On Poset Sandwich Problems.

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Abstract

A graph $G_s = (V, E_s)$ is a sandwich for a pair of graph $G_t = (V, E_t)$ and $G = (V, E)$ if $E_t \subseteq E_s \subseteq E$. Any poset, or partially ordered set, admits a unique graph representation which is directed and transitive. In this paper we introduce the notion of sandwich poset problems inspired by former sandwich problems on comparability graphs. In particular, we are interested in series-parallel and interval posets which are subclasses of 2-dimensional posets, we describe polynomial algorithms for these two classes of poset sandwich problems and then prove that the problem of deciding the existence of a 2-dimensional sandwich poset is NP-complete.

Keywords: analysis of algorithms, partially ordered sets, graph sandwich problems.

Résumé

Un graphe $G_s = (V, E_s)$ est un graphe sandwich de la paire de graphes $G_t = (V, E_t)$ et $G = (V, E)$ si $E_t \subseteq E_s \subseteq E$. Tout ordre partiel admet une représentation en graphe unique, ce graphe est orienté et transitif. Dans cet article, nous introduisons la notion de problèmes d'ordres sandwich inspirée des problèmes sandwich existants sur les graphes de comparabilité. En particulier, nous nous sommes intéressés aux ordres série-parallèles et aux ordres d'intervalle qui sont des sous-classes des ordres de dimension 2, nous décrivons des algorithmes polynomiaux pour ces deux classes de problèmes d'ordre sandwich puis nous démontrons que le problème de décider de l'existence d'un ordre sandwich de dimension 2 est NP-complet.

Mots-clés: analyse d'algorithmes, ordres partiels, problèmes de graphes sandwich.

1 Introduction

A graph $G_t = (V, E_t)$ is a *spanning subgraph* of $G = (V, E)$ if $E_t \subseteq E$. A graph $G_s = (V, E_s)$ is a *sandwich graph* for the pair G and G_t if $E_t \subseteq E_s \subseteq E$. The GRAPH SANDWICH PROBLEM for a property Π is defined as follows:

Problem 1: GRAPH SANDWICH PROBLEM FOR PROPERTY Π

Instance: Two graphs $G_t = (V, E_t)$ and $G = (V, E)$ such that $E_t \subseteq E$
Question: Does there exist a sandwich graph $G_s = (V, E_s)$ for the pair (G, G_t) satisfying property Π?

Notice that a sandwich problem for property Π can be seen a generalization of the recognition problem for property Π. Graph sandwich problems have been introduced by Golumbic, Shamir and Kaplan [7]. In their seminal paper, they study a wide range of sandwich problems for subfamilies of perfect graphs and a few ones for directed graphs. These families have important applications in diverse areas such as biology [5] [6], communication [8] or algebra [9].

In this paper we study poset (i.e. partially ordered sets) sandwich problems raised by natural questions on the original results. Sandwich problems on subfamilies of comparability or co-comparability graphs raise poset sandwich problems once G_t and G are assumed to be transitive acyclic digraphs (i.e. directed graphs), this is the *poset sandwich problem*. If they are not transitively oriented we then face a *directed sandwich problem*. It would be convenient to find a polynomial transformation from undirected to directed sandwich problems, we study which conditions make it possible.

If we are dealing with digraphs, a major distinction should be done depending on the transitivity of the supergraph of the instance. As a transitively oriented acyclic digraph $G = (V, E)$ represents a poset on V, a sandwich problem for a property Π defined on comparability or co-comparability graphs can directly be translated into a *poset sandwich problem* for the corresponding property $\overrightarrow{\Pi}$ on posets. An example of such dual properties on a graph or on a poset is the class of cographs with the class of series-parallel posets as a poset is series-parallel if and only if its comparability graph is a cograph.

Problem 2: POSET SANDWICH PROBLEM FOR POSET PROPERTY $\overrightarrow{\Pi}$

Instance: Two posets $P_t = (V, E_t)$ and $P = (V, E)$ such that $E_t \subseteq E$
Question: Does there exist a sandwich poset $P_s = (V, E_s)$ for the pair (P_t, P) satisfying $\overrightarrow{\Pi}$?

In the second case, digraphs D_t and D of the sandwich instance have an arbitrary orientation. Actually, for a property $\overrightarrow{\Pi}$ on posets, D_t is assumed to be transitively oriented without loss of generality given that a sandwich graph satisfying $\overrightarrow{\Pi}$ is transitively oriented and thus has to contain at least every D_t transitivity arc. Therefore the instance is written (P_t, D), P_t being a poset and D a digraph. This problem is the *directed sandwich problem*.

Problem 3: DIRECTED SANDWICH PROBLEM FOR POSET PROPERTY $\overrightarrow{\Pi}$

Instance: Two digraphs D_t and D such that $D_t \subseteq D$
Question: Does there exist a sandwich digraph D_s for the pair (D_t, D) satisfying $\overrightarrow{\Pi}$?

These two sandwich problems satisfy a very simple relation of complexity. Indeed, let Π be a property on directed graphs, $\overrightarrow{\Pi}$ the corresponding property on posets, if there exists a sandwich digraph for property Π for a pair of digraphs (D_t, D) arbitrarily oriented, then one exists for a transitively oriented pair, the reduction (1) follows, $\leq_K$ being the Karp reduction.

$$\text{Poset sandwich problem for } \overrightarrow{\Pi} \leq_K \text{ directed sandwich problem for } \Pi. \tag{1}$$

As most of the graph classes studied in [7] were included in comparability graphs class, a natural question is to look for a polynomial transformation between undirected and directed sandwich instances, in particular poset sandwich instance.

In the case where a property Π on comparability graphs is a comparability invariant, we would like that some results of undirected sandwich problems could still be used in the directed case. It could be tempting to say that if a property has sense both on posets and graphs, corresponding sandwich graphs should be related too. Unfortunately, the existence of a sandwich graph does not guarantee that the corresponding sandwich poset exists in the directed instance as depicts the example in Fig. 1 and Fig. 2.

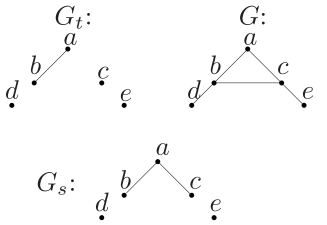


Figure 1: Sandwich graph G_s in undirected case

The undirected sandwich instance of the example is the pair (G_t, G) on vertex set $V = \{a, b, c, d, e\}$ for the property Π defined as follows: *a graph G on V satisfies Π if and only if $d_G(a) = 2$ and $d_G(b) = d_G(e) = 1$* where $d_G(a)$ is the degree of vertex a in G. Π is a comparability invariant because it does not depend on the orientation chosen for the graph. In Fig. 1 G_s is a sandwich graph satisfying Π for the undirected instance. But in Fig. 2, once G_t and G have been transitively oriented to get the pair of posets (P_t, P), it turns out that the corresponding sandwich is not transitively oriented, and thus is not a sandwich poset satisfying Π. Moreover, no sandwich poset for Π can be found with this poset instance. Finally directed and undirected sandwich problems are not equivalent and the results on undirected sandwich graphs don't give direct answers for poset or directed sandwich problems.

In the following, we study the complexity of some relevant classes of poset and directed sandwich problems: series-parallel, interval and permutation posets sandwich.

In certain scheduling problems, tasks are subject to a partial order. Whereas the scheduling problems for an arbitrary partial order are NP-complete, they have efficient algorithms if the partial order is series-parallel [4], these algorithms use a "divide-and-conquer" approach with the recursive structure of these posets. There exists a linear-time algorithm to recognize a series-parallel poset due to Valdes, Tarjan and Lawler [3].

Without loss of generality, we can assume P_s is connected and P_t does not have any isolated vertex.

2 Series-parallel posets

In this section, we are interested in series-parallel posets. This family of posets is obtained from the single vertex poset by the application of two composition rules. The *parallel composition* of posets P_1 and P_2 is the poset $P_1 + P_2 = (V_1 \cup V_2, <_+)$ such that $u <_+ v$ if and only if $u, v \in V_1$

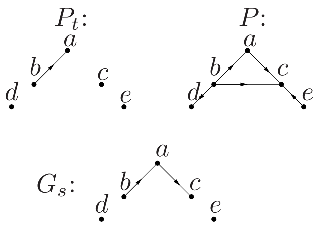


Figure 2: In the directed case, G_s is not an order, it is not a sandwich poset.

and $u <_1 v$ or $u, v \in V_2$ and $u <_2 v$. The *series composition* of posets P_1 and P_2 is the poset $P_1 * P_2 = (V_1 \cup V_2, <_*)$ such that $u <_* v$ if and only if $u, v \in V_1$ and $u <_1 v$ or $u, v \in V_2$ and $u <_2 v$ or $u \in V_1$ and $v \in V_2$. Therefore series-parallel posets are organized in a tree structure.

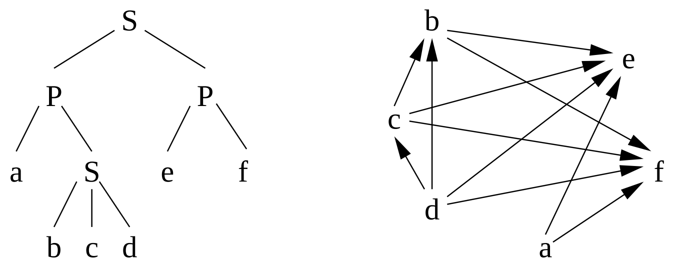


Figure 3: A series-parallel poset and its cannonical composition tree

The comparability graphs of the series-parallel posets are exactly the cographs [**?**]. The cograph sandwich problem has been proved to be polynomial [**?**]. Completing their proof with some argument on the transitiviy, we can show that their algorithm also applies to the series-parallel poset sandwich problem. By the way this problem is also polynomial. What about the series-parallel directed sandwich problem? We propose a polynomial time algorithm for this more general problem.

Problem 4: SERIES-PARALLEL DIRECTED SANDWICH PROBLEM

Instance: A poset $P_t = (V, E_t)$ and a digraph $D = (V, E)$ such that $E_t \subseteq E$
Question: Does there exist a series-parallel sandwich poset P_s for the pair (P_t, D)?

In the following, we note S_P the set of sources of a poset P, $N_P(x)$ the neighborhood of vertex x in P, $P_{|A}$ the poset induced by P on a vertex set A. The notation $Succ^P(x)$, pointing out the set of out-vertices (or successors) of x in P, is extended to sets: $Succ^P(A) = \bigcup_{a \in A} Succ^P(a)$. Without loss of generality we can assume that P_t is connected. Otherwise the problem can be applied on the sandwich instances induced by each connected component. If each sub-instance admits a solution ,then the results are gathered by a parallel composition.

Lemma 2.1 *If there is a series-parallel sandwich poset P_s for the instance (P_t, D), then there exists a set $S \subseteq S_{P_t}$ such that $V \setminus S \subseteq Succ^D(S)$.*

Proof: Suppose there exists P_s, a series-parallel sandwich poset for the instance (P_t, D). Since a series-parallel poset has a linearly ordered set of successors, a subset S of its set of sources satisfies $Succ^{P_s}(S) \supseteq V \setminus S_{P_t}$. As P_s is series-parallel, this set of source is composed in series with the rest of the poset and then $Succ^{P_s}(S) = V \setminus S$, as $Succ^{P_s}(S) \subseteq Succ^D(S)$ we have $V \setminus S \subseteq Succ^D(S)$. $\square$

The algorithm for solving the series-parallel directed sandwich problem recursively reduces the sandwich instance to a set $\tilde{V} \subset V$ by applying the following procedure: Look for a set A among P_t sources satisfying $V \setminus A \subseteq Succ^D(A)$. If none exits then by lemma 2.1 there is no series-parallel poset sandwich. Else remove from V the sets A and the set B of isolated vertices in $P_t|_{V \setminus A}$. Then solve the problem on the instance induced by $\tilde{V} = V \setminus (A \cup B)$. The correctness of this procedure is proved by Lemma 2.2. To compute the set A we iteratively apply the following $\mathcal{P}(V) \to \mathcal{P}(V)$ function f until a fixed point is obtained:

$$f(X) = \{x \in X \mid Succ^D(x) \supseteq V \setminus X\}$$

f^i will denote the i^{th} iteration of f. Let us consider the set $A = f^k(S_{P_t})$ where $k = min\{i \in \mathbb{N} \mid f^i(S_{P_t}) = f^{i+1}(S_{P_t})\}$. Remark that k is well defined as V is finite and f is a decreasing function in terms of inclusion. Note also that A is the unique subset of P_t satisfying $Succ^D(A) \supseteq V \setminus A$.

Lemma 2.2 *The following statements are equivalent:*

(i) There is a series-parallel sandwich poset for the instance (P_t, D).

(ii) There is a series-parallel sandwich poset for the instance $(P_t, D)|_{\tilde{V}}$ and $A \neq \emptyset$.

Proof: Suppose P_s is a series-parallel sandwich poset for the instance (P_t, D), from Lemma 2.1, A is not empty. By definition of A, there is a series composition between A and $V \setminus A$ in D, and thus in P_s as it is a series-parallel poset, then $P_s|_{V \setminus A}$ is a series-parallel poset. Finally B vertices are isolated in $P_s|_{V \setminus A}$. So $P_s|_{V \setminus A}$ is obtained from a parallel composition of $P_s|_{\tilde{V}}$ and S. It follows that $P_s|_{\tilde{V}}$ is still a series-parallel poset. Since no arc of $P_t|_{\tilde{V}}$ have been removed, $P_s|_{\tilde{V}}$ is a sandwich for the instance $(P_t, D)|_{\tilde{V}}$.

For the converse we assume there is $P_{\tilde{s}} = (\tilde{V}, E_{\tilde{s}})$ series-parallel sandwich poset for the instance $(P_t, D)|_{\tilde{V}}$ and $A \neq \emptyset$. As $V \setminus A \subseteq Succ^D(A)$, A can be composed in series with $V \setminus A$ in D. Let E_A be the set of series arcs between A and $V \setminus A$. Let us consider the arc set $E_s = E_{\tilde{s}} \cup E_A$ on vertex set V. Since there is no arc between B and $\tilde{V}$ in E_s, $P_s = (V, E_s)$ is obtained by the series-parallel decomposition : $(B + \tilde{V}) * A$. Indeed since $P_{\tilde{s}}$ is a series-parallel poset, also P_s does. Moreover P_s is a sandwich poset for the instance (P_t, D), as the definition of B implies that $E_t \subseteq E_s$ (i.e. no arc of E_t lies between B and $V \setminus A$). $\square$

Theorem 2.1 *The series-parallel directed sandwich problem is polynomial.*

Proof: The validity of the algorithm follows from Lemmas 2.1 and 2.2. Each iteration computes sets S_{P_t}, A, B in $\mathcal{O}(|V|^2)$ roughly, this step is repeated at most $|V|$ times if one vertex is eliminated each iteration, then the algorithm runs in time at most $\mathcal{O}(|V|^3)$. $\square$

Corollary 2.2 *The series-parallel poset sandwich problem is polynomial.*

3 Interval posets

This section deals with interval posets, posets that can be represented by assigning a real interval $I_v = [a_v, b_v]$ to each element v in P, such that $v \leq w$ if and only if $b_v \leq a_w$.

Problem 5: Interval directed sandwich problem

Instance:	A poset $P_t = (V, E_t)$ and a digraph $D = (V, E)$ such that $E_t \subseteq E$
Question:	Does there exist an interval sandwich poset P_s for the pair (P_t, D)?

Unlike the case where graphs of the instance are undirected (the interval graph sandwich problem has been proved to be NP-complete [7]), we prove the interval directed sandwich problem is polynomial. This problem therefore shows a complexity jump between oriented and non-oriented version of a sandwich problem. In that particular case, it means that fixing the transitive orientation of the interval graph widely simplify the problem (as there are many possible transitive orientations).

There exists a linear time recognition algorithm for interval poset [1] based on the following characterizations due to Fishburn [2].

Theorem 3.1 *[2] A poset P is an interval poset iff the set of successors $\{Succ^P(v) = \{u \in V, v \leq u\}\}_{v \in V}$ is linearly ordered by inclusion.*

Lemma 3.1 is very similar to Lemma 2.1. It is the basis of our algorithm that can be seen as a generalization of [1]'s recognition algorithm.

Lemma 3.1 *If there is an interval sandwich poset P_s for the instance (P_t, D), then there exists a set $S \subseteq S_{P_t}$ such that $V \setminus S_{P_t} \subseteq Succ^D(S)$.*

Proof: Let P_s be an interval sandwich poset for the instance (P_t, D). We can assume P_s is connected. From Theorem 3.1 P_s has its sets of successors linearly ordered by inclusion, then there exists a set S of P_s sources such that every non-source vertex of P_s is a successor of any vertex $x \in S$: $Succ^{P_s}(S) = V \setminus S_{P_s}$. Since any source of P_s is a source of P_t and $Succ^{P_s}(S) \subseteq Succ^{D}(S)$, it follows that $V \setminus S_{P_t} \subseteq Succ^{D}(S)$. □

Let $A' = \{x \in V / Succ^{D}(x) \supseteq V \setminus S_{P_t}\}$ and $B' = \{x \in V / N_{P_t|_{V \setminus A'}}(x) = \emptyset\}$. We denote by $\tilde{V} = V \setminus (A' \cup B')$. An example of sets A' and B' in a directed sandwich instance is given in Fig. 4.

Lemma 3.2 *The following statements are equivalent:*

(i) There is an interval sandwich poset for the instance (P_t, D).

(ii) There is an interval sandwich poset for the instance $(P_t, D)|_{\tilde{V}}$ and $A' \neq \emptyset$.

Proof: First assume P_s is an interval sandwich poset for the instance (P_t, D), from lemma 3.1, A' is not empty. Since being an interval poset is an hereditary property, the poset $P_s|_{\tilde{V}}$ is an interval poset. Moreover any arc of $P_t|_{\tilde{V}}$ is an arc of $P_s|_{\tilde{V}}$. It follows that $P_s|_{\tilde{V}}$ is an interval sandwich poset for $(P_t, D)|_{\tilde{V}}$.

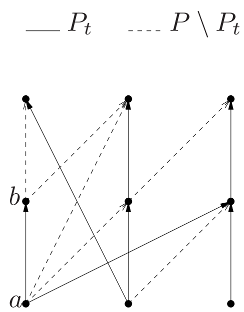


Figure 4: In this instance $b \in B'$ and $a \in A'$

Conversely assume there is $\tilde{P_s}$ interval sandwich poset for the instance $(P_t, D)|_{\tilde{V}}$ and $A' \neq \emptyset$. Then the vertices of $\tilde{V}$ are linearly ordered by the inclusion of their successors (Theorem 3.1). Adding the isolated vertices of B' preserve the existence of such linear ordering. An interval poset P_s on V is obtained if we connect the vertices of A' towards $\tilde{V} \cup B'$ (the A' vertices appear first in the inclusion order). P_s is a sandwich for the instance (P_t, D): first any arc xy such that $x \in A'$ and $y \in \tilde{V} \cup B'$ is an arc of the digraph D, finally there are no arc in P_t from vertices of $\tilde{V}$ towards vertices of B'. □

We can now describe an algorithm for the interval directed sandwich problem very similar to the series-parallel directed sandwich procedure: Look for a set A' among P_t sources satisfying $Succ^{D}(A') \supseteq V \setminus S_{P_t}$, if none exits then by lemma 3.1 there is no interval poset sandwich. Else remove from V the sets A' and B', the set of isolated vertices in $P_t|_{V \setminus A'}$, from lemma 3.1 there is an interval poset sandwich for the instance if and only if there is one for this reduced instance.

Theorem 3.2 *The interval directed sandwich problem is polynomial.*

Proof: Clearly, it is the same proof as Theorem 2.1 one's. □

Corollary 3.3 *The interval poset sandwich problem is polynomial.*

4 2-dimensional posets

Let L and P be respectively a total order and a poset on the same vertex set. If $x <_P y$ implies $x <_L y$, then L is a linear extension of P. The dimension of a poset P is the minimum number k of linear extensions such that $x <_P y$ iff $x <_{L_i} y$ for any i, $1 \leqslant i \leqslant k$. The comparability graphs of 2-dimensional posets are the permutation graphs. The permutation graph sandwich problem is knonw to be NP-complete [7]. Unlike the interval case where the directed problem is polynomial while the non-oriented problem is NP-complete, this section shows that for 2-dimensional poset sandwich problem is also NP-complete. In that case fixing the orientation of the edges does not help.

Problem 6: 2-DIMENSIONAL POSET SANDWICH PROBLEM

Instance: $P_t = (V, E_t)$ and $P = (V, E)$ two posets on the same ground vertex set such that $E_t \subseteq E$

Question: Does there exist a sandwich poset $P_s = (V, E_s)$ of dimension 2

Theorem 4.1 *The 2-dimensional poset sandwich problem is NP-complete.*

Proof: We give a reduction from BETWEENNESS problem as was done in [7] for permutation sandwich graphs. Since we are dealing with posets we need some additional arguments.

Problem 7: BETWEENNESS

Instance: A ground set S and a set $\mathcal{T} = \{T_1, \dots T_k\}$ of triples of S

Question: Does there exist a linear ordering λ such that for any triple $T_i = (a_i, b_i, c_i)$, either $a_i <_\lambda b_i <_\lambda c_i$ or $c_i <_\lambda b_i <_\lambda a_i$

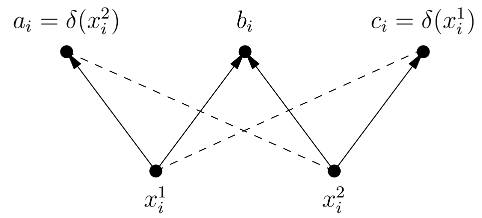


Figure 5: The gadget associated to a triple $T_i = (a_i, b_i, c_i)$. We denote $a_i = \delta(x_i^2)$ and $c_i = \delta(x_i^1)$

To any instance of BETWEENNESS we associate a pair of posets $P_t = (V, E_t)$ and $P = (V, E)$ with $E_t \subseteq E$ as follows:

$$V = S \cup X \text{ where } X = \{x_i^1 | 1 \leqslant i \leqslant k\} \cup \{x_i^2 | 1 \leqslant i \leqslant k\}$$
$$E_t = \bigcup_{1 \leqslant i \leqslant k} \{x_i^1 a_i, x_i^1 b_i, x_i^2 b_i, x_i^2 c_i\} \text{ and } E = X \times S \setminus \{uv | v = \delta(u)\}$$

Clearly P_t and P are polynomial time constructible.

First suppose there is a 2-dimensional sandwich poset P_s for (P_t, P). We note $u < v$ if uv is an arc of P_s, and $u||v$ if u and v are incomparable.

Let $L_s = (L_1, L_2)$ be a realizer of P_s. Then P_s is a suborder of the planar lattice $L = L_1 \times L_2$. We can define an ordering λ on it as follows:

$$u <_\lambda v \text{ iff } L_1(u) > L_1(v) \text{ and } L_2(u) < L_2(v)$$

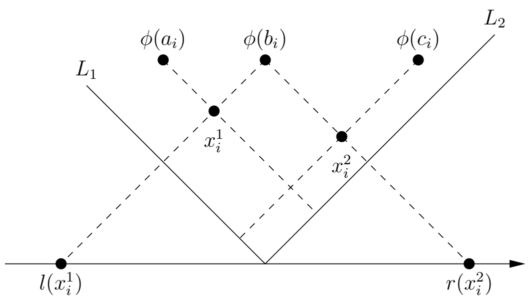
Since S is an antichain in P_s, the restriction of λ to S is a linear ordering. Let us consider a triple $T_i = (a_i, b_i, c_i)$. Without loss of generality we can assume that $a_i <_\lambda c_i$. Since $x_i^1 < a$ and $x_i^1 || c_i$, then $x_i^1 <_\lambda c_i$. Since $x_i^1 < b_i$ and $b_i || c_i$, then $b_i <_\lambda c_i$. Similarly, considering x_i^2 we can prove that $a_i <_\lambda b_i$. Therefore b_i is between a_i and c_i. Thus a solution to the 2-dimensional poset sandwich problem (P_t, P) implies a solution to the BETWEENNESS problem on S with triples $\mathcal{T}$.

For the converse suppose there exists a linear ordering λ on S that solves BETWEENNESS problem for the triples $\mathcal{T}$. By reversing some triples, we can assume that for an i, $a_i <_\lambda b_i <_\lambda c_i$.

Let us define a 2-dimensional poset P_s that is a sandwich for the pair (P_t, P). We embed S and X into $\{1\ldots,n\}^2$ as follows:

$$\phi \ : \ \begin{cases} a \to (a, n+1-a) \\ x_i^1 \to inf(\phi(a_i), \phi(b_i)) \\ x_i^2 \to inf(\phi(b_i), \phi(c_i)) \end{cases}$$

But in that construction four elements of X may be comparable in the lattice L. To X be an antichain, we move each element x of X, to $l(x)$ if $x < \lambda\delta(x)$, to $r(x)$ otherwise (see figure below).



Since P_s can be embedded in the planar lattice it is of dimension 2. It is easy to check that P_s is a 2-dimensional sandwich poset for the instance (P_t, P) previously defined.

$\square$

Conclusion

In the original paper on sandwich graphs [7], one directed sandwich problem was studied: the *directed Eulerian sandwich problem.* A directed graph is Eulerian iff the in-degree of every vertex equals its out-degree, it does not require transitivity, in this case the directed problem, as the undirected one are polynomial. The difficulty of poset sandwich problems comes from the NP-completeness of the comparability graph sandwich problem. Series-parallel and interval posets are sub-classes of 2-dimensional posets, despite a polynomial decision algorithm for the sandwich problems of these sub-classes, deciding if a 2-dimensional sandwich poset between 2 posets exists is NP-complete.

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