On edge tricolorations of triangulations of simply connected surfaces

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Research Report N° 2002-22

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Abstract

This paper studies the tricolorations of edges of triangulations of simply connected orientable surfaces such that the degree of each interior vertex is even. Using previous results on lozenge tilings, we give a linear algorithm of coloration for triangulations of the sphere, or of planar regions with the constraint that the boundary is monochromatic. We define a flip as a shift of colors on a cycle of edges using only two colors. We prove flip connectivity of the set of solutions for the cases seen above, and prove that there is no flip accessibility in the general case where the boundary is not assumed to be monochromatic. Nevertheless, using flips, we obtain a tiling invariant, even in the general case. We finish relaxing the condition, allowing monochromatic triangles. With this hypothesis, there exists some local flips. We give a linear algorithm of coloration, and strong structural results on the set of solutions.

Keywords: tiling, height function, flip

Résumé

Cet article etudie les tricolorations des aretes des triangulations des surfaces simplement connexes orientables, telles que le degre de chaque sommet interieur soit pair. A partir de resultats precedents sur les pavages par des losanges, nous donnons un algorithme lineaire de colorations pour les triangulations de la sphere, ou pour des regions du plan sous la contrainte que le bord est monochromatique. Nous definissons un flip comme etant une inversion des couleurs sur un cycle d'aretes n'utilisant que deux couleurs. Bous prouvons la connexite par flips de l'ensembles des solutions dans les cas vus ci-dessus, et montrons que la connexit n'est pas toujours obtenue quand le bord n'est pas monochromatique. Neanmoins, grace aux flips, nous avons un invariant de pavage, valable dans le cas general.

Nous terminons en relachant les conditions par l'introduction de tuiles monochromatiques. Dans ce cas, il existe des flips locaux. Nous donnons un algorithme linaire de colorations et des rsultats structurels fors sur l'espace des solutions.

Mots-clés: pavage, fonction de hauteur, flip

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Abstract

This paper studies the tricolorations of edges of triangulations of simply connected orientable surfaces such that the degree of each interior vertex is even.

Using previous results on lozenge tilings, we give a linear algorithm of coloration for triangulations of the sphere, or of planar regions with the constraint that the boundary is monochromatic.

We define a flip as a shift of colors on a cycle of edges using only two colors. We prove flip connectivity of the set of solutions for the cases seen above, and prove that there is no flip accessibility in the general case where the boundary is not assumed to be monochromatic. Nevertheless, using flips, we obtain a tiling invariant, even in the general case.

We finish relaxing the condition, allowing monochromatic triangles. With this hypothesis, there exists some local flips. We give a linear algorithm of coloration, and strong structural results on the set of solutions.

1 Introduction

In 1990, W. P. Thurston [13] gave an algebraic study of tilings problems, based on ideas from J. H. Conway and J. C. Lagarias [4]. Especially, in his paper, W. P. Thurston studies the tilings simply connected regions of the triangular lattice, with lozenges formed from two triangles of the lattice. Using a height function and local transformations (called flips), an algorithm of tiling is exhibited. Remark that such a tiling can be seen as a coloration of edges in two colors (blue and red) such that each triangle has two blue edges and a red edge and the boundary is blue.

In the present paper, we add a color for edges, and study tricolorations. We are focused in the most natural problem, where the constrains are that each triangle must have an edge of each color and the boundary has to be monochromatic.

Our framework is larger than simply connected bounded regions of the triangular lattice studied by W. P. Thurston : we work in finite triangulations of orientable compact surfaces, such the degree of each interior vertex is even. Our arguments hold in this framework, which points out the importance of topologic properties for this type of problem.

Using results about lozenge tilings, we first give an algorithm of coloration. Afterwards we introduce *flips* (which are shifts of colors on a cycle of edges using only two colors). At the opposite as for the classical cases as lozenge tilings, these flips are not local, which creates a strong difficulty. We prove that one can transform any coloration into any other one by a sequence of our flips. The flip connectivity is a main property in problems of uniformly random [8] or exhaustive [7] generation.

If we assume that the coloration of the boundary is fixed, but not monochromatic, then the flip connectivity is lost. Nevertheless flips can be used to prove that the number of direct rectangles (i. e. those whose colors yellow, blue, red are seen in this order turning clockwise around the rectangle) is a invariant which does not depends on the coloration.

We finish allowing monochromatic triangles : in this case, local flips appear, which permits to get a linear algorithm of coloration and strong structural properties of the set of solutions.

2 Previous results on lozenge tilings

Triangulations A *triangulation* (see [9] for details) of a compact surface S is a finite family $\{Tr_1, Tr_2, \ldots, Tr_p\}$ of closed subsets of S that cover S, and a family of homeomorphisms $\varphi_i : Tr'_i \to Tr_i$, where each Tr'_i is a a triangle of the plane $\mathbb{R}^2$. The subsets Tr_i are called "triangles", the subsets of Tr_i which are the images of the vertices and edges of the triangle Tr'_i under φ_i are also called *vertices* and *edges*. It is required that any two distinct triangles, Tr_i and Tr_j, either be disjoint, have a single vertex in common, or have an entire edge in common. Two vertices of a same triangle, or two triangles with an common edge, are said neighbors.

Let Φ be a triangulation. We say that Φ is *even* if Φ is a triangulation of a simply connected (i. e. each cycle can be contracted) orientable surface (i. e. homeomorphic to a sphere or a compact surface of the plane $\mathbb{R}^2$) such that each interior vertex has an even number of neighbors. This notion comes from the fundamental example of bounded subsets of the classical triangular lattice of the plane, without hole.

If Φ is even, then that there exists a bicoloration of triangles with white or black colors, in such a way that two neighbors triangles have not the same color. Now, we fix such a coloration. Edges of triangles of Φ can be directed in such a way that the three edges of any black triangle form a clockwise circuit, and the three edges of any white triangle form a counterclockwise circuit (see figure 1). For the sequel, we denote G_Φ as the directed graph defined by this way. For each pair (v, v') of neighbor vertices, we define $orient(v, v')$ by : $orient(v, v') = 1$ if (v, v') is a directed edge of G_Φ, and $orient(v, v') = -1$ otherwise.

The corresponding indirected edge is denoted by $[v, v']$.

Lozenge tilings A *lozenge* is a pair of neighbor triangles of Φ. The common (undirected) edge is called the *central axis* of the lozenge. A *lozenge tiling* of Φ is a set of lozenges which cover the whole surface with neither gap nor overlap. In other words, it is a perfect matching on the triangles of Φ.

There exists a very powerful tool to study lozenge tilings on even triangulations : it is the notion of *height function*, introduced by W. P. Thurston ([13]) and independently in the statistical physics literature (see [2] for a review)for simply connected regions of the triangular lattice, and precisely studied and generalized by several authors ([3], [11], [12], [7]). The main results of the study are summarized below (see especially [12], [7] for details). The extension to even triangulations is straightforwards. Notice that the notions below can be applied in a more general framework ([1], [3], [11]).

Height functions A lozenge tiling T of an even triangulation can be encoded by a height function h_T, defined as follows (see figure 1) : fix an origin vertex O of G_Φ (in the boundary of the surface, when it is not empty), for which $h_T(O) = 0$, and the following rule : if (v, v') is a directed edge of G_Φ such that $[v, v']$ is the central axis of a lozenge of T, then $h_T(v') = h_T(v) - 2$; otherwise $h_T(v') = h_T(v) + 1$. This definition is coherent, since it is coherent on each triangle and we have the simple connectivity.

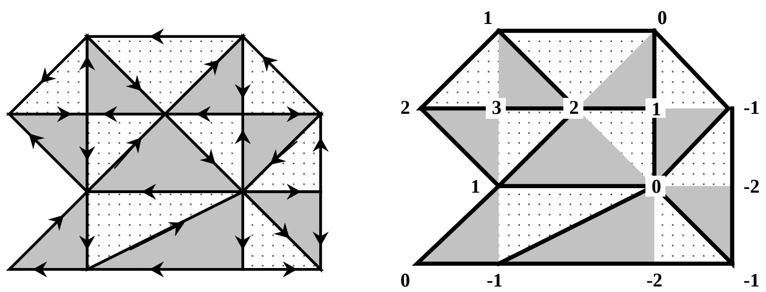

Figure 1: Left : An even triangulation and the induced orientation of edges. Right : A lozenge tiling and its height function

Order Let (T, T') be a pair of tilings of Φ. We say that $T \leq T'$ if for each vertex v of G_Φ, $h_T(v) \leq h_{T'}(v)$. The functions $h_{inf(T,T')} = min(h_T, h_{T'})$ and $h_{sup(T,T')} = max(h_T, h_{T'})$ are height functions of tilings, (which can be interpreted in order theory that the set of tilings has a structure of distributive lattice (see for example [5] for basis of lattice theory).

Flips Let v be an interior vertex such that all the directed edges of G_Φ ending in v correspond to the central axes of lozenges of a tiling T. A flip is the replacement of all these lozenges by lozenges whose central axis correspond to an edge starting in v (see figure 2). A new tiling T_{flip} is obtained by this way, and T and T_{flip} are comparable for the order defined above on tilings. Moreover $T \leq T'$ if and only if there exists an increasing sequence $(T = T_0, T_1, \ldots, T_p = T')$ of tilings such that for each integer i such that $0 \leq i < p$, T_{i+1} is deduced from T_i by a flip. As a corollary we get the *flip connectivity* : given any pair (T, T') of tilings of Φ, on can pass from T to T' by a sequence of flips and, more precisely, the minimal number of flips to pass from T to T' is $\sum_v |h_T(v) - h_{T'}(v')|/3$.

Construction There exists a minimal tiling which has a convexity property that no local maximum can exist, except on the boundary or in the origin vertex, since, otherwise,

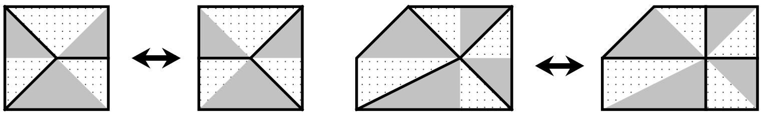

Figure 2: Examples of flips

a flip can be done around the local maximum.

From this property, there exists a linear algorithm of tiling, which construct the minimal tiling, when there exists a tilings, or, otherwise, claims that there is no tiling.

3 Construction of a tiling by trichromatic-edges triangles

In this section, we apply the previous results to solve the following problem : given a triangulation whose edges of the boundary are colored in yellow, does it exist a linear time algorithm to color the other edges (in either yellow, blue or red) in such a way that each triangle has exactly one edge of each color ? This is what we call a *tiling by trichromatic-edges triangles* (or trichromatic tiling for short) .

Theorem 1 *Let Φ be a triangulation and Φ_{inter} denote the new trangulation obtained removing triangles with an edge on the boundary of the surface.*

The triangulation Φ has a trichromatic tiling with yellow boundary if and only if Φ_{inter} has a lozenge tiling.

Moreover, we have a polynomial (linear for even triangulations) time algorithm to build a trichromatic tiling when there exists one (and claim that there is no tiling, otherwise).

Proof. We suppose that we have a tiling by trichromatic-edges triangles. Then, the red (resp. yellow or blue) edges are clearly the central axes lozenges of a tiling of Φ_{inter}.

Conversely, assume that we have a lozenge tiling T of Φ_{inter}. Color the central axes of the lozenges of T in yellow. Now, we denote by $H_{T,y}$ the symmetric graph on the cells of Φ and such that two cells are joined by an edge if and only if this two cells are adjacent by a non-yellow edge. It is clear that this graph is a disjoint union of elementary even cycles. Moreover, its edges are in one to one correspondence with the non-yellow edge of triangulation. It suffices to alternatively color on each cycle the edges in red and blue to obtain a tiling by trichromatic-edges triangles.

Moreover, given a tiling T of Φ_{inter}, the method above of alternatively coloring cycles of $H_{T,y}$ gives a trichromatic tiling in linear time. This gives the algorithmic part of the theorem, since such a lozenge tiling of Φ_{inter} can be obtained in polynomial time (and linear time for even triangulations, using height functions). □

4 Accessibility by cyclic flips

Let T be a tiling by trichromatic-edges triangles. We recall that we denote by $H_{T,y}$ (resp. $H_{T,b}$, $H_{T,r}$) the symmetric graph on the cells of Φ and such that two triangles are joined by an edge if and only if this two triangles are linked by a non-yellow (resp. non-blue, non-red) edge. This graphs are disjoint unions of elementary even cycles. We call an *anti-yellow cycle* (resp. anti-blue cycle, anti-red cycle), a cycle in $H_{T,y}$ (resp. $H_{T,b}$, $H_{T,r}$). An anti-yellow (resp. anti-red, anti-blue) *cyclic flip* id the inversion of the red and blue (resp. blue and yellow, yellow and red) edges in an anti-yellow (resp. anti-red, anti-blue) cycle. We obtain by this transformation a new tiling by trichromatic-edges triangles. A natural question is to know if we can obtain all the other tilings of Φ from T by a sequence of cyclic flips.

Theorem 2 *Let Φ be an even triangulation. All the tilings by trichromatic-edges triangles of Φ with yellow boundary are mutually flip accessible.*

Proof. We have seen below that we can associate to a tiling by trichromatic-edges triangles T a lozenge tiling T_y in considering the yellow edges of T as the central axes of lozenges of T_y.

Now, suppose that we have two tilings by trichromatic-edges triangles T_1 and T_2, then we know by the results of section 2 that there exists a sequence of flips to transform $T_{1,y}$ into $T_{2,y}$. We will prove by induction on the length $\Delta(T_1, T_2)$ of the minimum sequence of flips which transforms $T_{1,y}$ into $T_{2,y}$ that T_1 and T_2 are accessible by cyclic flips.

If $\Delta(T_1, T_2) = 0$, then $T_{1,y} = T_{2,y}$, and so, $T_{1,y}$ and $T_{2,y}$ have exactly the same anti-yellow cycles. It is easy to see that we can obtain T_2 from T_1 by a sequence of (anti-yellow) cycle flips.

Suppose that we have the property for any pair (T_0, T_0') such that $\Delta(T_0, T_0') = n$ and take a pair (T_1, T_2) such that $\Delta(T_1, T_2) = n + 1$. We will prove that there exists a trichromatic tiling T' such that T' can be deduced from T_1 by a sequence of cyclic flips and $\Delta(T', T_2) = n$. If we prove this point it is clear that we have achieved the proof by induction.

We denote by x the vertex where we do the first flip f_1. Consider the set ϕ_x of triangles a vertex of which is x. Each of these triangles has a unique edge which does not contain x, and ϕ_x is an even triangulation of a surface S_x, homeomorphic to a closed disk. We distinguish two cases according to the colors on the boundary of S_x.

The simple case is when this boundary is monochrome, for example red, in T_1. In this case, we have an anti-red cyclic flip around x, (which also is a lozenge flip) which transforms T_1 into T', and $\Delta(T', T_2) = n$, which gives the result.

The tricky case is when the boundary of S_x is not monochrome in T. Consider the lozenge tiling L_x of ϕ_x such that non-yellow edges issued from x are central axes of lozenges of L_x. Two neighbors triangles ϕ_x of are in the same lozenge of L_x if and only if they are in the same cycle. Notice that if triangles are ordered clockwise, all the first triangles of lozenges of L_x have the same color, which can be assumed to be white.

Then, take an anti-yellow cycle C of T which contains at least one of the lozenges of L_x. Let $(l_1, l_2, \ldots, l_p)$ be the sequence of lozenges of L_x in C, in a clockwise order around x (see figure 3). We direct C in such a way that C comes into l_1 by its white triangle and leaves l_1 by a black triangle. Notice that, following C, all the edges which are crossed when one passes from a white triangle to a black triangle have the same color.

What happens, following C, after having left l_1 ? Because of planarity (especially Jordan's theorem on loops of the plane), the next time that C comes back into S_x, then C necessarily comes into l_2, (since otherwise, l_2 cannot be visited, later in the cycle) by

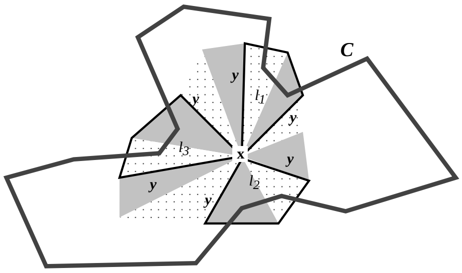

Figure 3: The intersection of the anti-yellow cycle C with S_x

its white triangle (since otherwise C cannot visit other lozenges without cutting itself). Thus, repeating the argument, C comes into all the lozenges of the sequence $(l_1, l_2, \ldots, l_p)$ by the white triangles.

The conclusion of this study is that all the edges of the boundary of ϕ_x which are edges of triangles of C have the same color. So, we can make cycle flips around some of the anti-yellow cycles which meet S_x to obtain a tiling T_1' which has a monochrome boundary of S_x. Moreover, observe that $T_{1,y} = T'_{1,y}$. So, we have gone back to the first simple case. This achieves the proof. □

A counter-example in the planar general case In the figure below, we present a example which shows that there is no general flip accessibility, when the boundary is not monochromatic.

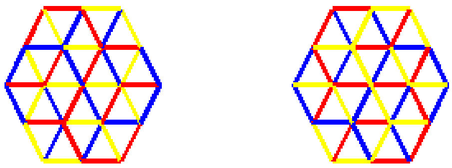

Figure 4: Two tricolorations, equal on he boundary, with no accessibility from one to the other one, since each of these colorations has no cycle.

5 Invariant of orientations of triangles

The goal of this section is to describe an invariant for the tilings. In other words, we look for some properties of the tilings which depend only on the triangulation.

We limit ourselves to even (consequently orientable) triangulations. In this case, remark that there exists to type of colored triangles : the *direct* colored ones and the *indirect*. If we turn counterclockwise around a direct triangle, we see consecutively a red, a blue and a yellow edge. Conversely, If we turn around a indirect triangle, we see a red, a yellow and a blue edge.

For each trichromatic tiling T, *direct*(T) is the number of direct triangles of T and *indirect*(T) is the number of indirect triangles of T. In this section, we will prove that these values are of tiling invariants of tilings.

We start with the simple case of the sphere, from which the result will be extended later.

Theorem 3 (invariant for the sphere) *All the tilings by trichromatic-edges triangles of an even triangulation of the sphere have the same number of direct (and indirect) triangles.*

Proof. As all tilings are accessible by cyclic flips, it suffices to show that the number of direct triangles in a cycle C in $H_{T,y}$ or $H_{T,b}$ or $H_{T,r}$ does not change when we make a flip on C. Indeed, when we make a flip on C, we transform all its direct triangles into indirect ones, and conversely.

So, we have to prove that, for any cycle, there is the same number of direct triangles than indirected ones.

Coding function To do it, we introduce the *coding functions* of trichromatic tilings. Let T such a tiling, the function f_T is defined from the set of vertices of G_Φ to the set $\mathbb{C}$ of complex numbers, by : $f_T(O) = 0$ (where O denotes the origin vertex), and for each directed edge (v, v') of G_Φ $f_T(v') - f_T(v)$ is equal to 1 (resp. j, j^2) if the edge $[v, v']$ is yellow (resp. blue, red) in T (where j denotes the unique complex number such that $j^3 = 1$ and the (purely) complex component of j is positive). This definition is coherent, since it is coherent for any triangle, and the sphere is simply connected.

Combinatorial boundaries of cycles Now, let C be an anti-yellow cycle of T, assumed to be directed clockwise. let Tr be a triangle of C, the *successor* of Tr is the first triangle Tr' of C, obtained after Tr in the cycle, such that the yellow edges of Tr and Tr' are not disjoint. By this definition, we define some (two in fact) disjoint circuits of triangles, whose union cover C. Each of these circuits induces a circuit $(v_0, v_1, \ldots, v_p = v_0)$ of vertices (called a combinatorial boundary of C) such that, for each integer i such that $0 \leq i \leq p$, $[v_i, v_{i+1}]$ is the yellow edge of the successor Tr' of the triangle Tr whose yellow edge is $[v_{i-1}, v_i]$.

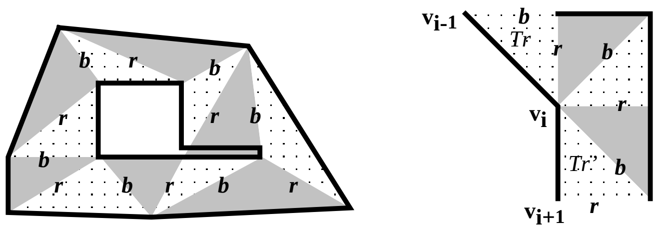

Figure 5: Left : the combinatorial boundaries of a cycle. Right : a triangle and its successor have the same orientation if and only if they have the same color.

Notice that Tr and Tr' have the same orientation if and only if Tr and Tr' have the same color, that is if and only if $orient(v_{i-1}, v_i) = orient(v_i, v_{i+1})$ (since indirected edges issued from v_i crossing the cycle C alternatively have red and blue colors) (see figure 5). Moreover, for each combinatorial boundary, $\sum_{i=1}^{p} orient(v_{i-1}, v_i) = \sum_{i=1}^{p} f_T(v_i) - f_T(v_{i-1}) = f_T(v_p) - f_T(v_0) = 0$, which means that, the number of edges of positive orientation is equal to the number of edges of negative orientation. Thus, for each circuit of triangles, the number of direct triangles is equal to the number of indirect triangles. The same argument can be done (up to the existence of a factor j or j^2 in the sequence of equalities) for blue or red cycles, which achieves the proof.

$\square$

We now prove the analog theorem for the triangulations of regions region of the plane. Notice that we do not necessarily assume that the boundary is monochromatic.

Theorem 4 (invariant for surfaces of the plane) *All the tilings by tri-chromatic-edges triangles of an even triangulation of a simply connected compact surface of the plane with the same fixed coloration of the boundary have the same number of direct (and indirect) triangles.*

Proof. We use the theorem 3 to prove that. Let T_1 and T_2 be two tilings of an even triangulation Φ of a simply connected compact surface of the plane with the same fixed coloration of the boundary. We denote by $[T_1, T_2]$ the tiling of the triangulation[1] of the sphere obtained gluing (by identification) the boundary of T_1 with the boundary of T_2. Notice that $[T_1, T_2]$ tiles an even triangulation.

Now, consider both tilings $[T_1, T_1]$ and $[T_1, T_2]$, by the theorem 3 they have the same number of direct (and indirect) triangles. Obviously, this involves that T_1 and T_2 have the same number of direct (and indirect) triangles. □

6 Related problems

In this section, we allow monochromatic triangles. We see that the problem becomes much easier since there exists local flips. In each case below, we use a height function and the ideas previously used in [10] for Wang tiles.

Tiling by trichromatic-edges and monochromatic-edges triangles Firstly, we relax our condition allowing all types of monochromatic triangles. Such a tiling T can be encoded using a function g_T from the set of vertices of G_Φ to the set $\mathbb{Z}/3\mathbb{Z}$ of integers modulo 3, such that $g_T(O) = 0$ (where O denotes the origin vertex) and for each directed edge (v, v') of G_Φ, $g_T(v') - g_T(v)$ is equal to 0 (resp. 1, 2) if the edge $[v, v']$ is yellow (resp. blue, red) in T. This definition is coherent, since it is coherent for any triangle, and we have the simple connectivity.

Notice that, conversely, each function g from vertices of G_Φ to $\mathbb{Z}/3\mathbb{Z}$ such that $g(O) = 0$ induces a coloration of Φ with trichromatic and monochromatic triangles. Thus there exists such a coloration if and only if the function g_T can be defined on the boundary with no contradiction (this condition is, of course satisfied in the sphere, which has an empty boundary, in the other cases, we just have to flow the boundary to construct g_T on the boundary). Moreover, given a coloration of the boundary, there exists an easy linear algorithm of tiling, and if a tiling exists, then there exists 3^N tilings satisfying the same boundary condition, where N denotes the number of free vertices (i. e. vertices which are not the origin for the spheres, vertices which are not on the boundary in the other cases).

We define a flip as the change the value $g_T(v)$ of a free vertex v (which changes all the colors of the edges with v as endpoint). We have the flip connectivity for each set of tilings satisfying the same boundary condition, and moreover the minimal number of

[1]Formally, we do not exactly have a triangulation when a triangle of Φ has two edges on the boundary (since two glued triangles share two edges), nevertheless our arguments can be obviously adapted to this case

necessary flips to pass from a tiling T to a tiling T' is $\sum_v (1 - \delta(g_T(v), g_{T'}(v))$ (where $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ otherwise).

Tiling by trichromatic-edges and one fixed monochromatic-edges triangle We now only allow monochromatic yellow triangles (and trichromatic triangles). This study is nearly the same as the one for lozenge tilings, so we sketch it.

Such a tiling T can be encoded using a height function g'_T from the set of vertices of G_Φ to the set $\mathbb{Z}$ of integers, such that $g_T(O) = 0$ (where O denotes the origin vertex) and for each directed edge (v, v') of G_Φ, $g_T(v') - g_T(v)$ is equal to 0 (resp. 1, -1) if the edge $[v, v']$ is yellow (resp. blue, red) in T. Conversely, for each function g' from vertices of G_Φ to $\mathbb{Z}$ such that $g'(O) = 0$ and for each edge (v, v') of G_Φ, $|g'(v) - g'(v')| \leq 1$, there exists a tiling T such that $g' = g'_T$. If g' and g'' are functions satisfying the above conditions, then $min(g', g'')$ and $max(g', g'')$ also satisfy the same conditions. This implies that each set of tilings with the same boundary has a structure of distributive lattice for the order defined by : $T \leq T'$ if, for each vertex v of Φ, $g_T(v) \leq g_{T'}(v)$.

We define a flip as, when it is possible, the change of the value $g'_T(v)$ of a free vertex v *of one unit* (which changes all the colors of the edges with v as endpoint). Let T and T' be tilings such that $T < T'$ and v_0 be a vertex such that $g_{T'}(v_0) - g_T(v_0)$ is maximal, and $g_T(v_0)$ is minimal with the previous condition. One easily sees that a flip can be done in v_0, which increases the height function. This yields, repeating the argument that there exists an increasing sequence $(T = T_0, T_1, \ldots, T_p = T')$ of tilings such that for each integer i such that $0 \leq i < p$, T_{i+1} is deduced from T_i by a flip. As a corollary, we get the flip formula : the minimal number of flips to pass from T to T' is $\sum_v |g'_T(v) - g'_{T'}(v')|$.

Finally, remark that the minimal tiling T_{min} of has no local maximum for free vertices (since, otherwise, a height decreasing flip can be done), which yields to a linear algorithm of tiling of the same kind as the algorithm of W. P. Thurston [13] for lozenges and dominoes.

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