > G(n)$. Thus, by the corollary 3,

$f \times g$ is Fischer's constructible. □

6 Relationships between Fischer's constructibility and related notions

We investigate relationship between Fischer's constructible functions and ratio of signals.

Proposition 10 *Let h be an increasing function. If there exists an ICA which sets up a signal of ratio $h(n)$, then h is Fischer's constructible.*

Proof From each site $(n, h(n))$, a signal of slope -1 is sent. This signal reaches the cell 0 at time $n + h(n)$. thus, the sites $(0, n + h(n)$ are distinguished and by the corollary 3, h is Fischer's constructible. □

Fact 2 *The converse is false.*

Proof We have seen (proposition 1) that there does not exist a signal of ratio $n + \theta(n)$ where $\theta(n)$ is sublogarithmic. But there exist Fisher's constructible functions which increase strictly faster than an exponential: factorial one (see paragraph 3.5), 2^{2^n} (by proposition 7), ... Thus, their complement functions, defined by $\gamma(n) = n + |\{i\ ;\ f(i) - i < n\}|$ are also Fischer's constructible. And $\gamma(n) - n$ are sublogarithmic and there do not exist signals of ratio $\gamma(n)$. □

Nevertheless, if the difference between $f(n)$ and n is, at least, linear, the converse is true.

Proposition 11 *Let f be a Fischer's constructible function. If there exists an integer k such that $(k-1)f(n) \geq kn$, then there exists a signal which characterizes the sites $(n, f(n))$*

Proof Assume that k is even: indeed if there exists an odd integer satisfying the condition, then an even one exists.
First, we mark the sites $(kn, kf(n))$. The figure 18 illustrates this construction.
From the site $(0, 0)$ are sent the following signals:

- a signal T of slope $\frac{k+1}{k-1}$,
- a signal $D_{\frac{k}{2}}$ which moves right of $\frac{k}{2}$ cells in $\frac{k}{2}$ units of time, and from time $\frac{k}{2}$ remains on the cell $\frac{k}{2}$,
- a signal D_k which moves right of k cells in k units of time and then remains on cell k.

From each site $(0, f(n))$, a signal E is sent.
Our ICA has the following behavior: at the meeting of a signal E and the signal T:

- the signal E dies,
- a signal E_{-1} of slope -1 is created,
- the signal T pursues its moves with the same slope $\frac{k+1}{k-1}$.

At the meeting of a signal E_{-1} with a signal $D_{\frac{k}{2}}$:

- the signal E_{-1} dies,
- a signal E_1 of slope 1 is created,
- the signal $D_{\frac{k}{2}}$ moves right of $\frac{k}{2}$ cells in $\frac{k}{2}$ units of time, and then remains on the same cell.

At the meeting of a signal E_1 and a signal D_k:

- the signal E_1 dies,
- the signal D_k moves right of k cells in k units of time and then remains on the same cell.

Now, we show that the signal D_k characterize the sites $(kn, kf(n))$. The n-th signal E reaches the signal T on the site $\left(\frac{(k-1)f(n)}{2}, \frac{(k+1)f(n)}{2}\right)$. From this site, the signal E_{-1} starts. As $(k-1)f(n) \geq kn$, the n-th signal E_{-1} reaches the signal $D_{\frac{k}{2}}$ on the cell $\frac{kn}{2}$ (note that signal $D_{\frac{k}{2}}$ moves of $\frac{k}{2}$ cells to the right when it meets a signal E_{-1}, thus after $n-1$-th meetings with signals E_{-1}, it is on the cell $\frac{kn}{2}$. So, the signal E_{-1} meets the signal $D_{\frac{k}{2}}$ on the site $(\frac{kn}{2}, \frac{kn}{2})$. From this last site, a signal E_{-1} starts. It meets the signal D_k which runs on the cell kn (by the same argument as previously), on the site $(kn, kf(n))$.
In order to characterizes the sites $(n, f(n))$, we group the cells k by k. It is sufficient to consider a new ICA such that the state of the site (i,j) represents the states of the sites $\{(ki+u, kj+v)\ ;\ 0 \leq u < k\ ;\ 0 \leq v < k\}$. □

Now, we consider the bijection between the set of increasing functions and the set of unary languages, defined by: at the function f, is associated the language $L_f = \left\{a^{f(n)}\ ;\ n \in \mathcal{N}\right\}$ which is the set of all words of length $f(n)$. We recall that a language L is recognizable in real time by a cellular automaton if on input ω in L, the CA enters an accepting state on cell 0 at time $|\omega|$. We observe that, if a CA recognizes the language L in real time, its working area on an input of length n, is bounded by the diagonal $\{(c,t)\ ;\ c+t=n\}$.

Proposition 12 *The function f is Fischer's constructible if and only if the language L_f is real time recognizable by a one dimensional cellular automaton.*

Proof If f is Fischer's constructible, then there exists a CA which marks the sites $(0, f(n))$. If, in addition, this CA creates a signal Σ of slope -1 on the last cell of the input word at time 0, then the CA knows if the length of the input word can be written $f(n)$ for some integer n when Σ reaches the cell 0 on a distinguished site.

Conversely, we suppose that there exists a CA ($\mathcal{A}$) which recognizes the words of length $f(n)$ in real time. We observe that, for any CA, each site (c,t) with $c+t<n$ is in the same state whatever the input word a^m for $m \geq n$. As our CA recognizes the language L_f in real time, the two space time diagrams on inputs a^n and a^m with $m>n$ are different only on the sites (c,t) with $c+t \geq n$ (in some way, recognition of a^n is done on the diagonal $D_n = \{(c,t)\ ;\ c+t=n\ ,\ c \geq 0\}$). Now we consider the CA ($\mathcal{A}^\star$) whose states are couple of states of $\mathcal{A}$. The first components correspond to the states of $\mathcal{A}$ on the input word $a^{infinity}$. On the diagonal D_n, the second components correspond to the states of $\mathcal{A}$ on the input word a^n. $\mathcal{A}^\star$ distinguishes a site on the first cell according to its second component. Clearly, $\mathcal{A}^\star$ marks the sites $(0, f(n))$. □

The next section shows a property of these functions on Turing machines.

Proposition 13 *If an increasing function is Fischer's constructible, it is Turing space constructible.*

Proof We construct a Turing machine.

- On its first tape, we consider the simulation of the one dimensional cellular automaton by a Turing machine as defined in [7]: the i-th cell of the tape of the Turing machine represents the i-th cell of the CA. On the CA, as at time 0 the cell 0 is the only one in a non quiescent state, at time t, only the $t+1$ first cells are in a non quiescent state. Thus, during the simulation of the step t of the CA, the head visits exactly the first $t+1$ cells of the tape.
- In addition, on a second tape, our Turing machine counts how many times the first cell of the CA has been distinguished.
- On a third tape, our Turing machine compares this number with the integer n, written on its input tape. If these numbers are equal, the machine halts.

So, the Turing machine halts, during the simulation of the $f(n)$-th step, when the head visits $f(n)+1$ cells and, thus, f is Turing space constructible. □

7 Conclusion

We have begun to investigate two sets of increasing computable (in some sense) functions: Fischer's constructible ones and another ones defined by ratios of signals. They correspond to possible moves of an elementary information. Some properties of stability have been shown. This work induces some open problems

- Are all Turing time constructible increasing functions Fischer's constructible?
- Do there exist another gaps in ratio of signals? In particular, for a signal of ratio $\rho(n)$, is there always a gap around it?

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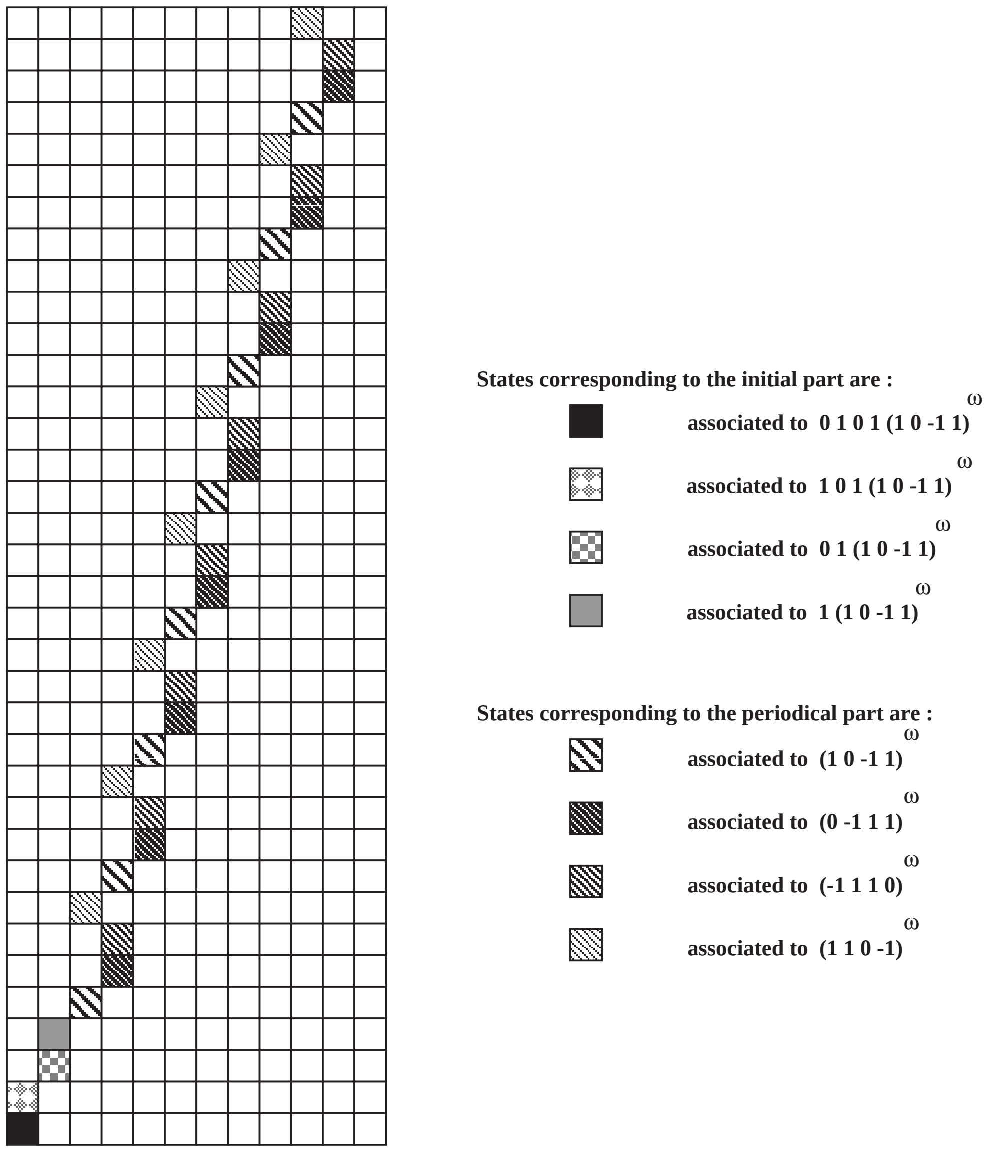

Figure 1: The basic signal, defined by the sequence of moves $0, 1, 0\ , (1, 0, -1, 1)^{\omega}$

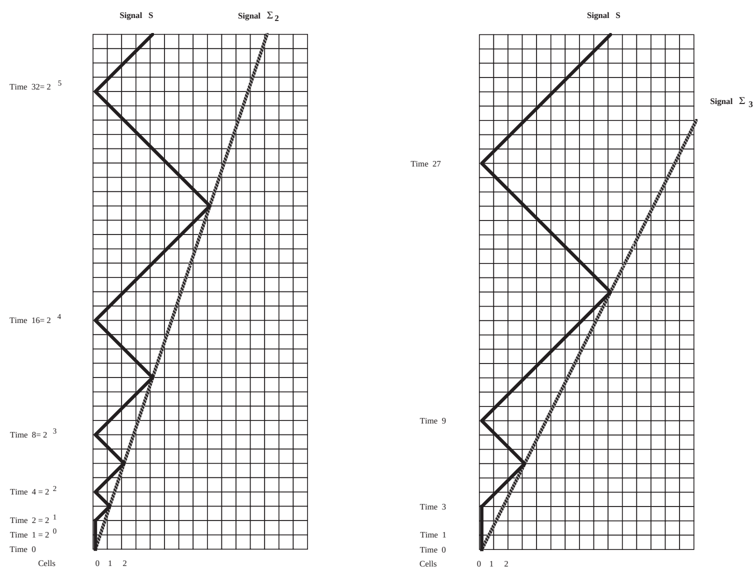

FIGURE 2. Fischer's constructibility of 2^n and 3^n

Figure 2: Fischer's production of 2^n and 3^n

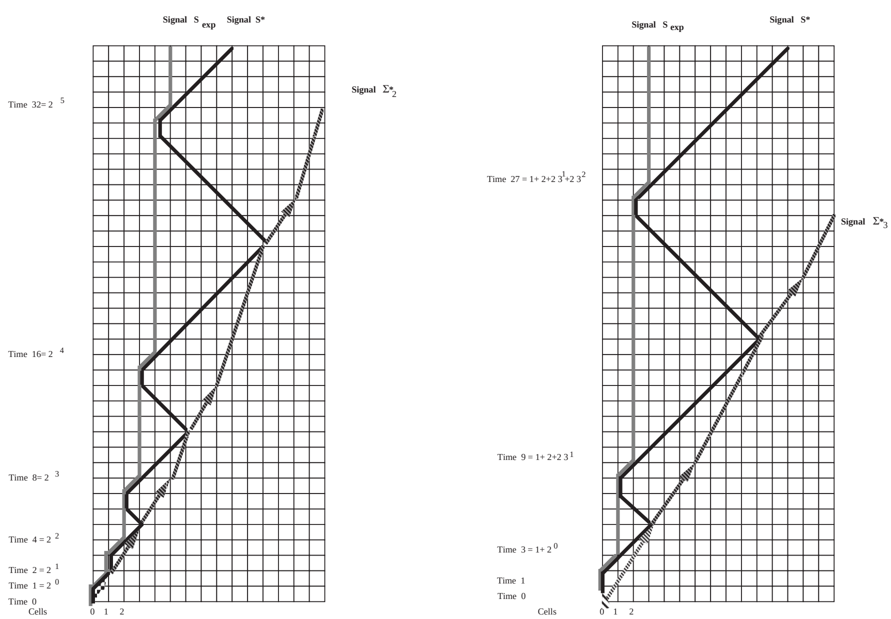

Figure 3: Signals of ratio 2^n and 3^n, on a geometric diagram

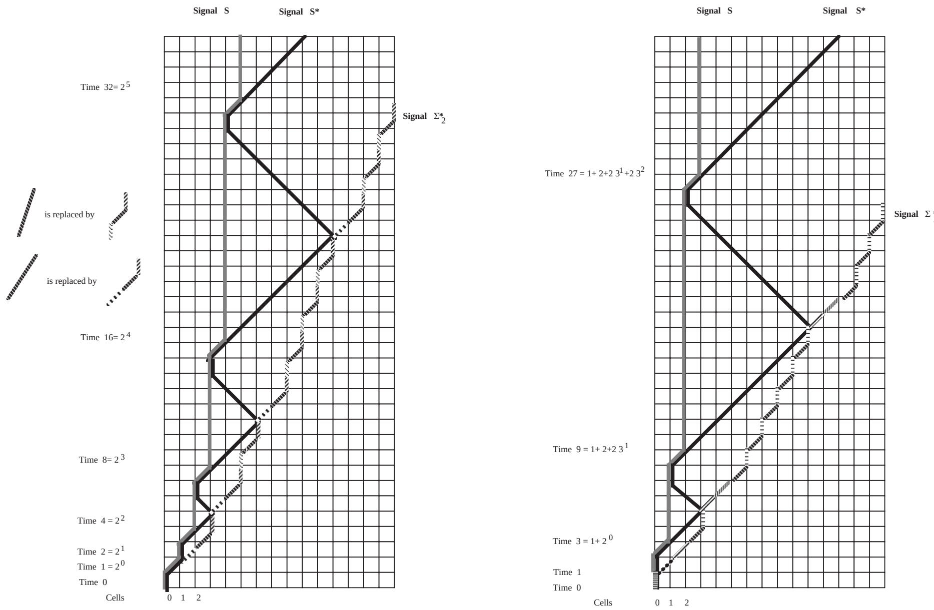

Figure 4: Signals of ratio 2^n and 3^n, on a communicational space time diagram

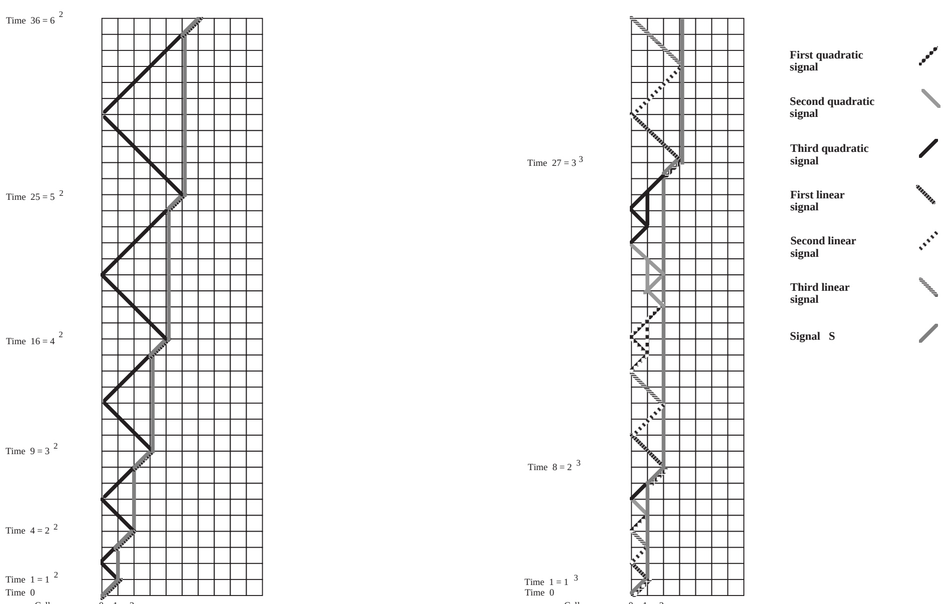

Figure 5: Signals of ratio n^2 and n^3, on a geometric diagram

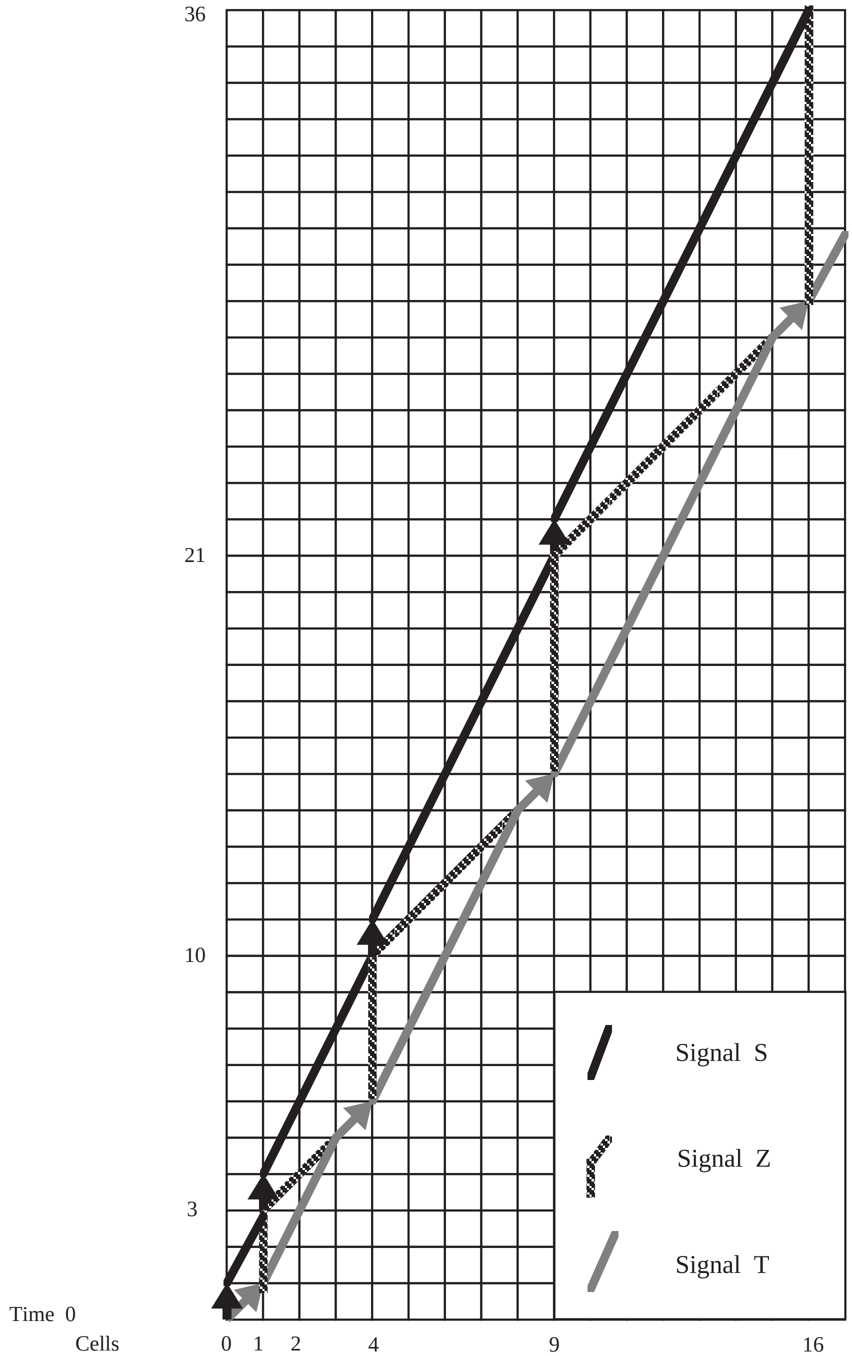

Figure 6: Signal of ratio $2n + \lfloor\sqrt{n}\rfloor$ on a geometric diagram

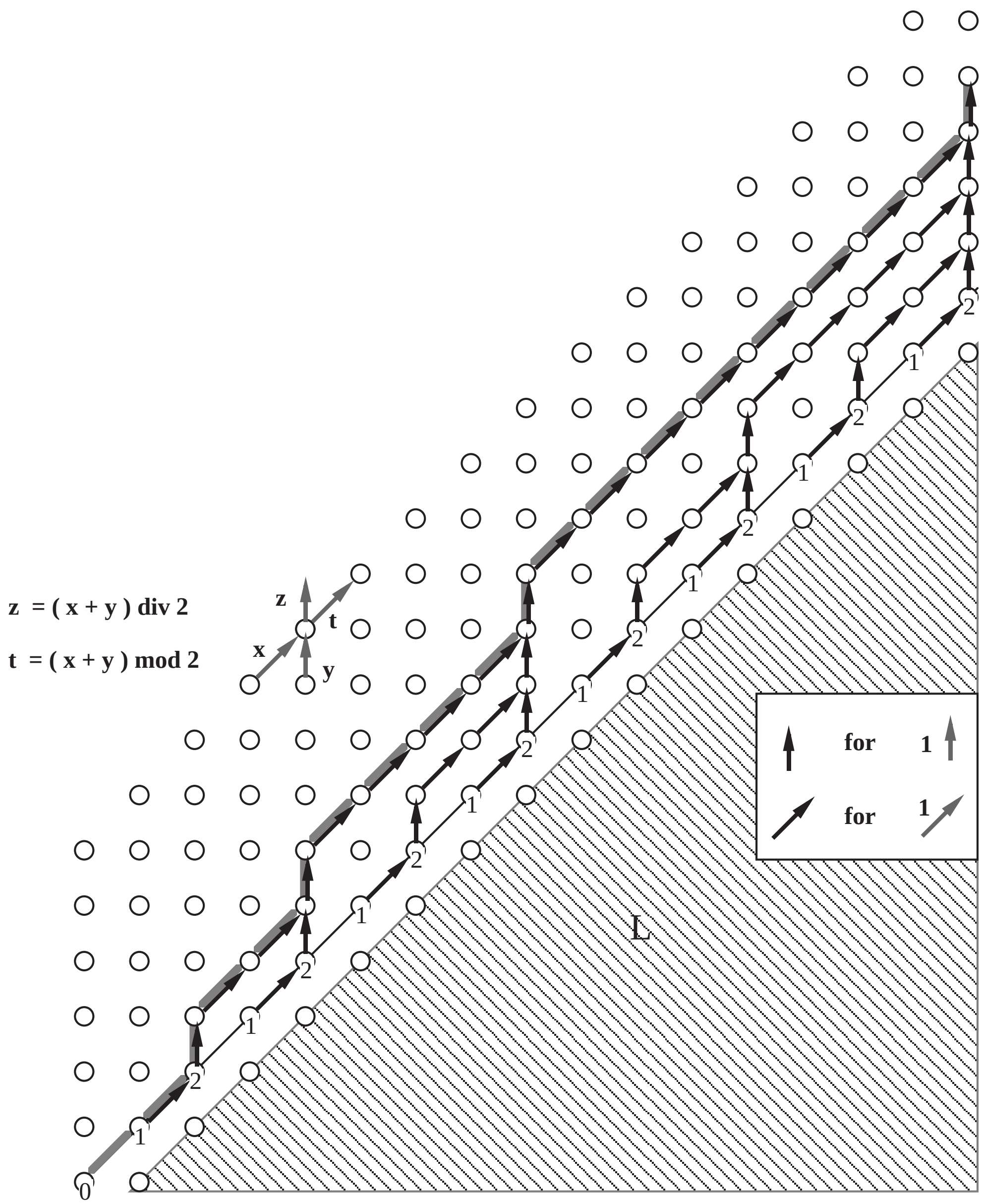

Figure 7: Signal of ratio $n + \lfloor \log_2 n \rfloor$ on a geometric diagram

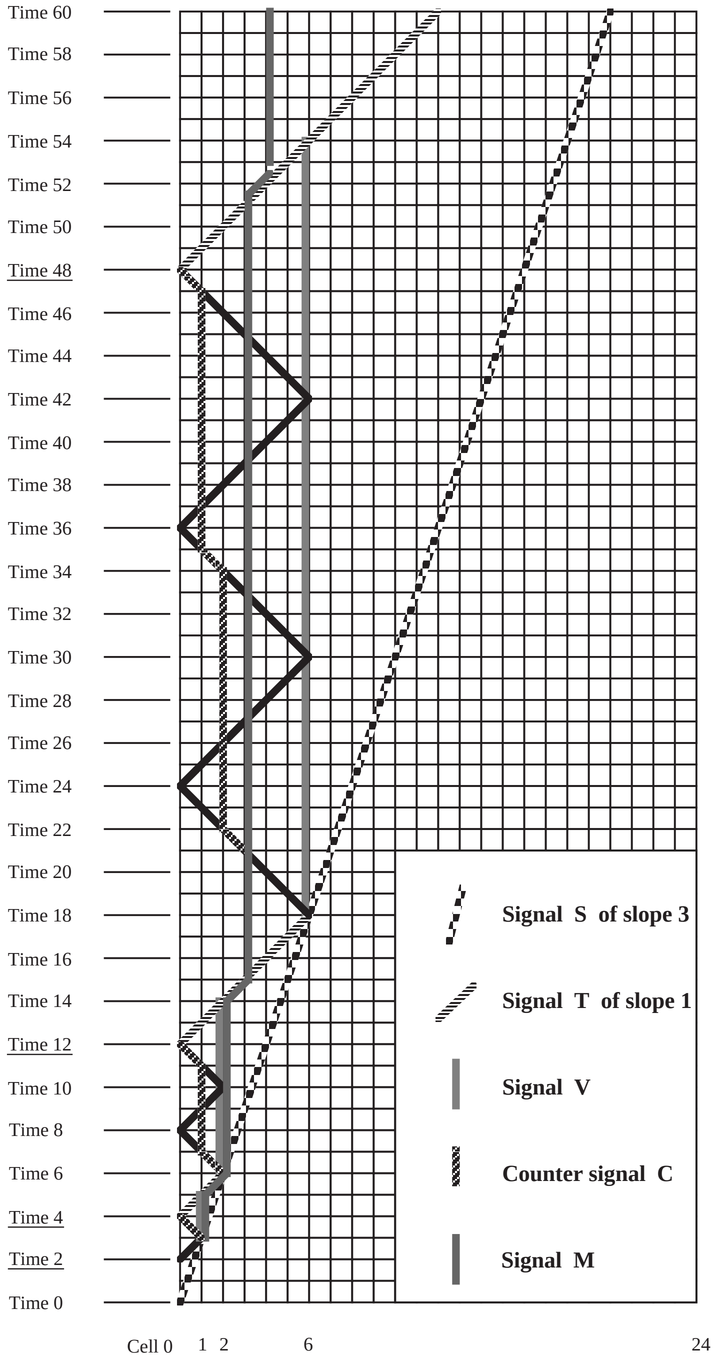

Figure 8: Fischer's construction of pf.

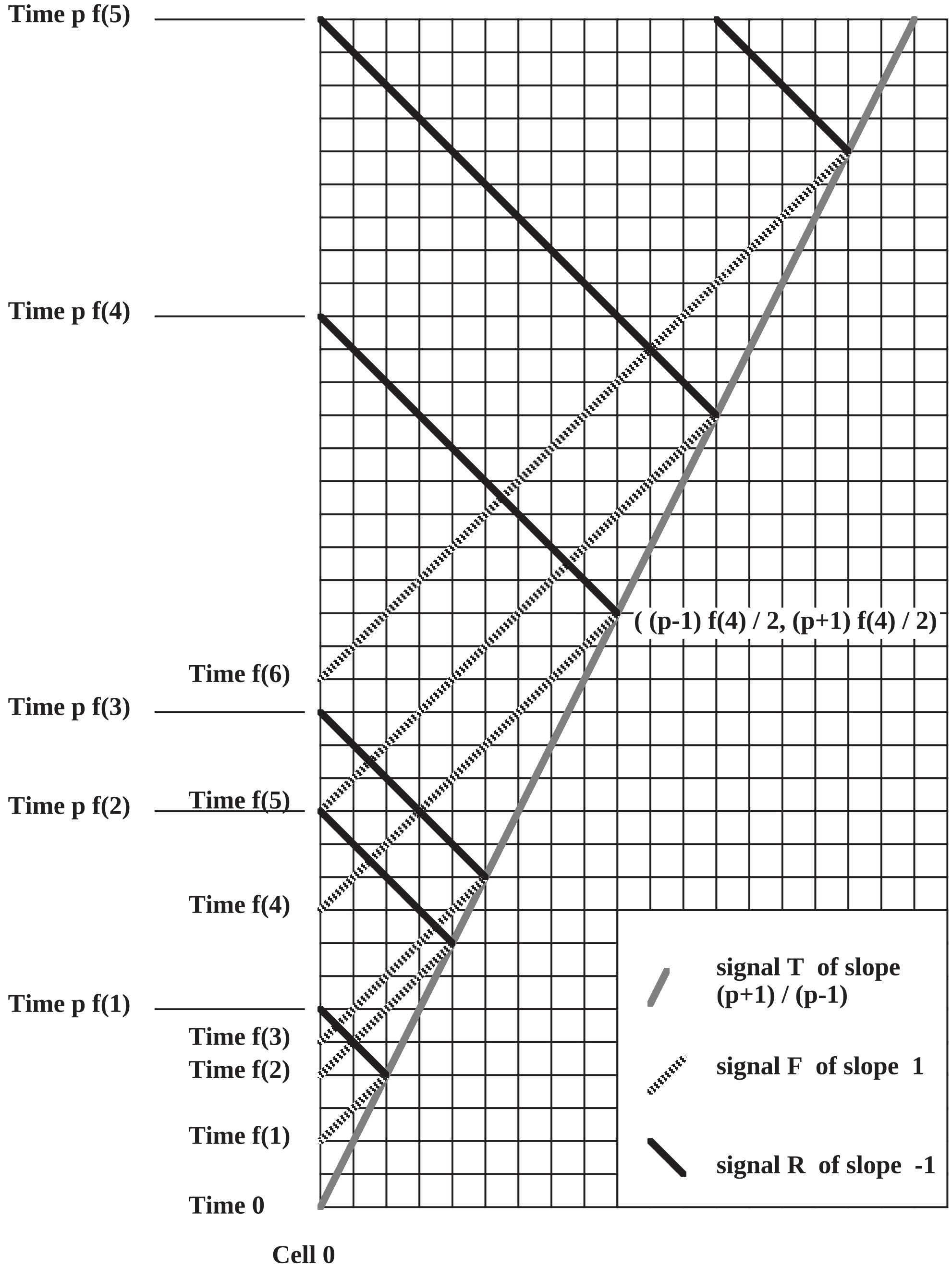

Figure 9: Fischer's construction of a factorial.

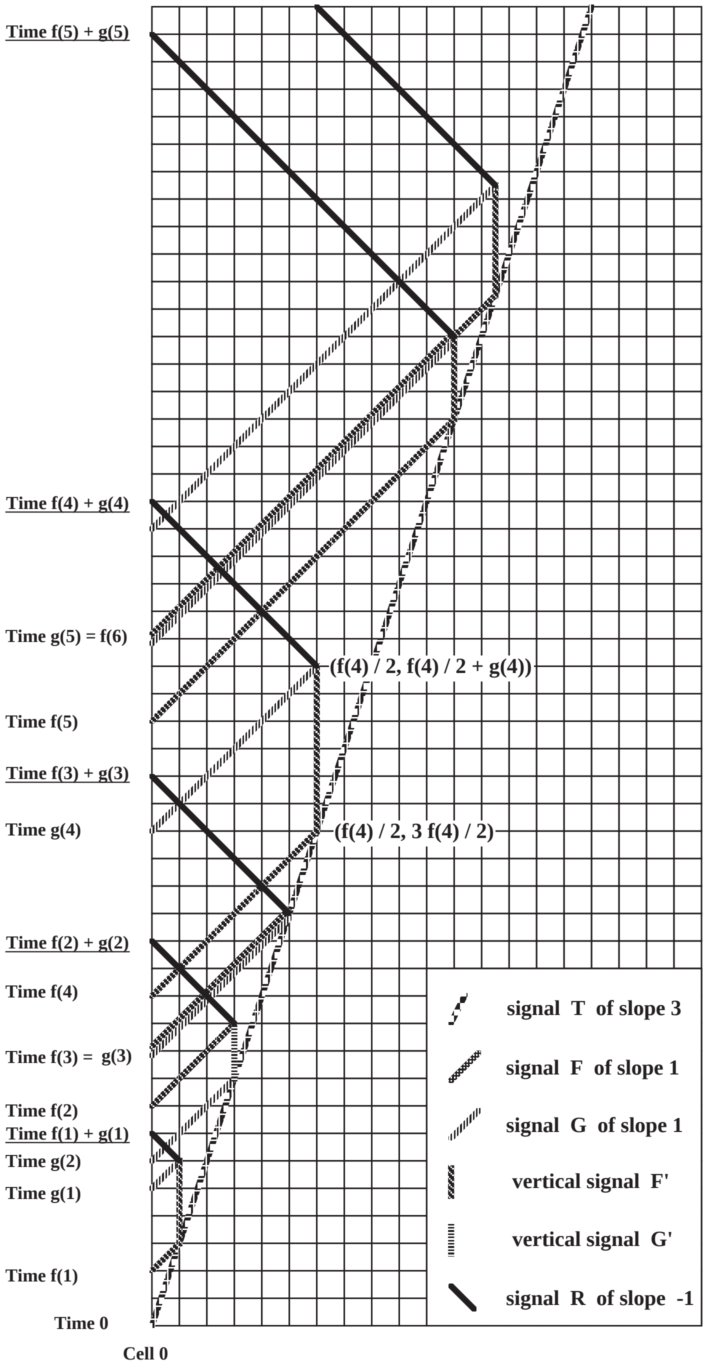

Figure 10: Fischer's construction of $f + g$.

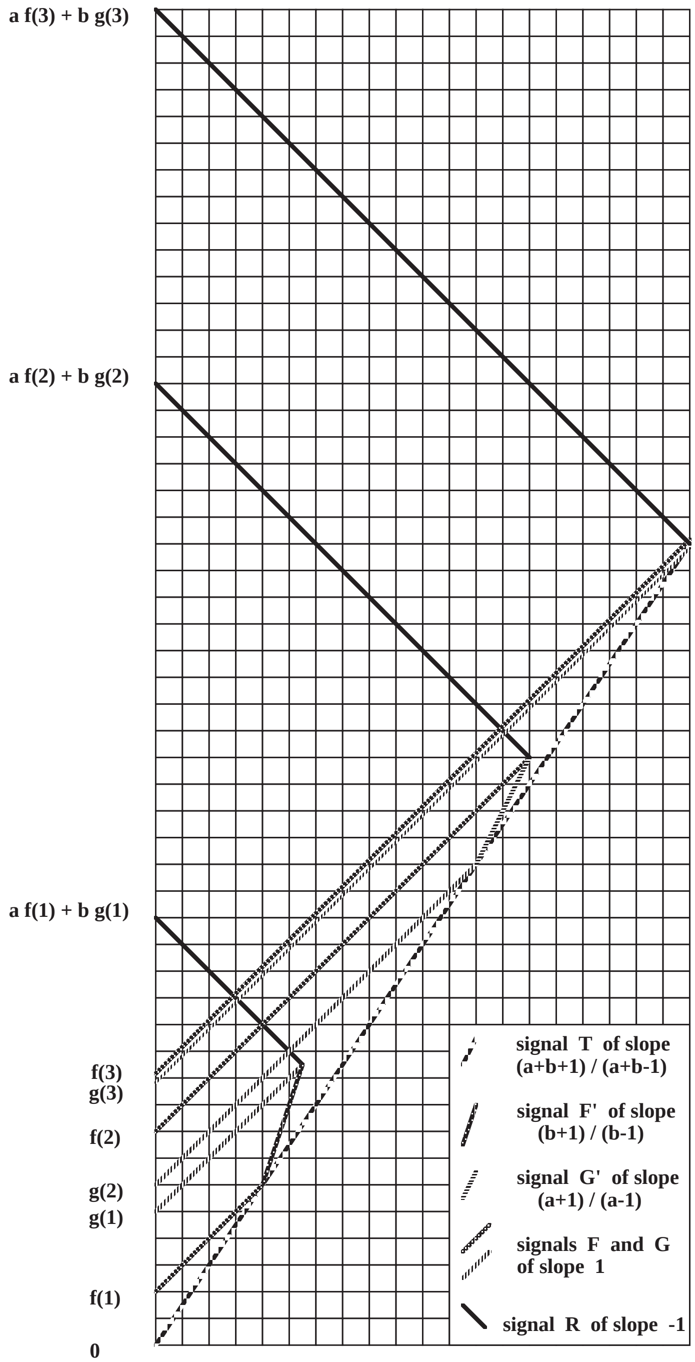

Figure 11: Fischer's construction of $af + bg$.

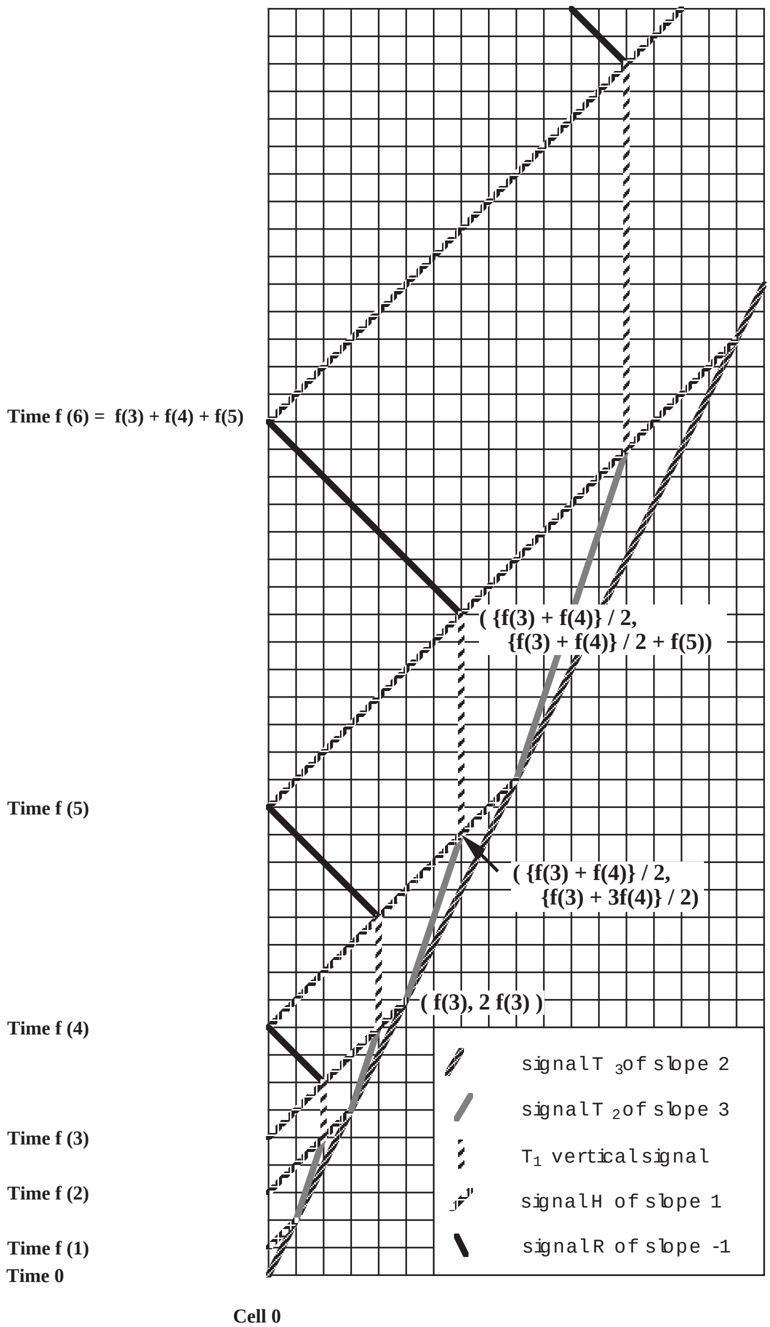

Figure 12: Fischer's construction of $f(n) = f(n-1) + f(n-2) + f(n-3) = 3f(n-3) + 2\left((f(n-2) - f(n-3)) + (f(n-1) - f(n-2)\right)$.

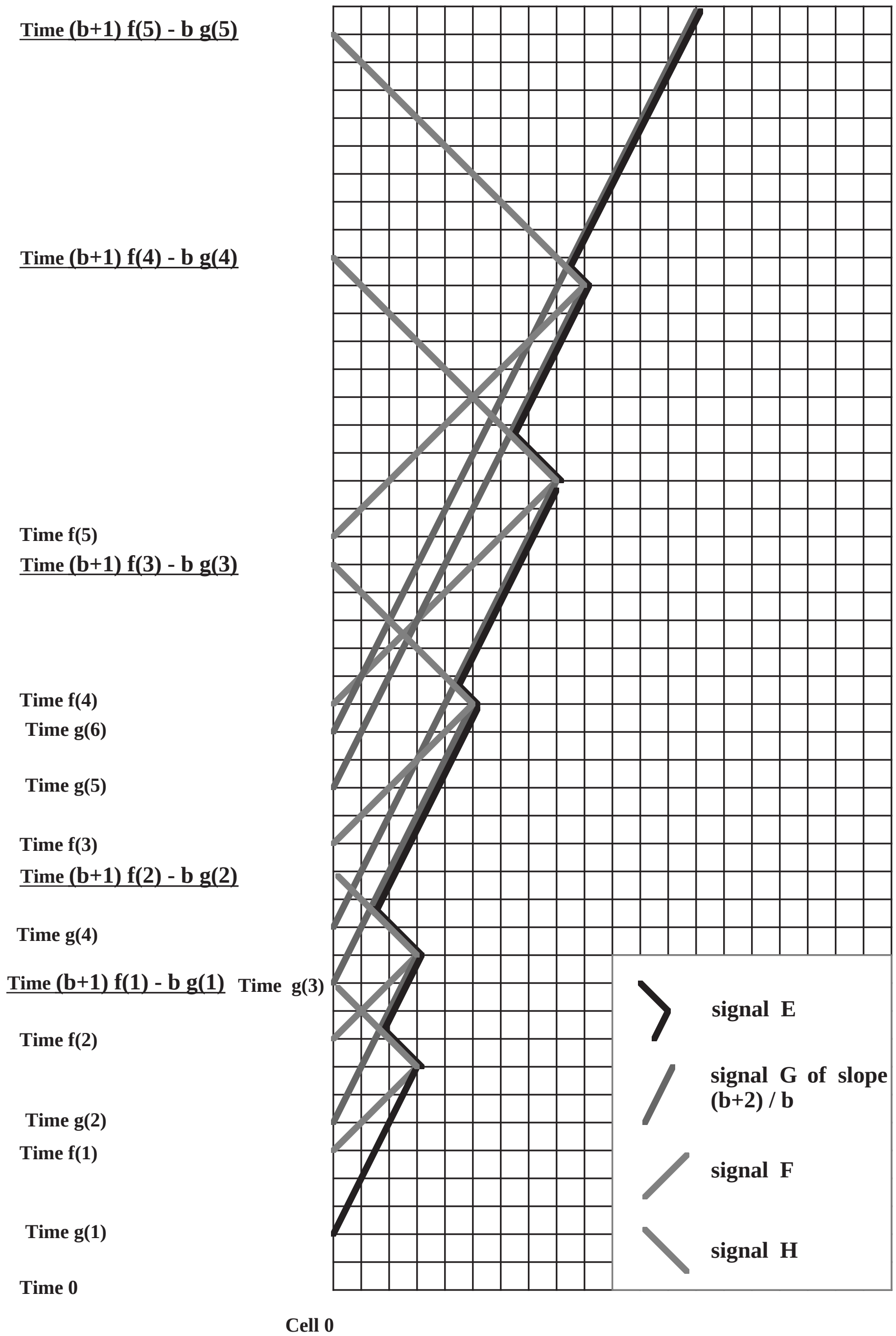

Figure 13: Fischer's construction of $(b+1)f - bg$.

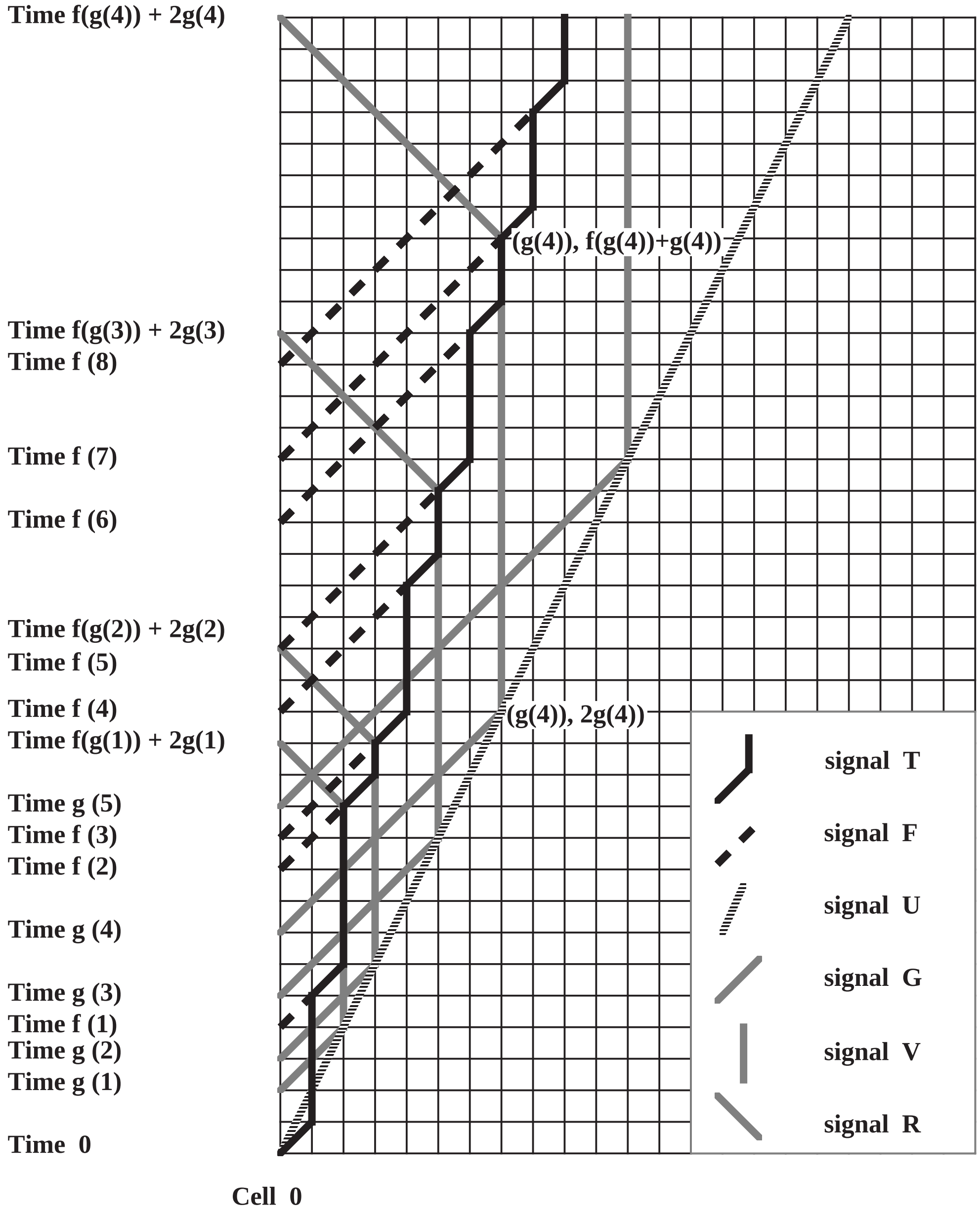

Figure 14: Fischer's construction of $f \circ g + 2g$.

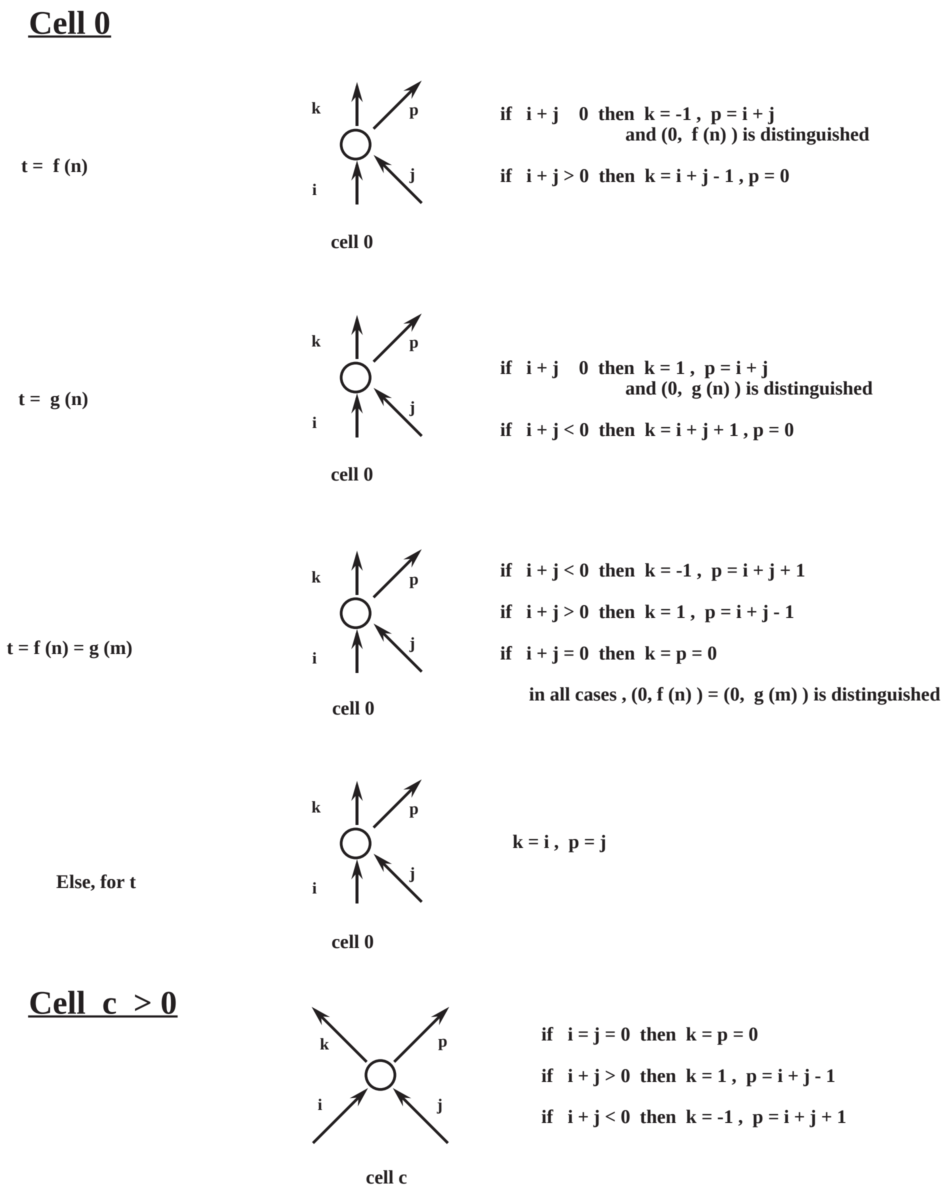

Figure 15: Transitions of the ICA constructing $min(f, g)$.

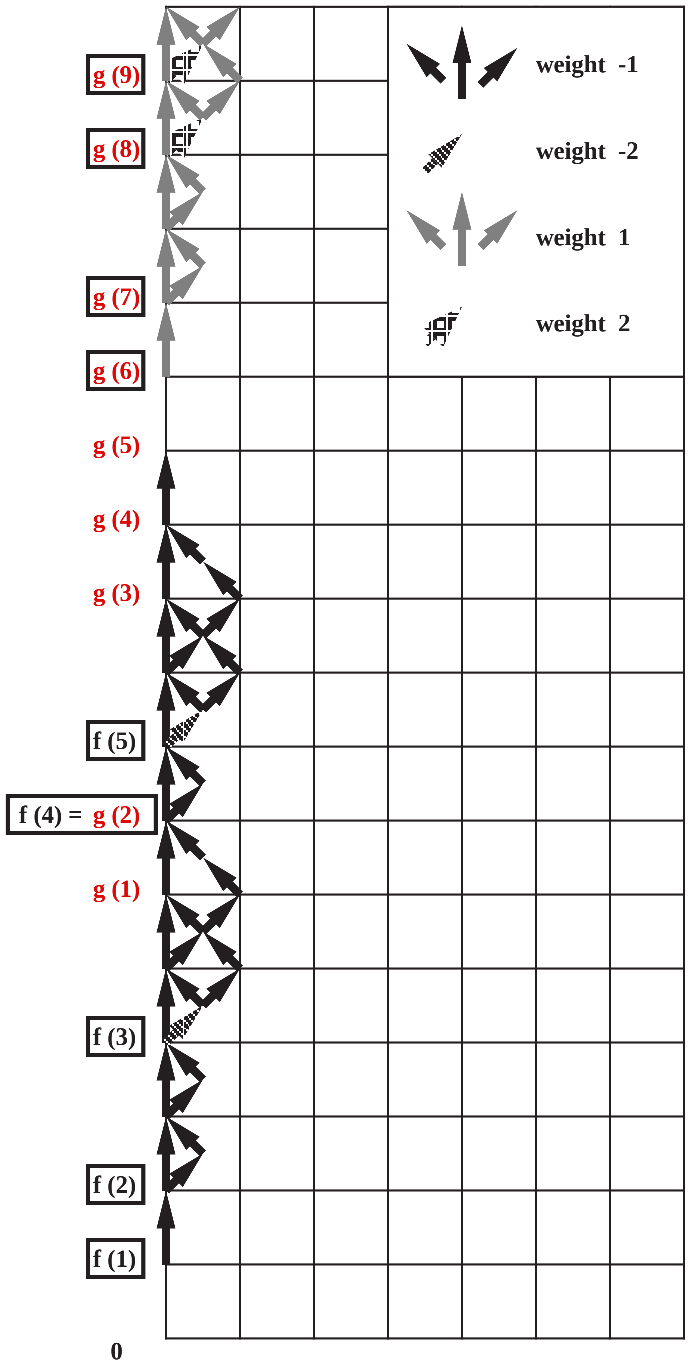

Figure 16: Fischer's construction of $min(f, g)$.

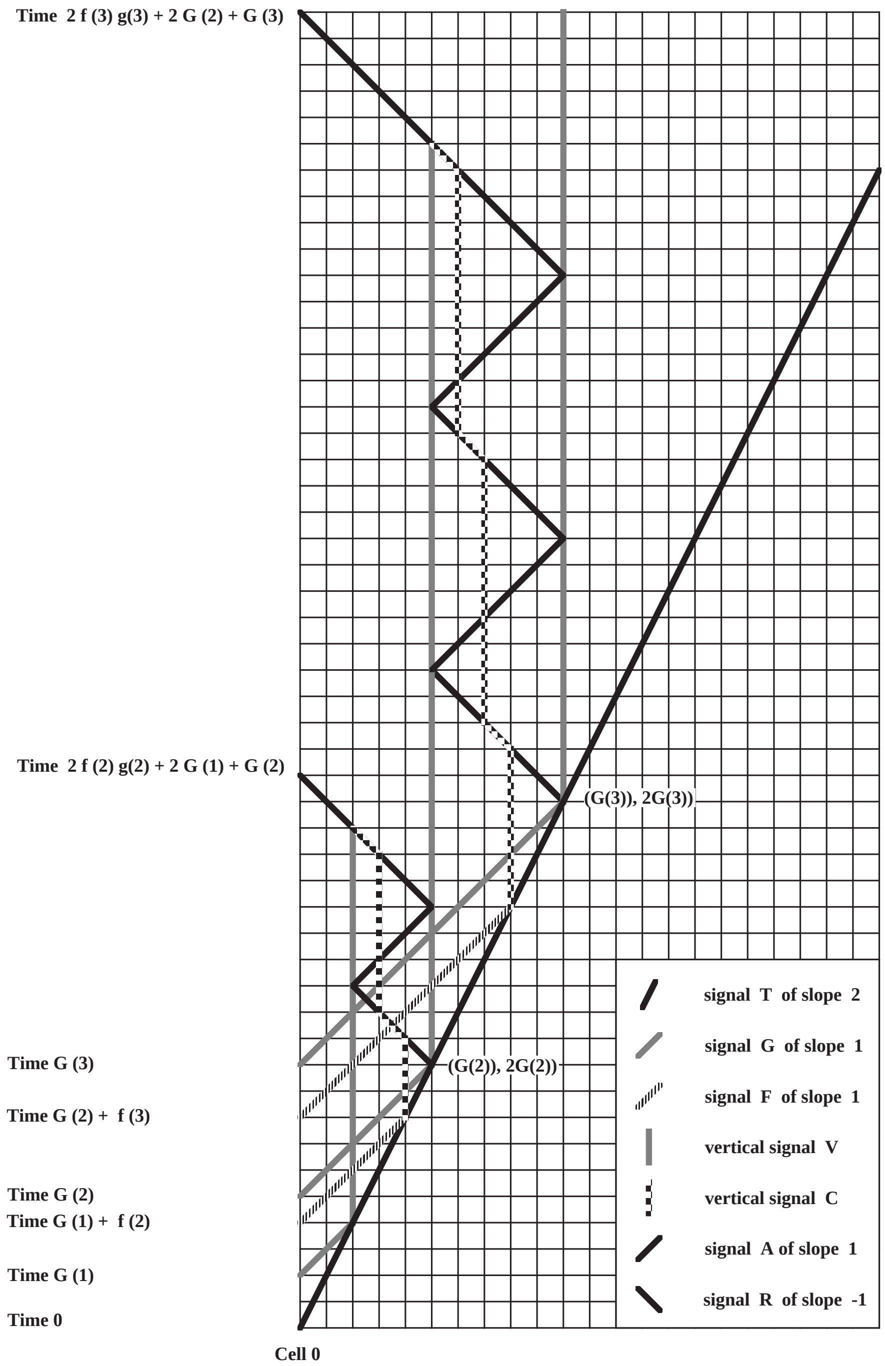

Figure 17: Fischer's construction of $2f(n)g(n) + 2G(n-1) + G(n)$.

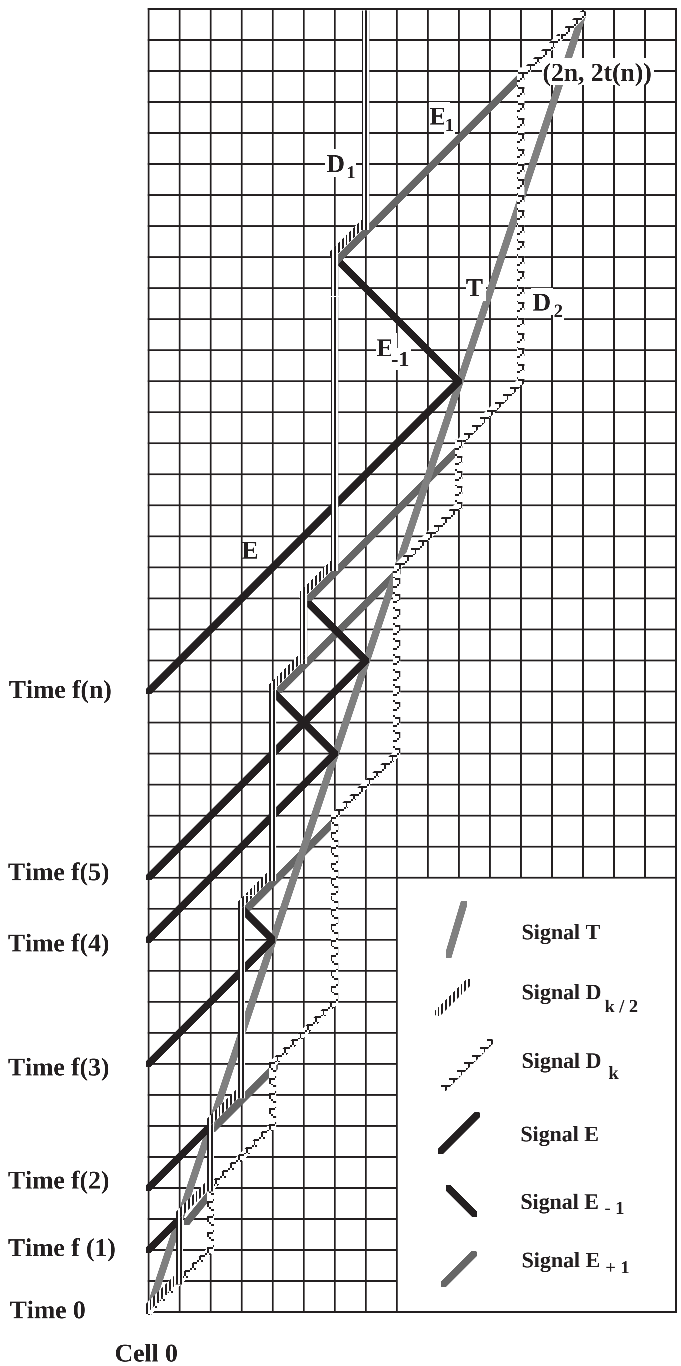

Figure 18: Characterization of the sites $(n, f(n))$.