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Abstract

The fused multiply accumulate instruction (fused-mac) that is available on some current processors such as the Power PC or the Itanium eases some calculations. We give examples of some floating-point functions (such as ulp($x$) or Nextafter($x, y$)), or some useful tests, that are easily computable using a fused-mac. Then, we show that, with rounding to the nearest, the error of a fused-mac instruction is exactly representable as the sum of two floating-point numbers. We give an algorithm that computes that error.

Keywords: Floating-point arithmetic, fused multiply accumulate, computer arithmetic

Résumé

L’instruction “fused-mac” (multiplication-addition regroupées), qui est disponible sur certains processeurs récents comme le Power PC ou l’Itanium facilite certains calculs. Nous donnons ici quelques exemples de fonctions virgule flottante (comme ulp($x$) ou Nextafter($x, y$)), ou de tests, qui sont facilement implantables avec un fused-mac. Nous montrons ensuite qu’en arrondi au plus proche, l’erreur d’une instruction fused-mac est exactement représentable comme somme de deux nombres virgule flottante. Nous donnons un algorithme calculant cette erreur.

Mots-clés: Arithmétique virgule flottante, multiplieur-additionneur, arithmétique des ordinateurs
Some Functions Computable with a Fused-mac

1 Introduction

The fused multiply accumulate instruction (fused-mac) is available on some current processors such as the IBM Power PC or the Intel/HP Itanium. That instruction computes an expression $ax + b$ or more generally $\pm ax \pm b$ with one final rounding error only. This makes it possible to perform correctly rounded division using Newton-Raphson division \cite{17, 7, 16} (the main idea behind that is that if $q$ approximates $x/y$ with enough accuracy, then the remainder $x - yq$ will be exactly computed with a fused-mac, allowing to correct the quotient estimation). Also, this makes evaluation of scalar products and polynomials faster and, generally, more accurate than with conventional (addition and multiplication) floating-point operations. This is important, since scalar products appear everywhere in linear algebra, and since polynomials are very often used for approximating functions.

It has been known for three decades \cite{9} that (assuming rounding to nearest) the error of a floating point addition or a floating-point multiplication in a given format is exactly representable as a floating-point number of the same format. This is also true for the remainder of a division or a square root with any rounding mode \cite{2, 3}. A natural question arises: is there a similar property for the fused-mac operation?

Also, expert floating-point programming sometimes requires the evaluation of functions such as $\text{Nextafter}(x, y)$, or the successor of a given floating-point number, or (for error estimation), $\text{ulp}(x)$. We may also, for some calculations, need to know if the last mantissa bit of a number is a zero \cite{4}. These various functions can always be computed at a low level, using masks and integer arithmetic: this results in software that is not portable, and sometimes quite slow, since the corresponding calculations are not performed in the floating-point pipeline. With conventional arithmetic, designing portable software for these functions is feasible \cite{5} but might be costly. We aim at showing that the availability of a fused-mac instruction facilitates portable yet efficient implementation of such functions.

2 Definitions and notations

Define $\mathbb{M}_n$ as the set of exponent-unbounded, $n$-bit mantissa, binary FP numbers (with $n \geq 1$), that is: $\mathbb{M}_n = \{ M \times 2^E, 2^{n-1} \leq M \leq 2^n - 1, M, E \in \mathbb{Z} \} \cup \{0\}$. It is an “ideal” system, with no overflows or underflows. We will show results in $\mathbb{M}_n$. These results will remain true in actual systems that implement the IEEE-754 standard \cite{6, 1}, provided that no overflows or underflows do occur. The mantissa of a nonzero element $M \times 2^E$ of $\mathbb{M}_n$ is the number $m(x) = M/2^{n-1}$, its integral mantissa, noted $M_x$ is $M$ and its corresponding exponent, noted $e_x$ is $E$.

We assume that the reader is familiar with the basic notions of floating-point (FP, for short) arithmetic: rounding modes, ulps, …. See \cite{10} for definitions. In the following $o(t)$ means $t$ rounded to the nearest even.

3 Previous results and preliminary properties

We will use the 2sum and Fast2Sum algorithm, presented below. These algorithms do not require the availability of a fused-mac. They make it possible to compute the error of a floating-point addition exactly, represented by a FP number. The first one \cite{14, 18} only assumes $a$ and $b$ are normalized FP numbers (i.e., elements of $\mathbb{M}_n$).
Property 1 (2Sum Algorithm) Let $a, b \in \mathbb{M}_n$. Define $x$ and $y$ as

$$
\begin{align*}
x &= \circ(a + b) \\
b' &= \circ(x - a) \\
a' &= \circ(x - b') \\
\epsilon_b &= \circ(b - b') \\
\epsilon_a &= \circ(a - a') \\
y &= \circ(\epsilon_a + \epsilon_b)
\end{align*}
$$

We have:

- $x + y = a + b$ exactly;
- $|y| \leq \frac{1}{2} \text{ulp}(x)$.

If we know in advance that $|a| \geq |b|$ (as a matter of fact, it suffices to have $\epsilon_a \geq \epsilon_b$), a much faster algorithm can be used [9, 14]:

Property 2 (Fast2Sum Algorithm) Let $a, b \in \mathbb{M}_n$, with $|a| \geq |b|$. Define $x$ and $y$ as

$$
\begin{align*}
x &= \circ(a + b) \\
b' &= \circ(x - a) \\
y &= \circ(b - b')
\end{align*}
$$

We have:

- $x + y = a + b$ exactly;
- $|y| \leq \frac{1}{2} \text{ulp}(x)$.

Although we have presented these properties assuming a radix-2 number system, it is worth being noticed that the 2Sum algorithm (property 1) works in any radix $\geq 2$, and that the Fast2Sum algorithm (property 2) works in radices 2 and 3. And yet, rounding to nearest is mandatory: with “directed” roundings it is possible [14] to exhibit cases where the difference between the computed value of $a + b$ and the exact value cannot be exactly expressed as a FP number.

The 2Sum algorithm satisfies the following property, that will be needed in Section 5.

Property 3 If $(x, y) = 2\text{Sum}(a, b)$ then $|y| \leq |b|$.

Proof. $x$ is the FP number that is closest to $(a + b)$. This implies that $x$ is closer to $(a + b)$ than $a$. Hence, $|(a + b) - x| = |y|$ is smaller than $|(a + b) - a| = |b|$. \hfill \Box

A well known and useful property of the fused-mac instruction, noticed by Karp and Markstein [13], is that it allows to very quickly compute the product of two FP numbers $x$ and $y$ exactly, expressed as the sum of two FP numbers $u$ and $v$. More precisely,

Property 4 (Fast2Mult Algorithm) Let $a, b \in \mathbb{M}_n$. Define $x$ and $y$ as

$$
\begin{align*}
x &= \circ(ab) \\
y &= \circ(ab - x)
\end{align*}
$$

we have:

- $x + y = ab$ exactly;
- $|y| \leq \frac{1}{2} \text{ulp}(x)$.

Without a fused-mac, computing $x$ and $y$ is possible, but requires much more computation [9] (the mantissas of $x$ and $y$ are splitted, then partial products are computed and summed up).
4 Basic functions computable with a fused-mac

4.1 Checking if the last mantissa bit of some number is a zero

Brisebarre, Muller and Raina [4] have suggested an algorithm for division by a constant that works when the last bit of the divisor mantissa is a zero. Checking that condition is easily done with a fused-mac.

Property 5 (Algorithm IsEven) The following algorithm on \( x \) checks if the last mantissa bit of \( x \) is a zero.

\[
\begin{align*}
\alpha &= \circ(3x) \\
\beta &= \circ(\alpha - 2x) \\
\text{IsEven} &= (\beta = x)
\end{align*}
\]

One may notice that the same algorithm also works with the usual (addition and multiplication) floating-point instructions. The availability of a fused-mac, here, only saves one operation.

4.2 Checking if a number is a power of 2.

The following algorithm requires storage of the constant

\( C = 2^n - 1 \).

Of course, \( C \in \mathbb{M}_n \); it is exactly representable as a floating-point number.

Property 6 (Algorithm IsAPowerOf2) The following algorithm on \( x \) returns “true” if \( x \) is a power of 2.

\[
\begin{align*}
y_h &= \circ(xC) \\
y_l &= \circ(xC - y_h) \\
\text{IsAPowerOf2} &= (y_l = 0).
\end{align*}
\]

Proof if \( x \) is not a power of 2 then \( M_x \) has at least a prime factor different from 2, thus \( M_xC \) is of the form \( P2^\alpha \), where \( P \) is odd and larger than \( 2^n \). Hence \( P \) cannot be exactly representable with \( n \) bits, hence \( y_h \neq xC \), hence \( y_l \neq 0 \). \( \square \)

Important remark The above given algorithm works in the “ideal” set \( \mathbb{M}_n \), which means that with “real world” floating-point arithmetic it will work provided that no overflow or underflow occur. To minimize the risk of overflow/underflow, one should choose

\( C = (2^n - 1)/(2^n) \),

instead of the previously given constant. The proof will be the same, overflow will never occur, and underflow will occur only where \( x \) is a subnormal FP number.

4.3 Floating-point successors

There are several notions of “floating-point successor” that can be defined. The IEEE-754 standard for FP arithmetic\(^1\) [1] recommends (but does not require) the availability of the function \texttt{Nextafter}. \texttt{Nextafter}(\( x, y \)) returns the next representable neighbor of \( x \) in the direction toward \( y \). If \( x = y \), then the result is \( x \) without any exception being signaled. If either \( x \) or \( y \) is a NaN, then the result is a NaN. Overflow is signaled when \( x \) is finite but \texttt{Nextafter}(\( x, y \)) is infinite; underflow is signaled when the result is subnormal or zero. Cody and Coonen [5] provide a portable C version of that function.

Let us show how such a function can be implemented using fused-mac instructions. First, define the following four functions.

\(^1\)See \url{http://754r.ucbtest.org/standards/754.txt}
Some Functions Computable with a Fused-mac

**Definition 1** The successor of a FP number $x$, denoted $x^+$ is the smallest FP number larger than $x$. The predecessor $x^-$ of $x$ is the largest FP number less than $x$. The symmetrical successor of $x$, denoted succ($x$) is $x^+$ if $x < 0$, and $x^+$ if $x > 0$. The symmetrical predecessor pred($x$) of $x$ is $x^+$ if $x < 0$ and $x^-$ if $x > 0$.

The following algorithm will use the constant
\[ s = 2^{-n} + 2^{-2n+1}. \]

Notice that $s \in \mathbb{M}_n$. Even on “real life” floating-point systems, $s$ will be representable: on all floating-point systems of current use, the number of mantissa bits is less than the absolute value of the smallest exponent. This is required by the IEEE-854 Standard for Floating-Point arithmetic [12], that says that $(E_{\max} - E_{\min})/n$ shall exceed 5 and should exceed 10, and that $b^{E_{\max} + E_{\min} + 1}$ should be the smallest integral power of $b$, where $b$ is the radix.

**Property 7** Computation of succ($x$) If $n \geq 2$, then
\[ \text{succ}(x) = o(x + sx) \]

**Proof** Assume $2^e \leq x < 2^{e+1}$ (i.e., the exponent of $x$ is $e$). Since, in that case, succ($x$) = $x + 2^{e-n} + 1$ and ulp($x$) = $2^{e-n+1}$, to show that $o(x + sx)$ is equal to succ($x$) it suffices to show that
\[ x + 2^{e-n} < x + sx < x + 3 \times 2^{e-n} \]

(i.e., that $x + sx$ is within $1/2$ulp from succ($x$)).

Thus, it suffices to show that
\[ 2^{e-n} < sx < 3 \times 2^{e-n}. \tag{1} \]

Since $x \geq 2^e$, $sx > 2^{e-n}$. Since $x < 2^{e+1}$, $sx < 2(1 + 2^{-n+1})2^{e-n}$, which is less than $3.2^{e-n}$ as soon as $n \geq 2$. \hspace{1cm} \Box

**Property 7** shows that succ($x$) can be computed with one fused-mac only.

Function pred($x$) is also computable with one fused-mac only. The proof is very similar to that of **Property 7**.

**Property 8** Computation of pred($x$) If $n \geq 2$, then
\[ \text{pred}(x) = o(x - sx) \]

Now, from functions succ and pred, one can very easily compute functions Nextafter, $x^+$ and $x^-$:

**Property 9**
\[
x^+ = o(x + s|x|) \\
x^- = o(x - s|x|) \\
\text{Nextafter}(x, y) = \begin{cases} 
  x^+ & \text{if } y > x \\
  x & \text{if } y = x \\
  x^- & \text{if } y < x 
\end{cases}
\]

**Important remark:** although we have proven these algorithms assuming an ideal FP arithmetic with unbounded exponents, they work well with “real life” arithmetic. From the definition of succ($x$), underflow is impossible. Also, if $|x|$ is equal to the largest representable FP number, then on a machine compliant with the IEEE 754 standard, $\pm\infty$ (depending on the sign of $x$) will be returned\(^2\), which is the right answer. If $x$ is a NaN, then the fused-mac operation will return

---

\(^2\)This is due to the definition of rounding to the nearest: the standard specifies that An infinitely precise result with magnitude at least $2^{E_{\max}}(2 - 2^{-n})$ shall round to $\infty$ with no change in sign.
Some Functions Computable with a Fused-mac

a NaN. Hence, our algorithm for \( \text{succ}(x) \) is always correct, unless \( x \) is a subnormal number. Function \( \text{pred}(x) \) cannot generate an overflow, correctly propagates NaNs, and correctly signal underflows, however, it does not work correctly if \( x \) is a subnormal number: that (rare) case should be handled separately.

If we use rounding to nearest, then the availability of a fused-mac instruction is mandatory for designing such algorithms. For example:

**Property 10** Apart from the “toy case” \( n \leq 2 \), there is no constant \( C \in M_n \) such that \( \circ(xC) \) always equals \( \text{succ}(x) \).

**Proof:** Suppose that there exists \( C \in M_n \) such that \( \circ(xC) \) always equals \( \text{succ}(x) \). Assume \( 1 \leq x < 2 \) (the other cases are easily deduced from this one). This implies

\[
x + 2^{-n} \leq Cx \leq x + 3.2^{-n}.
\]

Hence,

\[
2^{-n} \leq (C - 1)x \leq 3.2^{-n}
\]

for any \( x \in M_n, 1 \leq x < 2 \). For \( x = 1 \), this implies \( C \geq 1 + 2^{-n} \). Since the smallest element of \( M_n \) larger than or equal to \( 1 + 2^{-n} \) is \( 1 + 2^{-n+1} \), we then have \( C \geq 1 + 2^{-n+1} \). And yet, for \( x \) equal to the largest element of \( M_n \) less than \( 2 \) (i.e., \( 2 - 2^{-n+1} \)), \( C \geq 1 + 2^{-n+1} \) implies \( (C - 1)x \geq 2^{-n+1}(2 - 2^{-n+1}) = 4.2^{-n} - 2^{-2n+2} \). Therefore, in that case, \( (C - 1)x > 3.2^{-n} \), unless \( n \leq 2 \). \( \square \)

This may be different with other rounding modes. For instance, if rounding towards zero \( Z(x) \) is used, then \( Z(x\sigma) \) returns \( \text{pred}(x) \) for any \( x \in M_n \), with \( \sigma = 1 - 2^{-n} \). And yet, in practice, changing the rounding mode may be quite time consuming; this is why an algorithm that works in the default mode (i.e., round-to-nearest) is preferable.

4.4 Function \( \text{ulp}(x) \)

Function \( \text{ulp} \) (unit in the last place) is very frequently used for expressing the accuracy of a floating-point result. Several definitions have been given (see [11] for a discussion on that topic), they differ near the powers of 2. If we use as a definition, when \( x \) is a FP number:

\[
\text{ulp}(x) = |x|^+ - |x|
\]

then one can compute function \( \text{ulp} \) through the following sequence

\[
y = \circ(x + sx) \\
\text{ulp} = |y - x|
\]

where \( s \) is the same constant as in Section 4.3. If we define \( \text{ulp}(x) \) as

\[
\text{ulp}(x) = |x| - |x|^{-}
\]

then function \( \text{ulp} \) is computed through

\[
y = \circ(x - sx) \\
\text{ulp} = |y - x|
\]

The two functions differ only when \( x \) is a power of 2. The first one is compatible with Goldberg’s definition [10] (which is given for real numbers, not only for floating-point ones), the second is compatible with Kahan’s one\(^3\) and Harrison’s one [11] (they differ for real numbers but coincide on FP numbers).

\(^3\)Kahan’s definition is: \( \text{ulp}(x) \) is the gap between the two finite floating-point numbers nearest \( x \), even if \( x \) is one of them (But \( \text{ulp}(\text{NaN}) \) is \( \text{NaN} \)).
5 Computing the error term of a fused-mac

We require here that \( n \geq 3 \). The correcting term cannot be a single FP number, even in rounding to the nearest. We will therefore compute two FP numbers such that their sum is the exact correcting term of the fused-mac.

5.1 The algorithm \texttt{ErrFmac}

\textbf{Property 11 (Algorithm \texttt{ErrFmac})} Let \( a, x, y \in \mathbb{M}_n \). Define \( r_1, r_2 \) and \( r_3 \) as

\[
\begin{align*}
  r_1 &= \circ(ax + y) \\
  (u_1, u_2) &= \text{Fast2Mult}(a, x) \\
  (\alpha_1, \alpha_2) &= \text{2Sum}(y, u_2) \\
  (\beta_1, \beta_2) &= \text{2Sum}(u_1, \alpha_1) \\
  \gamma &= \circ(\beta_1 - r_1) + \beta_2) \\
  (r_2, r_3) &= \text{Fast2Sum}(\gamma, \alpha_2)
\end{align*}
\]

we have:

- \( ax + y = r_1 + r_2 + r_3 \) exactly;
- \( |r_2 + r_3| \leq \frac{1}{2} \text{ulp}(r_1) \);
- \( |r_3| \leq \frac{1}{2} \text{ulp}(r_2) \).

Figure 1 gives the idea behind the algorithm: we want to exactly add the 3 FP numbers \( y, u_1 \) and \( u_2 \). This is usually difficult, but as we know the correct answer \( (r_1) \) thanks to the fused-mac computation, we just have to get the two error terms. We first compute the “small” error, namely \( \alpha_2 \). Then the other terms \( u_1 \) and \( \alpha_1 \) are bigger than this value and can be combined with \( r_1 \) into a single value \( \gamma \).

![Figure 1: Intermediate values of the \texttt{ErrFmac} algorithm.](image)

5.2 Proof of the correctness of the \texttt{ErrFmac} algorithm

If \( \gamma = \circ(\beta_1 - r_1) + \beta_2 \) is equal to \( (\beta_1 - r_1) + \beta_2 \), then \( r_1 + r_2 + r_3 = r_1 + \gamma + \alpha_2 = r_1 + \beta_1 - r_1 + \beta_2 + \alpha_2 = u_1 + \alpha_1 + \alpha_2 = u_1 + u_2 + y = y + ax \). If this equality holds, we easily also have that \( |r_2 + r_3| \leq \frac{1}{2} \text{ulp}(r_1) \) and \( |r_3| \leq \frac{1}{2} \text{ulp}(r_2) \).
There is left to prove that $\beta_1 - r_1$ and $(\beta_1 - r_1) + \beta_2$ are in $\mathbb{M}_n$. If they are, then they are exactly computed and the algorithm is correct. To guarantee that a value $v$ is in $\mathbb{M}_n$, we just have to find an exponent $e$ such that $v2^-e$ is an integer and $|v2^-e| < 2^n$. There may exist more than one suitable $e$, but the existence of one is enough. We split the proof into two subcases.

If we have $\beta_2 = 0$,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Intermediate values of of the ErrFmac algorithm when $\beta_2 = 0$.}
\end{figure}

Figure 2 reminds the compared positions of the FP numbers involved. As $\beta_2 = 0$, we have left to prove that $\beta_1 - r_1$ is in $\mathbb{M}_n$. If $\beta_1 = 0$, then this is correct. Let us assume that $\beta_1 \neq 0$. We then know that $r_1 = o(\beta_1 + \alpha_2)$ as $\beta_2 = 0$.

But we also have that $|\alpha_2| \leq \frac{1}{2}\ulp(\alpha_1)$ from Property 1 and that $|\alpha_2| \leq |u_2| \leq \frac{1}{2}\ulp(u_1)$ from Property 3 and by definition $\beta_1 = o(u_1 + \alpha_1)$. This means that $|\alpha_2| \ll |\beta_1|$. More precisely, we either have:

- the general case: $|\beta_1| \geq 4|\alpha_2|$;
- the special case where $\beta_1$ is a result of a near-total cancellation: $\beta_1 = 2^{\min(e_{\alpha_1}, e_{\alpha_1})}$ and $|\beta_1| \geq 2|\alpha_2|$.

In the general case, we are in the conditions of Sterbenz’s theorem [19]: $r_1$ and $\beta_1$ share the same sign and

\[
|r_1| \leq \frac{|\beta_1 + \alpha_2|}{1 - 2^{-n}} \leq \frac{5}{4} \frac{1}{1 - 2^{-n}} |\beta_1| \leq 2 |\beta_1|
\]

\[
|r_1| \geq \frac{|\beta_1 + \alpha_2|}{1 + 2^{-n}} \geq \frac{3}{4} \frac{1}{1 + 2^{-n}} |\beta_1| \geq \frac{1}{2} |\beta_1|
\]

In the special case, we have $4|\alpha_2| > |\beta_1| \geq 2|\alpha_2|$. As $\beta_1$ is a power of 2, we know that $e_{\beta_1} - 1 \leq e_{r_1} \leq e_{\beta_1}$, so $e_{r_1}$ is a suitable exponent for $\beta_1 - r_1$ and

\[
|\beta_1 - r_1|2^{-e_{r_1}} = |\beta_1 - o(\beta_1 + \alpha_2)|2^{-e_{r_1}}
\]

\[
\leq \left(\frac{1}{2}\ulp(r_1) + |\alpha_2|\right)2^{-e_{r_1}}
\]

\[
\leq \frac{1}{2} + |\beta_1|2^{-e_{r_1} - 1}
\]

\[
\leq \frac{1}{2} + (2^n - 1)2^{e_{r_1} + 1 - e_{r_1} - 1} < 2^n.
\]

If we have $\beta_2 \neq 0$,

Figure 3 reminds the compared positions of the FP numbers involved. In the general case, we have here that $\beta_1 = r_1$, then of course $\beta_1 - r_1 = 0$ and $(\beta_1 - r_1) + \beta_2 = \beta_2$ are in $\mathbb{M}_n$. If not, as $\beta_2 \neq 0$, the only possibility for $\beta_1 = o(\beta_1 + \beta_2)$ not to be equal to $o(\beta_1 + \beta_2 + \alpha_2) = r_1$ is that either $|\beta_2| = \frac{1}{2}\ulp(\beta_1)$ or $\beta_2 = -\frac{1}{4}\ulp(\beta_1)$ if $\beta_1$ is a power of 2.
We also deduce that the exponent of \( r_1 \) and of \( \beta_1 \) differ from at most 1. Lastly, we know that 
\[ |\alpha_2| \leq |\beta_2| \leq 2^{e_\beta_1 - 1}. \]
The value \( \min(e_{r_1}, e_{\beta_1}) \) is a suitable exponent for \( \beta_1 - r_1 \) and 
\[ |\beta_1 - r_1| 2^{-\min(e_{r_1}, e_{\beta_1})} = |\beta_1 - \alpha_1 - \beta_1 + \alpha_2| 2^{-\min(e_{r_1}, e_{\beta_1})} \]
\[ \leq \left( \frac{1}{2} \ulp(r_1) + |\beta_2| + |\alpha_2| \right) 2^{-\min(e_{r_1}, e_{\beta_1})} \]
\[ \leq (2^{e_{r_1} - 1} + 2^{e_{\beta_1} - 1} + 2^{e_{\beta_1} - 2}) 2^{-\min(e_{r_1}, e_{\beta_1})} \leq 4 \]

So \( \beta_1 - r_1 \in \mathbb{M}_n \), as \( n \geq 3 \). There is left to prove that \( (\beta_1 - r_1) + \beta_2 = u_1 + \alpha_1 - r_1 \) is in \( \mathbb{M}_n \). We know that \( e_{\beta_1} + 1 \geq e_{r_1} \geq e_{\beta_1} - 1 \) and that \( \beta_2 \) is either \( 2^{e_{\beta_1} - 1} \) or \( 2^{e_{\beta_1} - 2} \), so \( e_{\beta_1} - 2 \) is a suitable exponent for \( (\beta_1 - r_1) + \beta_2 \) and 
\[ |(\beta_1 - r_1) + \beta_2| 2^{-e_{\beta_1} + 2} = |u_1 + \alpha_1 - \alpha_1 + \alpha_1 + \alpha_2| 2^{-e_{\beta_1} + 2} \]
\[ \leq \left( \frac{1}{2} \ulp(r_1) + |\alpha_2| \right) 2^{-e_{\beta_1} + 2} \]
\[ \leq (2^{e_{r_1} - 1} + 2^{e_{\beta_1} - 1}) 2^{-e_{\beta_1} + 2} \leq 6 \]

So \( (\beta_1 - r_1) + \beta_2 \in \mathbb{M}_n \), as \( n \geq 3 \).

### 5.3 With other rounding modes

Such correcting terms for the fused-mac are only representable when the rounding is to the nearest. For example, when rounding up, if \( a = x = 2^n - 1 \) and \( y = 2^{3n} \) then \( ax + y = 2^{4n} + 2^{2n} - 2^{n+1} + 1 \) and therefore \( r_1 \) must be strictly greater than \( 2^{4n} \) so \( r_1 = \triangle(ax + y) = 2^{4n} + 2^{3n+1} \). So \( r_2 + r_3 \) should be exactly equal to \(-2^{3n+1} + 2^{2n} - 2^{n+1} + 1 \) that cannot be represented as the sum of two FP numbers in \( \mathbb{M}_n \).

### 5.4 Cost of the algorithm

The basic cost of the algorithm is 20 cycles, but this can be tremendously reduced.

The first enhancement is when we know that \( |y| \geq |ax| \) or that \( |y| \geq |u_1| \). Then, the first 2Sum is useless as \( \alpha_1 = y \) and \( \alpha_2 = u_2 \). This is typically the case in range reduction [8, 15].

The second enhancement is to get rid of the final Fast2Sum: this means that the result will not be compressed. It means that we only have:

- \( ax + y = r_1 + r_2 + r_3 \) exactly;
- \( |r_2 + r_3| \leq \frac{1}{2} \ulp(r_1) \);
- \( r_2 = 0 \) or \( |r_2| > |r_3| \).
The last enhancement is if the processor can use several floating-point units (FPUs) in parallel. There are indeed several computations that can be done either at the same time or at consecutive steps in a pipe-line, as there is no dependence between them. For example, the computations of $a'$ and $\epsilon_b$ in the 2Sum algorithm (Property 1) can be done in parallel.

If 3 FPUs are available, the algorithm only costs 12 cycles. The tasks given to each processor are given in Figure 4. More FPUs are useless to speed up the algorithm.

$$\begin{array}{c|c|c|c|c|c|c} P_1 & u_1 & u_2 & \alpha_1 & \alpha_2 & \gamma & r_2 & r_3 \\ \hline P_2 & \beta_1 & \beta_2 \\ \hline P_3 & \end{array}$$

Figure 4: Task repartition when 3 FPUs are available.

If only 2 FPUs are available, the algorithm costs 14 cycles. The tasks given to each processor are shown in Figure 5.

$$\begin{array}{c|c|c|c|c|c|c} P_1 & u_1 & u_2 & \alpha_1 & \alpha_2 & \gamma & r_2 & r_3 \\ \hline P_2 & \beta_1 & \beta_2 \\ \end{array}$$

Figure 5: Task repartition when 2 FPUs are available.

The following table gives the cost of the $\text{ErrFmac}$ algorithm depending on the conditions (number of FPUs, final compression and knowledge that the inequality $|y| \geq |ax|$ holds):

<table>
<thead>
<tr>
<th>Cost (in cycles)</th>
<th>1 FPU</th>
<th>2 FPUs</th>
<th>3 FPUs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given algorithm</td>
<td>20</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>Without the final compression</td>
<td>17</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>When $</td>
<td>y</td>
<td>\geq</td>
<td>ax</td>
</tr>
<tr>
<td>When $</td>
<td>y</td>
<td>\geq</td>
<td>ax</td>
</tr>
</tbody>
</table>

### 6 Conclusion

We have shown that the fused-mac instruction makes it possible to implement efficiently and in a portable way many functions that are useful for expert floating-point programming. We also have shown that the error of a fused-mac operation in a given format is exactly representable as a sum of two floating-point numbers of the same format. We have given a fast and portable algorithm that returns that error. We can take advantage of this algorithm for implementing a very accurate range reduction.

### References


Some Functions Computable with a Fused-mac


