Dynamics of the Picking transformation on integer partitions

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April 2003

Research Report N° 2003-26

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Abstract

This paper studies a conservative transformation defined on families of finite sets. It consists in removing one element from each set and adding a new set composed of the removed elements. This transformation is conservative in the sense that the union of all sets of the family always remains the same.
We study the dynamical process obtained when iterating this transformation on a family of sets and we focus on the evolution of the cardinalities of the sets of the family. This point of view allows to consider the transformation as an application defined on the set of all partitions of a fixed integer (which is the total number of elements in the sets).
We show that iterating this particular transformation always leads to a heterogeneous distribution of the cardinalities, where almost all integers within an interval are represented.
We also tackle some issues concerning the structure of the transition graph which sums up the whole dynamics of this process for all partitions of a fixed integer.

Keywords: discrete dynamical system, integer partitions.

Résumé

Ce papier étudie une transformation sur les familles d'ensembles finis qui consiste à enlever un élément de chaque ensemble et ajouter un nouvel ensemble composé des éléments qui viennent d'être retirés. Cette transformation est conservative au sens où l'union de tous les ensembles de la famille reste toujours la même.
Nous étudions le processus dynamique obtenu en itérant cette transformation sur une famille d'ensembles et nous nous intéressons à l'évolution du cardinal de chacun des ensembles de la famille. Avec ce point de vue, la transformation devient une application definie sur l'ensemble des partitions d'un entier fixé (qui est le nombre total d'éléments dans la famille initiale).
Nous montrons qu'itérer cette transformation converge toujours vers une distribution hétérogène des cardinaux, où persque tous les entiers d'un intervalle sont représentés.
Nous abordons aussi d'autres questions concernant la structure du graphe des transitions qui représente la dynamique de ce processus pour toutes les partitions d'un entier fixé.

Mots-clés: systèmes dynamiques discrets, partitions des entiers.

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Abstract

This paper studies a conservative transformation defined on families of finite sets. It consists in removing one element from each set and adding a new set composed of the removed elements. This transformation is conservative in the sense that the union of all sets of the family always remains the same.

We study the dynamical process obtained when iterating this transformation on a family of sets and we focus on the evolution of the cardinalities of the sets of the family. This point of view allows to consider the transformation as an application defined on the set of all partitions of a fixed integer (which is the total number of elements in the sets).

We show that iterating this particular transformation always leads to a heterogeneous distribution of the cardinalities, where almost all integers within an interval are represented.

We also tackle some issues concerning the structure of the transition graph which sums up the whole dynamics of this process for all partitions of a fixed integer.

1 Introduction

The aim of this work is to study the iteration of a transformation defined for families of sets. Let P be a family of disjoint and nonempty sets S_1,...,S_k, then pick from each set an element and remove it, suppress the sets that become empty and add one new set composed of all the picked elements. It leads to a new family of sets, and our concern is the way the cardinalities of the sets evolve for such a transformation. Figure 1 shows an example of this transformation iterated several times. At each step, the new set is represented with a dotted border. The right part of the figure shows the evolution of the cardinalities of the sets.

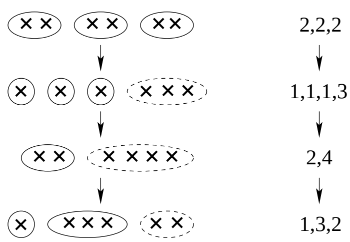

Figure 1: Three iterations of the transformation

First note that this transformation preserves the total number of elements in the family, which will be denoted by n. It is also clear that if we focus on the cardinalities of the sets in the family, the choice of the picked elements does not matter. Thus each family of sets P may be seen as a *partition* $p_1, \dots, p_k$ of the integer n, where p_i is the cardinality of S_i. A *partition* of n is a set of integers $p_1, \dots, p_k$ satisfying $\forall i$, $1 \leq i \leq k$, $p_i \geq 1$ and $\sum_{i=1}^{k} p_i = n$. Then the image of P by the transformation is a partition of n, uniquely defined, which will be denoted by $f(P)$.

The question is now to understand for a fixed $n \geq 1$ the behaviour of this application f on $\mathcal{P}(n)$ the set of all partitions of n, especially when this application is iterated. The ith iteration of f on P, i.e. $(\underbrace{f \circ \ldots \circ f}_{i \text{ times}})(P)$, will be denoted by $f^{(i)}(P)$.

The directed graph of Figure 2 illustrates the dynamics associated to f when $n = 10$. For commodity, the integers composing a partition $p_1, p_2, ..., p_k$ are given in decreasing order, i.e. $p_1 \geq p_2 \geq \ldots \geq p_k$. This graph is called the *graph of transitions*. The vertices are the partitions of n and a directed edge between two partitions corresponds to the application of f. These directed edges are also called *transition edges*. On this example, we see that starting from any partition, by iterating f, we always reach a fixed point which is the partition $4, 3, 2, 1$.

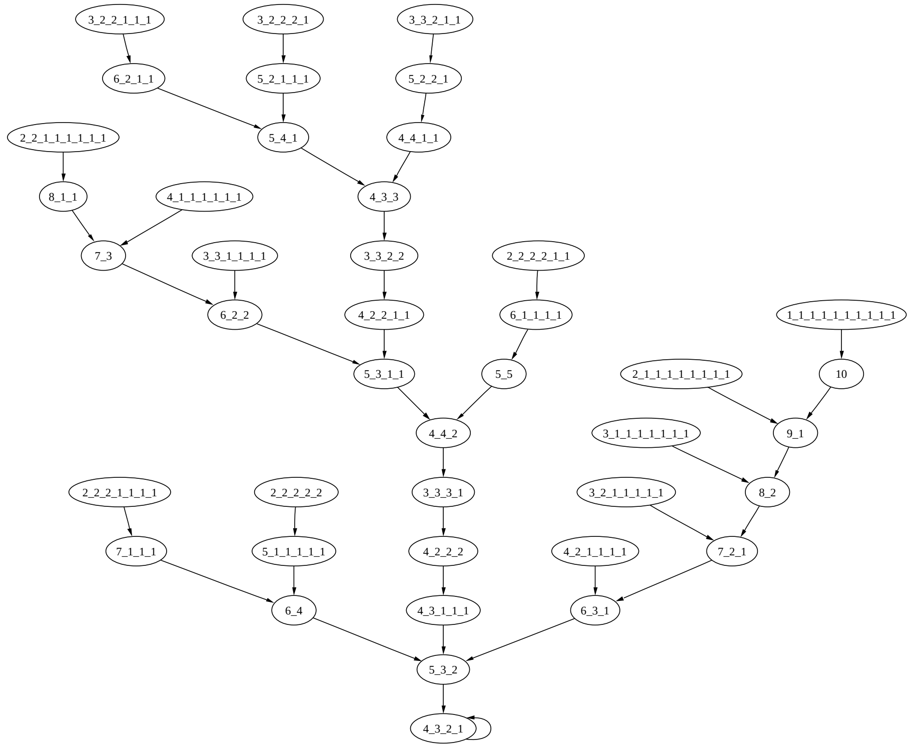

Figure 2: Dynamics induced by f on $\mathcal{P}(10)$

The set of all partitions of an integer is associated to a large number of discrete dynamical systems: see for instance sandpile models [3], chip firing games [2], tiling models [5]. Note that these systems often have an underlying topology (falling directions of sand grains, firing directions of chips ...).

On the contrary, in the description of the application f, there is no underlying topology. For instance the order you follow to enumerate the integers of a partition P does not affect the definition of $f(P)$ (nevertheless we will see later that this order has some importance in the study of the dynamics).

Since the set of all partitions of an integer n is finite, the dynamics associated to the application f has a well-defined structure. Given a partition of n, iterating the application of f eventually leads to a finite circuit (possibly reduced to one single partition) that the sequence of iterated images will follow indefinitely. Asymptotically the sequence of iterated images has a periodic behaviour, and the graph of all transitions between partitions of n has a forest-like structure where paths eventually leads to finite elementary circuits.

The general structure of the dynamics is thus obvious. The main problem is now to describe the precise features of this dynamics. Here are some questions aiming at this:

- Characterize the compositions, sizes and distribution of the circuits. What are the conditions of existence of fixed points ? What about unicity ? (study of the *asymptotic behaviour*)

- Quantify the convergence speed to the circuits. Study the complexity of predicting the asymptotic behaviour: given a partition P, which circuit the iterated sequence $(f^{(i)}(P))_{i \geq 0}$ will reach ? (study of the *trajectories*)

This article presents some answers to these questions and some conjectures. The main result is the complete characterization of the circuits of the transition graph for any $n \geq 1$. It is presented in Section 2 where the existence of fixed points is discussed in Subsection 2.1 and the main theorem concerning the convergence to identified circuits is proved in Subsection 2.2. Some further quantitative features of circuits are exposed in Subsection 2.3. In Section 3, we deal with the structure of paths in the transition graph. In particular, we present a conjecture concerning the convergence speed, i.e. the number of iterations, to reach a circuit.

The drawings of directed graphs in the body of the paper have been generated with the open source graph drawing software *Graphviz* from AT&T [1].

2 The structure of circuits

2.1 Fixed points

First of all, studying the existence of fixed points for f, namely partitions P such that $f(P) = P$, gives some indications about the dynamics. It is a first step towards a classification of the possible asymptotic behaviours.

Proposition 1 *All fixed points for f are partitions of the form $k, k-1, \ldots, 2, 1$, where k is an integer ≥ 1.*

Proof. First it can be checked easily that any partition of the form $k, k-1, \ldots, 2, 1$ is a fixed point for f.

Conversely, let $p_1, p_2, \ldots, p_k$ be a fixed point for f, $k \geq 1$. This partition P is composed of k integers (corresponding to k sets), thus $f(P)$ contains the integer k (cardinality of the new set composed of the picked elements). Since $f(P) = P$, P contains k. Moreover if P contains an integer $i > 1$, then $f(P) = P$ contains the integer $i - 1$ (element removed from set(s) of cardinality i). It implies that P contains the sequence of integers $k, k-1, \ldots, 2, 1$. But P is composed of exactly k integers, which means that $P = k, k-1, \ldots, 2, 1$. $\square$

Corollary 1 *Given $n \geq 1$, the application f admits a fixed point on $\mathcal{P}(n)$ if and only if there exists $k \geq 1$ such that $n = k(k+1)/2$. In that case, there exists a unique fixed point, which is the partition $k, k-1, \ldots, 2, 1$.*

However if $n = k(k+1)/2$, this corollary does not imply that for all partition $P \in \mathcal{P}(n)$ the sequence $f^{(i)}(P)$ will converge to the fixed point $k, k-1, \ldots, 1$. There might exist some other circuits in the transition graph. In fact, a more careful analysis of the dynamics will show that if $n = k(k+1)/2$, then for all $P \in \mathcal{P}(n)$, the sequence $(f^{(i)}(P))_{i \geq 0}$ actually always converges to the fixed point $k, k-1, \ldots, 1$, as it can be seen on Figure 2. The arguments are provided in the next subsection.

2.2 Convergence to circuits

The previous subsection has presented some particular circuits (loops) in case n has a special value. In fact, in the general case of any arbitrary n, we can fully describe the circuits of the transition graph on $\mathcal{P}(n)$, thanks to a particular representation of partitions.

Ferrer's diagrams

Any partition of an integer may be represented by its *Ferrer's diagram.* This diagram is an stair-shaped stacking of squares. If the partition P is $p_1, p_2, \ldots, p_k$, where the p_i are given *in decreasing order*, then its Ferrer's diagram is obtain by placing side by side, from left to right, a column with p_1 squares, a column with p_2 squares, ..., and a column with p_k squares. Figure 3 shows the Ferrer's diagram of the partition $P = 4, 3, 1, 1, 1$. There is a clear one-to-one correspondence between Ferrer's diagrams with n squares and partitions of n. From now, to describe a partition, we will indifferently use the sequence of integers $p_1, p_2, \ldots, p_k$, given *in decreasing order* or its Ferrer's diagram.

Reinterpretation of the application f

Considering this representation of partitions, the study of f leads us to distinguish two cases of transformation.

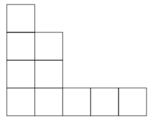

Figure 3: Ferrer's diagram of $P = 4, 3, 1, 1, 1$

Let $P = p_1, p_2, \ldots, p_k$ be a partition where $k \geq 1$ and the p_i are given in decreasing order, $p_1 \geq p_2 \geq \ldots \geq p_k$.

Transformation of Type I

If $k \geq p_1 - 1$, then $f(P) = k, p_1 - 1, p_2 - 1, \ldots, p_k - 1$ (we maintain the decreasing order) where some of the last $p_j - 1$ may be equal to zero.

The effect of f on the Ferrer's diagram is a move where the first line becomes the first column, as shown on Figure 4.

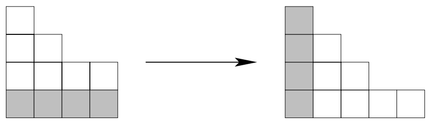

Figure 4: Type I transformation on Ferrer's diagrams

Transformation of Type II

If $k < p_1 - 1$, then there exists a index j, $1 \geq j < k$, such that $p_j - 1 > k \geq p_{j+1} - 1$ or we have $p_k - 1 > k$. It implies that $f(P) = p_1 - 1, p_2 - 1, \ldots, p_j - 1, k, p_{j+1}, \ldots, p_k - 1$ where some of the last $p_j - 1$ may be equal to zero (or $f(P) = p_1 - 1, p_2 - 1, \ldots, p_k - 1, k$ if $p_k - 1 > k$).

The effect of f on the Ferrer's diagram is a move where the first line is inserted as a column which is not the first one, as shown on Figure 5.

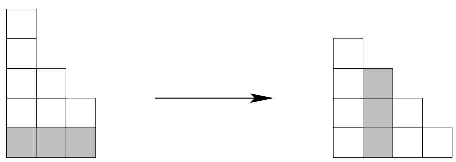

Figure 5: Type II transformation on Ferrer's diagrams

Observation of the effects of f on diagrams

Looking at Figure 4 and 5, it seems that the application f tends to concentrate the squares in the right-bottom corner of Ferrer's diagrams. These effects can also be noticed on examples such as the one on Figure 6.

After several iterations of f, the Ferrer's diagrams of partitions tend to a regular stair shape. We call *r-regular stair* the Ferrer's diagram of the partition $r, r-1, \ldots, 2, 1$ (such as the first diagram of Figure 7).

But this diagram is reachable only if $n = r(r+1)/2$. In the general case, we see on examples (such as Figure 6) that the sequence of iterated images always reaches partitions whose diagrams are as close as possible to regular stairs. For all r, s with $r \geq 1$ and $0 \leq s \leq r$, we call *(r, s)-quasi-regular stair* the Ferrer's diagram of a partition $P = p_1, p_2, \ldots, p_k$ of the integer $r(r+1)/2 + s$ such that for all $r \geq j \geq 1$, we have $j + 1 \geq p_{r+1-j} \geq j$. In other words, these quasi-regular stairs are regular stairs where some extra squares have been put but only on one extra layer.

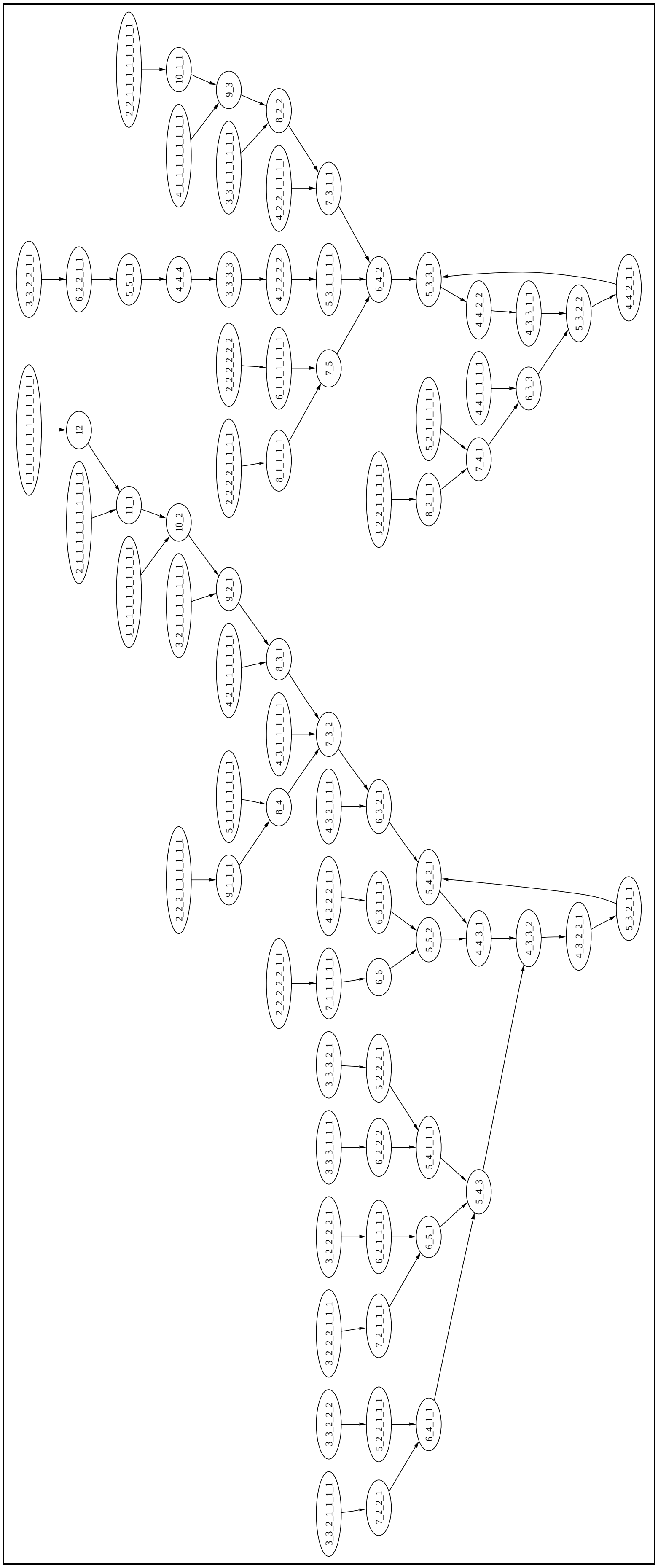

Figure 6: Transition graph on $\mathcal{P}(12)$

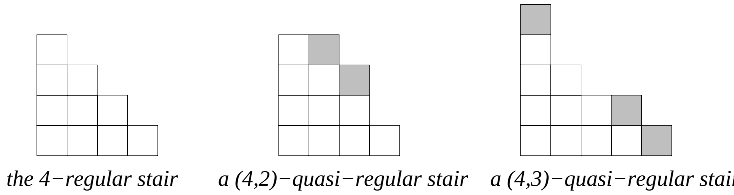

Figure 7: Regular and quasi-regular stairs

A careful look at Figure 6 shows that on this example all circuits are composed of quasi-regular stairs and inversely all quasi-regular stairs belong to a circuit. This is the result we will prove in the general case, using an appropriate weight function.

A weight function on the partitions of integers

Given a partition P, to each square of the Ferrer's diagram we associate a positive weight which is the sum of its distance to the bottom border (i.e. the line index) plus its distance to the left border (i.e. the column index) minus 1. The *weight* $\pi(P)$ of the partition P is the sum of the weights of all the squares of its Ferrer's diagram.

The following example on Figure 8 shows the weights of the squares of the diagram of $P = 5, 4, 2, 2, 2, 1$. Summing these weights, we obtain $\pi(P) = 62$. Another presentation of the way we assign weights to squares is to draw *diagonal levels* whose indexes give the value to each square on them as shown on the right on Figure 8.

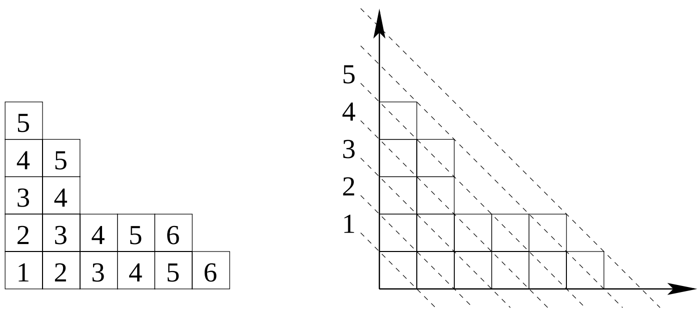

Figure 8: Weights for Ferrer's diagrams

The weight of a partition may also be given by a formula: let $P = p_1, p_2, \ldots, p_k$, the calculation of $\pi(P)$ gives

$$\pi(P) = s(p_1) + s(p_2) + p_2 + \ldots + s(p_k) + (k-1)p_k = \sum_{j=1}^{k} [s(p_j) + (j-1)p_j]$$

where $s(x) = \sum_{m=1}^{x} m = x(x+1)/2$.

For this weight function on partitions, we can characterize the partitions of minimum weight.

Proposition 2 *Let $n \geq 1$, there exists a unique decomposition $n = r(r+1)/2 + s$ such that $r \geq 1$ and $0 \leq s \leq r$. Then the partitions of $\mathcal{P}(n)$ of minimum weight are the (r,s)-quasi-regular stairs.*

Proof. The existence and unicity of r and s is clear.

Let P be a partition given by its Ferrer's diagram and m the maximum weight of a square of P (implying that there exist at least one square on level m). If P is not a quasi-regular stair, then level $m-1$ is not completely filled. Then we can take one square at level m and move it to an empty space on a level $\leq m-1$. The new partition has now a weight strictly lower than $\pi(P)$, which means that P is not a partition of minimum weight. Figure 9 illustrates such a move.

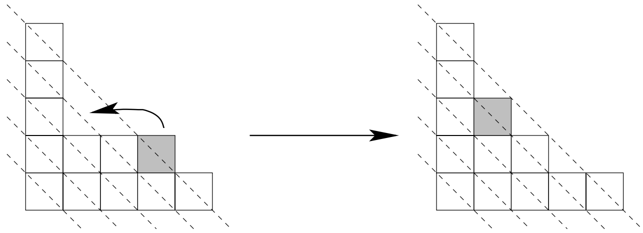

Figure 9: Decreasing the weight of a non quasi-regular stair

On the other hand, all (r, s)-quasi-regular stairs have the same weights (all levels from 1 to r are filled and the s remaining squares are on level $r + 1$). This proves that they correspond to the partitions of minimum weight. $\square$

The general dynamics when iterating f

Theorem 1 *Let $n \geq 1$ and its unique decomposition into $n = r(r+1)/2 + s$ with $r \geq 1$ and $0 \leq s \leq r$. The set of partitions composing the circuits is exactly the set of (r, s)-quasi-regular stairs.*

As a result, let P be a partition of n, then the sequence of partitions $(f^{(i)}(P))_{i \geq 0}$ obtained when iterating f always reaches a circuit only composed of (r, s)-quasi-regular stairs.

Proof. The proof has three steps:

1. First we prove that applying f decreases the weight of a partition, i.e. for all partition P, $\pi(f(P)) \leq \pi(P)$, noticing that we only have a strict decrease in case of Type II transformations.
2. It implies that on circuits, the transformation f is always of Type I. We prove then that such a sequence of Type I transformations only occurs when the partitions are all quasi-regular stairs.
3. Conversely, we prove that any quasi-regular stair belongs to a circuit.

First, the weight function of a partition decreases when applying f. We consider two cases: transformations of Type I and Type II. Let $P = p_1, p_2, \ldots, p_k$ a partition of n, where the p_i are given in decreasing order.

<u>Type I transformation:</u> $k \geq p_1 - 1$, thus $f(P) = k, p_1 - 1, p_2 - 1, \ldots, p_k - 1$.

Then calculating $\pi(f(P))$ gives:

$$\begin{aligned}
\pi(f(P)) &= s(k) + s(p_1 - 1) + (p_1 - 1) + \ldots + s(p_{k-1} - 1) + (k-1)(p_{k-1} - 1) + s(p_k - 1) + k(p_k - 1) \\
&= \pi(P) - (p_1 + \ldots + p_{k-1} + p_k) + (p_1 + \ldots + p_{k-1} + p_k) + s(k) - (1 + \ldots + (k-1) + k) \\
&= \pi(P)
\end{aligned}$$

The weight remains constant in case of Type I transformation.

<u>Type II transformation:</u> $k < p_1 - 1$, then there exists $1 \leq j < k$ such that $p_j - 1 > k \geq p_{j+1} - 1$, or otherwise $p_k - 1 > k$.

In the first case, $f(P) = p_1 - 1, p_2 - 1, \ldots, p_j, k, p_{j+1}, \ldots, p_k$. Then:

$$\begin{aligned}
\pi(f(P)) &= s(p_1 - 1) + \ldots + s(p_j - 1) + (j-1)(p_j - 1) + s(k) + j.k + s(p_{j+1} - 1) + (j+1)(p_{j+1} - 1) \\
&\quad + \ldots + s(p_k - 1) + k(p_k - 1) \\
&= \pi(P) - (p_1 + \ldots + p_k) + j.k + s(k) + (p_{j+1} + \ldots + p_k) - (1 + \ldots + (j-1) + (j+1) + \ldots + k) \\
&= \pi(P) - (p_1 + \ldots + p_k) + j(k+1) \\
&< \pi(P)
\end{aligned}$$

In case that $p_k - 1 > k$, the same kind of calculations also provides:

$$\pi(f(P)) = \pi(P) - (p_1 + \ldots + p_k) + k(k+1) < \pi(P)$$

The weight strictly decreases in case of Type II transformation.

We now prove that any circuit of the transition graph on $\mathcal{P}(n)$ is only composed of (r,s)-quasi-regular stairs. Let $P_1 \to P_2 \to \ldots \to P_m \to P_1$ be a circuit of the transition graph. Then each transition edge $\to$ necessarily corresponds to a Type I transformation, since Type II transformations strictly decrease the weight of partitions which would contradict the existence of the circuit. Now we prove that all these transition edges are Type I transformations implies that all the partitions of the circuit are quasi-regular stairs. This proof uses a new interpretation of Type I transformations: the Ferrer's diagram of $f(P)$ is also obtained from the diagram of P with a circular permutation of the squares on each diagonal level. Figure 10 illustrates these movements of the squares. Each square does not leave its level: if it was not on the bottom line, it decreases its line index by 1 and increases its column index by 1 (diagonal move), if it was on the bottom line, it is moved to the left column on the same diagonal level.

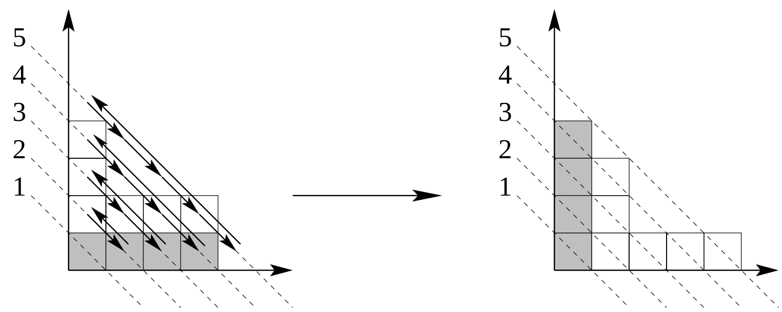

Figure 10: Type I transformations: circular permutation of each diagonal level.

Now suppose for instance that in the circuit $P_1 \to P_2 \to \ldots \to P_m \to P_1$, the partition P_1 is not a (r,s)-quasi-regular stair. Then level r is not completely filled, which means that there exists at least one hole (empty square) on level r. We denote by c_{hole} its column index. On the other hand, there is at least one square on level $r+1$. We denote by c_{square} its column index. On the circuit, we only have Type I transformation, meaning circular permutations on levels. On each level, it is clear that holes follow the same permutation as squares.

Level r admits r positions, thus after i iterations of f on P_1, the hole on level r has a column index equal to $(c_{hole}+i-1) \bmod r+1$. Level $r+1$ admits $r+1$ positions, thus after i iterations of f on P_1, the square on level $r+1$ has a column index equal to $(c_{square}+i-1) \bmod (r+1)+1$.

When iterating f, the hole and square respectively slide on level r and level $r+1$. Then there exists a number i of iterations such that at the same time the hole reaches column 1 and the square reaches column 2, since the following system where i is the unknown always has a solution as r and $r+1$ are prime together:

$$\begin{cases} [(c_{hole}+i-1) \bmod r]+1 & = & 1 \\ [(c_{square}+i-1) \bmod (r+1)]+1 & = & 2 \end{cases}$$

Since the hole is on level r and column 1 and the square is on level $(r+1)$ and column 2, they are exactly on the same line but the hole is on the left of the square. But it is impossible in a Ferrer's diagram.

Figure 11 shows how the contradiction arises when starting from a non-quasi-regular stair and supposing that from there all transformations will be of Type I (the hole and square are indicated by a cross). It can be seen that it would lead to a configuration which is not a Ferrer's diagram.

Consequently, all partitions on circuits are quasi-regular stairs.

Conversely, any quasi-regular stair belongs to a circuit. Let $P = p_1, p_2, \ldots, p_k$ be a (r,s)-quasi-regular stair, all diagonal levels indexed from 1 to r are completely filled and level $r+1$ contains s squares. Given the shape of the diagram of P, we have $r \geq p_1 - 1$ and thus the transformation f is of Type I. Consequently, $f(P)$ is obtained by a circular permutation of the squares on each level of P, and $f(P)$ keeps the shape of a quasi-regular stair. Levels 1 to r remain filled and iterating f just changes the positions of squares and holes of level $r+1$ according to a circular permutation. This circular permutation of the $r+1$ positions on

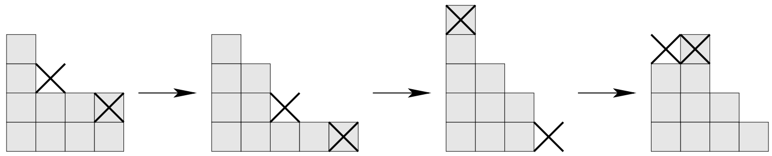

Figure 11: The sequence of iterations from a non-quasi-regular stair can not all be of Type I.

level $r+1$ admits $r+1$ as a period, this implies that $f^{(r+1)}(P) = P$ and P belongs to a circuit of partitions. Such a circuit is represented on Figure 12, where we illustrate how the transformation may be seen as a movement of the squares on level 4.

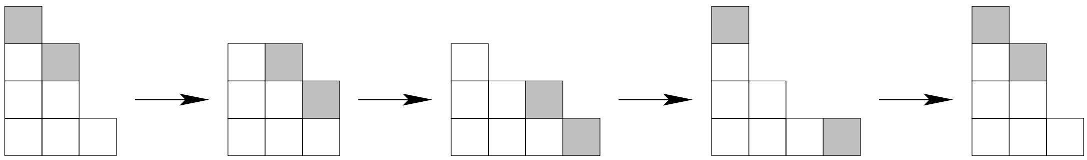

Figure 12: A circuit of $(3, 2)$-quasi-regular stairs on $\mathcal{P}(8)$.

The three steps of the proof have been completed. □

As a corollary, we prove the remark concerning the convergence of all partitions to the fixed point when it exists and illustrated on Figure 2. When $n = r(r+1)/2$, there exists a unique $(r, 0)$-quasi-regular stair which is $r, r-1, \ldots, 2, 1$. It implies Corollary 2.

Corollary 2 *If $n = r(r+1)/2$, $r \geq 1$, then for any partition P of n, the sequence $(f^{(i)}(P))_{i\geq 0}$ of partitions obtained when iterating f always converges to the fixed point $r, r-1, \ldots, 2, 1$.*

2.3 Quantitative description of the circuits

Thanks to Theorem 1, we have a qualitative description of the circuits on $\mathcal{P}(n)$ for any $n \geq 1$. From this result, we can quantify the number of circuits.

Proposition 3 *Let $n \geq 1$ and its unique decomposition into $n = r(r+1)/2 + s$ with $r \geq 1$ and $0 \leq s \leq r$. Then the number of circuits of the transition graph on $\mathcal{P}(n)$ is*

$$\frac{1}{r+1} \sum_{i|gcd(r+1,s)} \phi(i) \binom{(r+1)/i}{s/i}$$

where ϕ is the Euler function.

Proof. Any (r, s)-quasi-regular stair is completely characterized by the positions of the s squares on level $r+1$. For these partitions, the application of f corresponds to a circular permutation of the squares on this level.

Thus each (r, s)-quasi-regular stair can be seen as a 2-coloration of an undirected graph which a cycle on $r+1$ vertices indexed from 1 to $r+1$ round the cycle. Vertex indexes correspond to the column indexes of squares and the colors correspond to the presence or absence of square at the position. One color appears s times (squares) and the other one $r+1-s$ times (holes). A circuit of the transition graph corresponds to a class of 2-colorations which are identical up to circular permutations round the cycle.

To enumerate such classes, we use Pólya's Enumeration Theorem [4] in the case of cycles: in a $(r+1)$-cycle, the number of classes of 2-colorations identical up to circular permutations with s occurences of one color is given by the coefficient of x^s in the polynomial

$$Z(C_{r+1}) = \frac{1}{r+1}\sum_{i|r+1} \phi(i)(1+x^i)^{(r+1)/i}$$

where ϕ is the Euler function. We recall that this function is defined for any integer q by $\phi(q) = q(1 - 1/q_1)(1-1/q_2)\ldots(1-1/q_m)$ where the q_i are the distinct prime factors of q and $\phi(1) = 1$.

By developping this polynomial, we get the coefficient of x^s which is

$$\frac{1}{r+1}\sum_{i|gcd(r+1,s)} \phi(i)\binom{(r+1)/i}{s/i}$$

□

This formula can be checked on the example of Figure 6. For $n = 12$, we have $12 = 4 \times (4+1)/2 + 2$, thus $r = 4$, $s = 2$ and the number of circuits is equal to

$$\frac{1}{5}\sum_{i|1} \phi(i)\binom{5/i}{2/i} = \frac{1}{5}\binom{5}{2} = 2$$

This is what we observe on Figure 6.

3 The structure of paths

3.1 Predecessors of a partition

Given a partition P of the integer n, it is possible to enumerate in a simple way the *predecessors* of P in the transition graph, i.e. the partition(s) Q such that $f(Q) = P$. As a corollary, it provides a simple characterization of the partitions without any predecessor, which appear as *leaves* in the transition graph.

Proposition 4 *Let $P = p_1, p_2, \ldots, p_k$ be a partition of the integer n. Then the number of predecessors of P in the transition graph is the number of indexes j such that $p_j \geq k-1$*

Proof. If P is the image by f of a partition, it means that one of the p_j corresponds to the new added set. A predecessor of P is thus necessarily of the form

$$p_1+1, p_2+1, \ldots, p_{j-1}+1, p_{j+1}+1, \ldots, p_k+1, \underbrace{1, 1, \ldots, 1}_{m \text{ times}}$$

where $p_j = k-1+m$ (with $m \geq 0$). Such an integer m exists if and only if there exists an index j such that $p_j \geq k-1$. And consequently, the number of predecessors of P is exactly the number of indexes j such that $p_j \geq k-1$. □

For instance, a clear consequence of Proposition 4 is that if $n = r(r+1)/2$, then the fixed point $P = r, r-1, \ldots, 1$ has an unique predecessor (different from itself). It can be checked on Figure 2.

Corollary 3 *Let P be a partition with at least one predecessor and let $P_1, \ldots, P_h$ be its predecessors. Then there is exactly one P_{i_0} such that $P_{i_0} \to P$ is a transition of Type I and if $h > 1$ then for all other P_i, $i \neq i_0$, $P_i \to P$ is a transition of Type II and P_i has at least one predecessor.*

Proof. Let $P = p_1, p_2, \ldots, p_k$ admitting at least one predecessor. It is clear that it is possible to construct one predecessor P_{i_0} of P by considering that p_1 corresponds to the new added set and the transition from P_{i_0} to P is a Type I transformation. It is also clear that this is the only predecessor corresponding to a Type I transformation. Then any other predecessor $P_i = p'_1, \ldots, p'_l$ is constructed as in Proposition 4 by considering that some p_j, $p_j < p_1$, corresponds to the new added set. For P_i, we have $p'_1 = p_1 + 1$ and $l = p_j$. It implies that $p'_1 = p_1 + 1 > p_j + 1 = l + 1 > l - 1$ and P_i has at least one predecessor. □

3.2 Maximum convergence speed

Given a partition $P \in \mathcal{P}(n)$, the *convergence speed* $sp(P)$ is the least integer i such that $f^{(i)}(P)$ belongs to a circuit of the transition graph on $\mathcal{P}(n)$. This is the length of the path starting at P and ending at the first encountered partition which belongs to a circuit. Then for any $n \geq 1$, we can define the *maximum convergence speed* as $sp(n) = \max\{sp(P) | P \in \mathcal{P}(n)\}$.

On figure 13, the maximum convergence speed has been plotted for $1 \leq n \leq 70$, and some of the values are presented in Table 1.

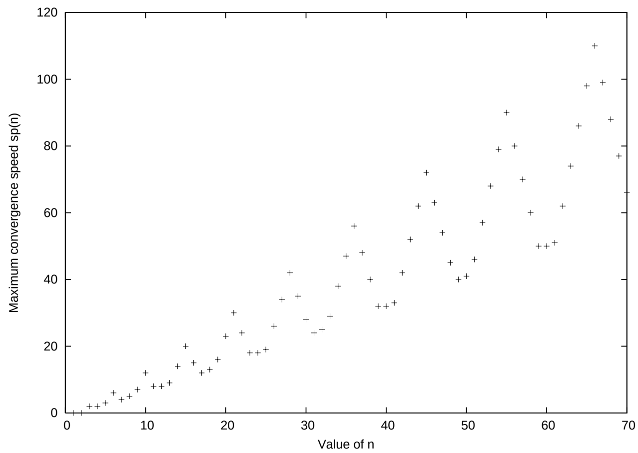

Figure 13: Maximum convergence speed $sp(n)$ for $1 \leq n \leq 70$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$sp(n)$	0	2	2	3	6	4	5	7	12	8	8	9	14	20	15	12	13	16	23	30

Table 1: Some values of $sp(n)$.

Several features may be observed from these computations. First, Figure 13 shows a strong regularity of the values. Then the peaks always correspond to values of n equal to $r(r+1)/2$ for some $r \geq 1$. Moreover in case $n = r(r+1)/2$, the computed maximum speed is always $r(r-1)$ (see for instance $n = 1, 3, 6, 10, 15$ or 21). Indeed it seems that there exist two numbers α and β such that for all $n \geq 1$, $\alpha n \leq sp(n) \leq \beta n$. We shall prove some partial results about these observed bounds and state a conjecture.

Proposition 5 *Let $n \geq 1$ where $n = r(r+1)/2 + s$ with $0 \leq s \leq r$. Then $sp(n) \geq n - r$.*

Proof. The partition $P = \underbrace{1, 1, \ldots, 1}_{n \text{ times}}$ achieves this lower bound and we can explicitly describe its sequence of iterated images. It is easy to check that $f(P) = n$ and then, by induction, that for all $1 \leq i \leq n - r$, which is uniquely decomposed into $i = m(m+1)/2 + l$, $0 \leq l \leq m$, $f^{(i+1)}(P) = p_0, p_1, ..., p_m$ where

$$p_j = \begin{cases} n - i & \text{if } j = 0, \\ i + 2 - j & \text{if } 1 \leq j \leq l, \\ i + 1 - j & \text{if } l + 1 \leq j \leq m. \end{cases}$$

It can be checked that while $i < n - r$, $f^{(i+1)}(P)$ is not a quasi-regular stair. If $s > 0$ then when $i = n - r - 1$, $f^{(i+1)}(P)$ becomes a (r, s)-quasi-regular stair and thus belongs to a circuit. If $s = 0$ then when $i = n - r - 1$ is not a regular stair yet, but when $i = n - r$, $f^{(i+1)}(P)$ becomes the r-regular stair.

Figure 14 illustrates this path for $n = 11$ where it can be seen that the dynamics is equivalent to pick squares from the left column and fill the diagonal levels one by one, from left to right.

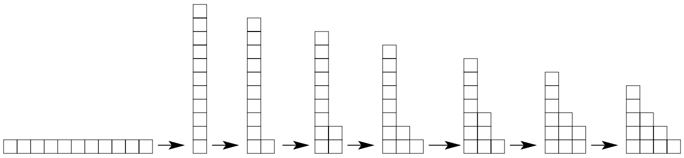

Figure 14: Path from $P = 1, 1, \ldots, 1$ to a partition in a circuit in $\mathcal{P}(11)$.

Note that all the transformations along this path have Type II. $\square$

Computations show that this lower bound is reached for many values of n (see for instance $n = 4, 5, 7, 8, 12, 13, 17$ or 18). The way these values are distributed suggests the following conjecture.

Conjecture 1 *Let $n = r(r+1)/2 + s$, $0 \leq s \leq r$, then $sp(n) = n - r$ if and only if $s = \lfloor \frac{r}{2} \rfloor$ or $s = \lfloor \frac{r}{2} \rfloor + 1$.*

Here is another lower bound for a special case.

Proposition 6 *If $n = r(r+1)/2$, $r \geq 1$, then $sp(n) \geq r(r-1)$.*

Proof. As in the former proposition, we can exhibit a partition P such that $sp(P) = r(r-1)$. To construct this partition, consider the r-regular stair, remove one square from the first column and add it to the $(r+1)$th column (which was previously empty). We obtain the partition $P = r-1, r-1, r-2, \ldots, 2, 1, 1$ where the rth diagonal level has exactly one hole in column 1 and the $(r+1)$th diagonal level has exactly one square in column $(r+1)$. Figure 15 shows this partition for $n = 15 = 5 \times (5+1)/2$.

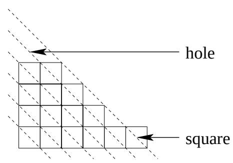

Figure 15: A partition P of $n = 15 = 5 \times (5+1)/2$ such that $sp(P) = 5 \times (5-1) = 20$.

It can be shown easily that while $f^{(i)}(P)$ is not the fixed point, the transformation $f^{(i-1)}(P) \to f^{(i)}(P)$ has Type I: by induction by comparing the length of the first column to the length of the first line of the Ferrer's diagram, or by noticing that the potential function defined in Subsection 2.2 always decreases but remains the same for Type I transformations and in our case strictly decreases only if it has reached the fixed point. Then we follow exactly the reasoning of Theorem 1 concerning column indexes of the hole on level r and the square on level $(r+1)$: the number of iterations needed to reach the fixed point is the smallest integer i such that

$$\begin{cases} [(1+i-1) \bmod r] + 1 & = & 1 \\ [(r+1+i-1) \bmod (r+1)] + 1 & = & 2 \end{cases}$$

This system is equivalent to

$$\begin{cases} i \bmod r & = & 0 \\ (r+i) \bmod (r+1) & = & 1 \end{cases}$$

By writing $i = r.m$, $m \in \mathbb{N}$, we obtain $(r-m) \bmod (r+1) = 1$, with smallest solution $m = r-1$. Thus the smallest integer i satisfying the system is $r(r-1)$. $\square$

Conjecture 2 *Let $n \geq 1$ where $n \leq r(r+1)/2$. Then $sp(n) \leq r(r-1)$ and with equality if and only if $n = r(r+1)/2$.*

This conjecture sets an upper bound. However the main question remains open. Is it possible to provide a closed formula or a fast algorithm which gives the maximum convergence speed $sp(n)$ for any $n \geq 1$?

Concerning the leaves P of the transition graph such that $sp(P) = sp(n)$, Corollary 3 enables to explain another feature of the graph.

Proposition 7 *Let $n \geq 1$ and P be a leaf of the transition graph on $\mathcal{P}(n)$, such that $sp(P) = sp(n)$. Then $f(P)$ has an unique predecessor.*

Proof. This is a direct application of Corollary 3 since if P has at least two predecessors, one of them also has a predecessor which is in contradiction with the fact that $sp(n)$ is the maximum convergence speed. $\square$

3.3 Complexity of prediction

More generally, given a partition P, what is the complexity of predicting its convergence speed $sp(P)$ and the circuit it will reach after iterating f ? There is a chance that it is not necessary to simulate the whole iteration process and that it is even possible to find some closed formulas thanks to a good decomposition of the initial partition P.

It seems that there should exist a more accurate potential function associated to the dynamics. The lack of the potential function described in Subsection 2.2 is that it does estimate the number of squares and holes on each level but it does not take into account the exact positions of holes and squares. These positions are important regarding the convergence speed as it was seen in Proposition 6.

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