Fully asynchronous behavior of double-quiescent elementary cellular automata.

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Abstract

In this paper we propose a probabilistic analysis of the fully asynchronous behavior (i.e., two cells are never simultaneously updated, as in a continuous time process) of elementary finite cellular automata (i.e., $\{0,1\}$ states, radius 1 and unidimensional) for which both states are quiescent (i.e., $(0,0,0) \mapsto 0$ and $(1,1,1) \mapsto 1$). It has been experimentally shown in previous works that introducing asynchronism in the global function of a cellular automata was perturbing its behavior, but as far as we know, only few theoretical work exists on the subject. The cellular automata we consider live on a ring of size n and asynchronism is introduced as follows: at each time step one cell is selected uniformly at random and the transition is made on this cell while the others stay in the same state. Among the sixty-four cellular automata belonging to the class we consider, we show that nine of them diverge almost surely on all non-trivial configurations while the fifty-five other converge almost surely to a random fixed point. We show that the exact convergence time of these fifty-five automata can only take the following values: either 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, or $\Theta(n2^n)$. Furthermore, the global behavior of each of these cellular automata is fully determined by reading its code.

Keywords: cellular automata, discrete dynamical systems, convergence, stochastic process

Résumé

Cet article présente une analyse du comportement des automates cellulaires élémentaires doublement quiescents (i.e., les deux états sont stables) suivant avec une dynamique totalement asynchrone. Nous montrons que parmi les soixante-quatre règles considérées, neuf d'entre elles divergent presque surement alors que les cinquante-cinq autres convergent presque sûrement. Nous montrons que l'espérance du temps de convergence de ces règles ne peut prendre que les valeurs suivantes : 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, ou $\Theta(n2^n)$. De plus, nous démontrons que le comportement global de l'automate cellulaire en régime totalement asynchrone est entièrement déterminé par la donnée de son code de transition.

Mots-clés: automates cellulaires, systèmes dynamiques discrets, convergence, processus stochastiques

1 Introduction

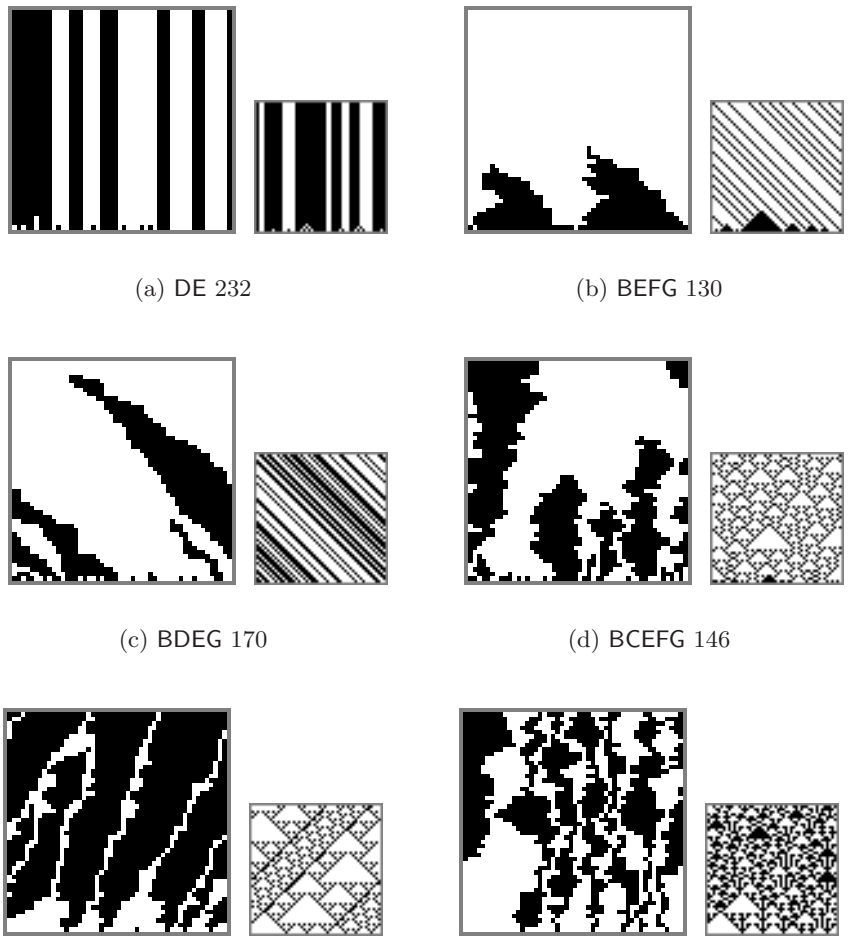
The aim of this article is to analyze theoretically the asynchronous behavior of unbounded finite cellular automata. During the last two decades, several empirical studies [1, 2, 3, 4, 5, 6] have shown that certain cellular automata behavior change drastically under asynchronous behavior. In particular, [4, 7] observe that finite size Game of Life space-time diagrams under synchronous and asynchronous updating differ qualitatively. For instance, fixed size Game of Life exhibits convergence to cycles of arbitrary length under synchronous updating, while appears to converge towards a random fixed point under asynchronous dynamics [4].

Cellular automata are widely used to model systems involving a huge number of interacting elements such as agents in economy, particles in physics, proteins in biology, etc. In most of these applications, in particular in many real system models, agents are not synchronous. Interestingly enough, in spite of this lack of synchronism, real living systems are very resilient over time. One might then expect the cellular automata used to model these systems to be robust to asynchronism and other kind of failure as well (such as misreading the state of the neighbors). Surprisingly enough, it turns out that the resilience to asynchronism widely varies from one automata to another (e.g., [4, 6]). In particular, the aspect of asynchronous space-time diagrams of cellular automata may differ radically from their synchronous ones.

As far as we know, the question of the importance of perfect synchrony on the behavior of a cellular automaton is not yet understood theoretically. To our knowledge, only Gács shows in [8] undecidability results on the invariance with respect to the update history. Studies have also been led in the more general context of probabilistic cellular automata regarding the question of the existence of stationary distribution on infinite configurations (see [9] for a state of the art).

In this paper, we quantify the convergence time and describe the space-time diagrams for a class of cellular automata under fully asynchronous updating, where two cells are not updated simultaneously. This asynchronous regime, also known as step-driven asynchronous dynamics [5], arises for instance in continuous time updating processes. We focus on double-quiescent elementary automata. We show that among these sixty-four automata, nine diverge on all non-trivial configurations (see Theorem 17), and the fifty-five other converge almost surely to a random fixed point (see Theorem 1). Furthermore, the convergence time of these fifty-five automata on (spatially) periodic configurations, can only take the following values: either 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, or $\Theta(n2^n)$, where n is the size of the configurations. One of the most striking results is that the fully asynchronous global behavior of double quiescent elementary automata is obtained simply by reading the code of their local transition rules (see Tab. 1), which is known to be a difficult problem in general. Moreover, the asynchronous behavior of all automata is in a certain sense *characterized* by this convergence time: all automata within the same convergence time present the same kind of space-time diagrams (see Tab. 1 and Fig. 1). Remark that the asynchronous behavior of some very simple automata like the shift (Wolfram rule code 170) actually contains intricate stochastic processes that are currently under investigation in mathematics and physics, such as annihilating random walks, studied for instance in [10]. Our results rely on coupling the automata with a proper random process. Indeed, we were able to couple all automaton of each class with the same random process.

Definitions and our main result are given in Section 2. In section 3, we present basic but useful properties of the automata we consider. Section 4 is a technical section that develops probabilistic tools used to analyze the automata. Section 5 finally analyzes in details the asynchronous behavior of each automaton. Section 6 concludes the paper with open questions.



(a) DE 232

(b) BEFG 130

(c) BDEG 170

(d) BCEFG 146

(e) BCEF 210

(f) BCFG 150

Figure 1: Examples of space-time diagrams under fully asynchronous and synchronous dynamics for each type of convergence, with $n = 50$. For each automaton, the larger left and the smaller right diagrams are respectively examples of asynchronous and synchronous dynamics. White and black pixels respectively stand for states 0 and 1. The k-th line from bottom is the configuration at time $t = 50\,k$ for the asynchronous dynamics, and at time $t = k$ for the synchronous one. Note that automata (a) and (c) are respectively the classic Majority and Shift rules. Each automata is described by two codes: a number, which is the classic Wolfram's number, and a sequence of letters, which will be introduced later in the paper.

2 Definitions, Notations and Main Results

In this paper, we consider two-state cellular automata on finite size configurations.

Definition 1 *An* Elementary Cellular Automata (ECA) *is given by its* transition function $\delta : \{\mathtt{0},\mathtt{1}\}^3 \to \{\mathtt{0},\mathtt{1}\}$. *We denote by* $Q = \{\mathtt{0},\mathtt{1}\}$ *the set of* states. *A state* q *is* quiescent *if* $\delta(q,q,q) = q$. *An ECA is* double-quiescent (DQECA) *if both states* $\mathtt{0}$ *and* $\mathtt{1}$ *are quiescent.*

A finite configuration *with periodic boundary conditions* $x \in Q^{\mathbb{Z}/n\mathbb{Z}}$ *is a word indexed by* $\mathbb{Z}/n\mathbb{Z}$ *with letters in* Q. *We denote by* $U = \mathbb{Z}/n\mathbb{Z}$ *the set of* cells. *For a given pattern* $w \in Q^{\mathbb{Z}/n\mathbb{Z}}$, *we denote by* $|x|_w = \#\{i \in \mathbb{Z}/n\mathbb{Z} : x_{i+1} \dots x_{i+|w|} = w\}$ *the number of occurrences of* w *in configuration* x.

We consider two kinds of dynamics for ECAs: the synchronous dynamics and the fully asynchronous dynamics. The synchronous dynamics is the classic dynamics of cellular automata, where the transition function is applied at each (discrete) time step on each cell simultaneously.

Definition 2 (Synchronous Dynamics) *The* synchronous dynamics $S_\delta : Q^U \to Q^U$ *of an ECA* δ, *associates to each configuration* x *the configuration* y, *such that for all* i *in* U, $y_i = \delta(x_{i-1}, x_i, x_{i+1})$.

The asynchronous regime studied here can be seen as the most extreme asynchronous regime as two cells are never updated simultaneously.

Definition 3 (Fully Asynchronous Dynamics) *The* fully asynchronous dynamics AS_δ *of an ECA* δ *associates to each configuration* x *a random configuration* y, *such that* $y_j = x_j$ *for* $j \neq i$, *and* $y_i = \delta(x_{i-1}, x_i, x_{i+1})$, *where* i *is uniformly chosen at random in* U. AS_δ *could equivalently be seen as a function with two arguments, the configuration* x *and the random index* $i \in U$. *For a given ECA* δ, *we denote by* x^t *the random variable for the configuration obtained by* t *applications of the asynchronous dynamics function* AS_δ *on configuration* x, *i.e.,* $x^t = (AS_\delta)^t(x)$.

Definition 4 (Fixed point) *We say that a configuration* x *is a* fixed point *for* δ *under fully asynchronous dynamics if* $AS_\delta(x) = x$ *whatever the choice of* i *(the cell to be updated) is.* $\mathfrak{F}_\delta$ *denotes the set of fixed points for* δ.

The set of fixed points of the asynchronous dynamics is clearly identical to $\{x : S_\delta(x) = x\}$ the set of fixed points of the synchronous dynamics. Note that every DQECA admits two *trivial fixed points*, $\mathtt{0}^n$ and $\mathtt{1}^n$.

Definition 5 (Worst Expected Convergence Time) *Given an ECA* δ *and a configuration* x, *we denote by* $T_\delta(x)$ *the random variable for the time to reach a fixed point from configuration* x *under fully asynchronous dynamics, i.e.,* $T_\delta(x) = \min\{t : x^t \in \mathfrak{F}_\delta\}$. *The* worst expected convergence time T_δ *of ECA* δ *is :*

$$T_\delta = \max_{x \in Q^U} \mathbb{E}[T_\delta(x)].$$

We can now state our main theorem.

Theorem 1 (Main result) *Under fully asynchronous dynamics, among the sixty-four DQECAs,*

- *fifty-five converge almost surely to a random fixed point on any initial configuration, and the worst expected convergence times of these fifty-five convergent DQECAs are* 0, $\Theta(n \ln n)$, $\Theta(n^2)$, $\Theta(n^3)$, *and* $\Theta(n2^n)$*;*

- *the nine others diverge almost surely on any initial configuration that is neither* $\mathtt{0}^n$*, nor* $\mathtt{1}^n$ *nor, when* n *is even,* $(\mathtt{01})^{n/2}$*.*

Furthermore, the exact behavior of the different DQECAs is the same within each class, and is obtained by simply reading its code as illustrated in Tab. 1.

Figure 1 gives examples of the asynchronous space-time diagrams of a representative of each class (but Identity). It is interesting to notice that except for the first diagram (Fig. 1(a)), the asynchronous space-time diagrams (the larger ones) considerably differ from the corresponding synchronous ones (the smaller ones).

3 Basic properties of DQECAs

The transition function δ of an ECA is given by the set of its eight transitions $\delta(\mathtt{000}), \delta(\mathtt{001}), \ldots, \delta(\mathtt{111})$, traditionally written $\frac{\mathtt{000}}{\delta(\mathtt{000})}, \ldots, \frac{\mathtt{111}}{\delta(\mathtt{111})}$. The following code describes each ECA by its differences to the Identity automaton. We use this notation rather than the classic Wolfram's one [11] since it is not immediate to infer the local behavior of the cellular automaton just by looking at its Wolfram code. In order to allow comparison with other work we still indicate the classic Wolfram number in Tab. 1.

Notation 1 We say that a transition is *active* if it changes the state of the cell where it is applied. Each ECA is fully determined by its active transitions. We label each active transition by a letter as follow:

A	B	C	D	E	F	G	H
000 1	001 1	100 1	101 1	010 0	011 0	110 0	111 0

We label each ECA by the set of its active transitions.

Note that with these notations, the DQECAs are exactly the ECAs having a label containing neither A nor H. By 0/1 and horizontal symmetries of configurations, we shall w.l.o.g. only consider the 24 DQECAs listed in Tab. 1 among the 64 DQECAs. For each of these 24 DQECAs, the number of the equivalent automata under symmetries is written within parentheses after their classic ECA code in the table.

From now on, we only consider the fully asynchronous dynamics (with uniform choice); this will be implicit in all the following propositions. Our results rely on the study of the evolution of the "regions" in the space-time diagram (i.e., of the intervals of consecutive 0s or 1s in configuration x^t). The key observation is that for DQECAs, under fully asynchronous dynamics, the number of regions is non-increasing since no new region can be created; furthermore, only regions of length one can disappear (see Fig. 1). We denote by $Z(x) = |x|_{\mathtt{01}}$ $(= |x|_{\mathtt{10}})$ the *number of alternations* from 0 to 1 in configuration x, which will be our counter for the number of regions.

Table 1: Behavior of DQECA under fully asynchronous dynamics. WECT stands for worst expected convergence time. See Section 2 for explanations.

Behavior	**ECA** (#)	**Rule**	**01**	**10**	**010**	**101**	**WECT**
Identity	204 (1)	∅	·	·	·	·	0
Coupon collector	200 (2)	E	·	·	+	·	$\Theta(n \ln n)$
	232 (1)	DE	·	·	+	+	
Monotonic	206 (4)	B	←	·	·	·	$\Theta(n^2)$
	222 (2)	BC	←	→	·	·	
	234 (4)	BDE	←	·	+	+	
	250 (2)	BCDE	←	→	+	+	
	202 (4)	BE	←	·	+	·	
	192 (4)	EF	→	·	+	·	
	218 (2)	BCE	←	→	+	·	
	128 (2)	EFG	→	←	+	·	
Biased Random Walk	242 (4)	BCDEF	⇜⇝⇝	→	+	+	
	130 (4)	BEFG	⇜⇝⇝	←	+	·	
Random Walk	226 (2)	BDEF	⇜⇝⇝	·	+	+	$\Theta(n^3)$
	170 (2)	BDEG	←	←	+	+	
	178 (1)	BCDEFG	⇜⇝⇝	⇜⇝⇝	+	+	
	194 (4)	BEF	⇜⇝⇝	·	+	·	
	138 (4)	BEG	←	←	+	·	
	146 (2)	BCEFG	⇜⇝⇝	⇜⇝⇝	+	·	
Biased Random Walk	210 (4)	BCEF	⇜⇝⇝	→	+	·	$\Theta(n2^n)$
Divergent	198 (2)	BF	⇜⇝⇝	·	·	·	Divergent
	142 (2)	BG	←	←	·	·	
	214 (4)	BCF	⇜⇝⇝	→	·	·	
	150 (1)	BCFG	⇜⇝⇝	⇜⇝⇝	·	·	

Fact 2 *For any DQECA, $Z(x^t)$ is a non-increasing function of time. Furthermore, $Z(x^{t+1}) < Z(x^t)$ if and only if x^{t+1} is obtained from x^t by applying a transition* D *or* E *at time t, and then $Z(x^{t+1}) = Z(x^t) - 1$.*

On the one hand, transitions D and E are thus responsible for decreasing the number of regions in the space-time diagram: D "erases" the 1-regions and E the 0-regions. On the other hand, transitions B and F act on patterns 01. Intuitively, transition B moves a pattern 01 to the left, and transition F moves it to the right. In particular, patterns 01 perform a kind of random walk for DQECA with both transitions B and F. Similarly, transitions C and G act on patterns 10. Transition C moves a pattern 10 to the right, and transition G moves it to the left. The arrows in Tab. 1 represent the different behavior of the patterns: $\leftarrow$ or $\rightarrow$, for left or right moves of the patterns 01 or 10; $\leftrightsquigarrow$, for random walks of these patterns.

The following lemma characterizes the fixed points of a given DQECA according to its code.

Fact 3 *If a DQECA δ admits a non-trivial fixed point x, then:*

- *if δ contains transition* B *or* C*, then all* 0*s in x are isolated;*
- *if δ contains transition* F *or* G*, then all* 1*s in x are isolated;*
- *if δ contains transition* D*, then none of the* 0*s in x is isolated;*
- *if δ contains transition* E*, then none of the* 1*s in x is isolated.*

The next section is a technical section that analyzes particular random walk-like processes that will be used as tools to obtain our bounds on the convergence time.

4 Probabilistic toolbox

Notation 2 *For a given random sequence $(X_t)_{t\in\mathbb{N}}$, we denote by $(\Delta X_t)_{t>0}$ the random sequence $\Delta X_t = X_t - X_{t-1}$.*

4.1 Quadratic DQECA toolbox

Consider $\epsilon > 0$, a non-negative integers m and m', and $(X_t)_{t\in\mathbb{N}}$ a sequence of random variables with values in $\{-m, \dots, m'\}$ given with a suitable filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$. In probability theory, $\mathcal{F}_t$ represents intuitively the σ-algebra (the "set") of the events that happened up to time t and is the formal tool to condition relatively to the past (see [12, Chap. 7]). In the sequel, $\mathcal{F}_t$ will either be the values of the previous random variables $X_0, \dots, X_t$, or in some cases, the set of past configurations $x^0, \dots, x^t$. The following lemma bounds the convergence time of a random variable that decreases by a constant on expectation.

Lemma 4 *Assume that if $X_t > 0$, then $\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] \leqslant -\epsilon$. Let $T = \min\{t : X_t \leqslant 0\}$ denote the random variable for the first time t where $X_t \leqslant 0$. Then, if $X_0 = x_0$,*

$$\mathbb{E}[T] \leqslant \frac{m + x_0}{\epsilon}.$$

Proof. First we prove that $\mathbb{E}[T] < \infty$ under these assumptions. For all $t < T$, we have

$$\begin{aligned}
-\epsilon &\geqslant \mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] \\
&= \mathbb{E}\left[\Delta X_{t+1}|\left(\Delta X_{t+1} < -\epsilon/2\right), \mathcal{F}_t\right] \cdot \Pr\{\Delta X_{t+1} < -\epsilon/2|\mathcal{F}_t\} \\
&\qquad + \mathbb{E}\left[\Delta X_{t+1}|\left(\Delta X_{t+1} \geqslant -\epsilon/2\right), \mathcal{F}_t\right] \cdot \Pr\{\Delta X_{t+1} \geqslant -\epsilon/2|\mathcal{F}_t\} \\
&\geqslant -(m+m')\Pr\{\Delta X_{t+1} < -\epsilon/2|\mathcal{F}_t\} - \epsilon/2,
\end{aligned}$$

since $|\Delta X_{t+1}| \leqslant m + m'$ for all t.

Thus, for all $t < T$,

$$\Pr\{\Delta X_{t+1} < -\epsilon/2|\mathcal{F}_t\} \geqslant \frac{\epsilon}{2(m+m')}.$$

This implies that from any time t and any starting value X_t, the process reaches a value below 0 after $2m'/\epsilon$ steps with a positive probability, independent of $\mathcal{F}_t$. More precisely,

$$\Pr\{X_{t+2m'/\epsilon} \leqslant 0|\mathcal{F}_t\} \geqslant \left(\frac{\epsilon}{2(m+m')}\right)^{2m'/\epsilon},$$

which implies that the expected time to reach a value below 0 satisfies

$$\mathbb{E}[T] \leqslant \frac{2m'}{\epsilon} + \left(\frac{2(m+m')}{\epsilon}\right)^{2m'/\epsilon}.$$

Then, let $Y_t = X_t + \epsilon\, t$. For all $t < T$,

$$\mathbb{E}[Y_{t+1}|\mathcal{F}_t] \leqslant X_t - \epsilon + \epsilon\,(t+1) = Y_t.$$

Since T is almost surely finite, with finite expectation, and since $|\Delta Y_{t+1}| \leqslant m + m' + \epsilon$, the Optional Stopping Theorem for the supermartingale (Y_t) (see [12]) gives:

$$\mathbb{E}[X_0] = \mathbb{E}[Y_0] \geqslant \mathbb{E}[Y_T] = \mathbb{E}[X_T] + \epsilon \cdot \mathbb{E}[T].$$

Thus, if $X_0 = x_0$, we have

$$\mathbb{E}[T] \leqslant \frac{x_0 - \mathbb{E}[X_T]}{\epsilon} \leqslant \frac{m + x_0}{\epsilon}.$$

□

4.2 Cubic DQECA toolbox

Let $\epsilon > 0$ and $(X_t)_{t\in\mathbb{N}}$ a sequence of random variables with values in $\{0, \ldots, m\}$, given with a suitable filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$.

Definition 6 *The following two types of process will be extensively used in the next section:*

- *We say that* $(X_t)_{t\in\mathbb{N}}$ *is* of type I *if for all* t*:*
  - $\mathbb{E}[X_{t+1}|\mathcal{F}_t] = X_t$ *(i.e.,* (X_t) *is a martingale), and*
  - *if* $0 < X_t < m$*, then* $\Pr\{\Delta X_{t+1} \geqslant 1|\mathcal{F}_t\} = \Pr\{\Delta X_{t+1} \leqslant -1|\mathcal{F}_t\} \geqslant \epsilon$.

- *We say that* $(X_t)_{t\in\mathbb{N}}$ *is* of type II *if for all* t*:*
  - *if* $X_t < m$*, then* $\mathbb{E}[X_{t+1}|\mathcal{F}_t] = X_t$ *(i.e.,* (X_t) *behaves as a martingale when* $X_t < m$*), and*
  - *if* $0 < X_t < m$*, then* $\Pr\{\Delta X_{t+1} \geqslant 1|\mathcal{F}_t\} = \Pr\{\Delta X_{t+1} \leqslant -1|\mathcal{F}_t\} \geqslant \epsilon$*, and*
  - *if* $X_t = m$*, then* $\Pr\{X_{t+1} \leqslant m-1|\mathcal{F}_t\} \geqslant \epsilon$ *(i.e.,* X_t *"bounces" on the value* m*).*

Note that when (X_t) is of type I, if for some t, $X_t \in \{0, m\}$, then $X_{t'} = X_t$ for all $t' \geqslant t$, because (X_t) is a martingale bounded between 0 and m. Thus, $\{0, m\}$ are the (only) fixed points of any type I sequence. When (X_t) is of type II, if for some t, $X_t = 0$, then $X_{t'} = X_t$ for all $t' \geqslant t$, because (X_t) is a martingale lower bounded by 0. Thus, 0 is the (only) fixed point of any type II sequence.

Definition 7 *The* convergence time *of a type I sequence* (X_t) *is defined as the random variable* $T = \min\{t : X_t \in \{0, m\}\}$*. The convergence time of a type II sequence* (X_t) *is similarly defined as the random variable* $T = \min\{t : X_t = 0\}$*.*

Lemma 5 *For both types of sequences,* T *is almost surely finite:*

$$\Pr\{T < \infty\} = 1.$$

Proof. The proof is similar to the beginning of Lemma 4; we just need to prove that $\mathbb{E}[T] < \infty$. For a type I sequence, for all t, we clearly have $\Pr\{X_{t+m} \in \{0, m\}|\mathcal{F}_t\} \geqslant \epsilon^m$, which implies that the expected time to reach $\{0, m\}$ satisfies $\mathbb{E}[T] \leqslant 1/\epsilon^m + m$. Replace $\{0, m\}$ by $\{0\}$ to obtain the same bound for type II sequences. □

The following lemmas bound the convergence time of these two types of random processes.

Lemma 6 *For any type I sequence* (X_t)*, if* $X_0 = x_0$*, the expectation of* T *satisfies:*

$$\mathbb{E}[T] \leqslant \frac{x_0(m - x_0)}{2\epsilon}.$$

Proof. The sequence (X_t) is a martingale with respect to $(\mathcal{F}_t)$. According to Lemma 5, the convergence time T is a stopping time with respect to $(\mathcal{F}_t)$ such that $\Pr\{T < \infty\} = 1$, and $|X_t|$ is bounded by m for all t. We can then apply the Optional Stopping Theorem (see [12]) which gives: $\mathbb{E}[X_T] = \mathbb{E}[X_0] = x_0$.

But, since $X_T \in \{0, m\}$, we have

$$x_0 = \mathbb{E}[X_T] = 0 \cdot \Pr\{X_T = 0\} + m \cdot \Pr\{X_T = m\}.$$

Thus, $\Pr\{X_T = m\} = x_0/m$.

Now let $Y_t = X_t^2 - 2\epsilon t$, the sequence (Y_t) is a submartingale with respect to $(\mathcal{F}_t)$ as shown below. Then :

$$\begin{aligned}\mathbb{E}[X_{t+1}^2 - 2\epsilon(t+1)|\mathcal{F}_t] &= X_t^2 + 2X_t\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] + \mathbb{E}[\Delta X_{t+1}^2|\mathcal{F}_t] - 2\epsilon - 2\epsilon t \\ &\geqslant X_t^2 - 2\epsilon t,\end{aligned}$$

since $\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] = 0$ and

$$\mathbb{E}[\Delta X_{t+1}^2|\mathcal{F}_t] \geqslant \Pr\{\Delta X_{t+1} \leqslant -1|\mathcal{F}_t\} + \Pr\{\Delta X_{t+1} \geqslant 1|\mathcal{F}_t\} \qquad \geqslant 2\epsilon.$$

Since $\mathbb{E}[T] < \infty$ and $|Y_{t+1} - Y_t| \leqslant m^2 + 2\epsilon$, we can apply the Optional Stopping Theorem to the submartingale $Y_t = X_t^2 - 2\epsilon t$ which leads to

$$\mathbb{E}[X_T^2 - 2\epsilon T] \geqslant \mathbb{E}[x_0^2] = x_0^2.$$

We conclude as follows :

$$\begin{aligned}\mathbb{E}[X_T^2 - 2\epsilon T] &= 0^2 \Pr\{X_T = 0\} + m^2 \Pr\{X_T = m\} - 2\epsilon \mathbb{E}[T] \\ &= m x_0 - x_0^2.\end{aligned}$$

□

Lemma 7 *For any type II sequence (X_t) with $X_0 = x_0$, the expectation of T satisfies:*

$$\mathbb{E}[T] \leqslant \frac{x_0(2m + 1 - x_0)}{2\epsilon}.$$

Proof. By definition of T, we have $\Pr\{X_T = 0\} = 1$. We now introduce the sequence $Y_t = X_t^2 - (2m+1)X_t - 2\epsilon t$ instead of $X_t^2 - 2\epsilon t$. We can easily check that this sequence is a submartingale by considering the two cases: $0 < X_t < m$ and $X_t = m$. Indeed:

$$\begin{aligned}&\mathbb{E}[X_{t+1}^2 - (2m+1)X_{t+1} - 2\epsilon(t+1)|\mathcal{F}_t] \\ &\qquad = X_t^2 - (2m+1)X_t - 2\epsilon t + 2X_t \mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] \\ &\qquad\qquad + \mathbb{E}[\Delta X_{t+1}^2|\mathcal{F}_t] - (2m+1)\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] - 2\epsilon.\end{aligned}$$

If $0 < X_t < m$, then $\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] = 0$ and $\mathbb{E}[\Delta X_{t+1}^2|\mathcal{F}_t] \geqslant 2\epsilon$.

If $X_t = m$, then $\mathbb{E}[\Delta X_{t+1}|\mathcal{F}_t] \leqslant -\epsilon$ and $\mathbb{E}[\Delta X_{t+1}^2|\mathcal{F}_t] \geqslant \epsilon$.

We conclude that in both cases,

$$\mathbb{E}[X_{t+1}^2 - (2m+1)X_{t+1} - 2\epsilon(t+1)|\mathcal{F}_t] \geqslant X_t^2 - (2m+1)X_t - 2\epsilon.$$

As in lemma 6, we can apply the Optional Stopping Theorem which gives

$$\mathbb{E}[X_T^2 - (2m+1)X_T - 2\epsilon T] \geqslant \mathbb{E}[X_0^2 - (2m+1)X_0] = x_0^2 - (2m+1)x_0.$$

Since $\mathbb{E}[X_T^2] = 0^2 \cdot \Pr\{X_T = 0\} = 0$ and $\mathbb{E}[X_T] = 0$, we get the result. □

5 Convergence

In this section, we evaluate the worst expected convergence time for each of the twenty-four representative automata in Tab. 1. Our results rely on studying the evolution of quantities computed on the random configurations (x^t), whose convergence implies the convergence of the automaton. The upper bounds on the convergence time of these quantities are obtained by coupling them with one of the integer random processes analyzed in the previous section. The lower bounds are obtained by analyzing the exact expected convergence time for a particular initial configuration (most of the time, a configuration with a single `0`-region and a single `1`-region). This involves building suitable variants measuring progress towards fixed points. One

of the main difficulties is to handle correctly the mergings of the regions, i.e., the applications of transitions D and E.

We introduce the following convenient functions that simplify the evaluation of the quantities that are used to bound the convergence time. These function will spare us tedious parsings of the patterns in the configurations. For a given configuration x, we denote by $a(x), \ldots, h(x)$ the number of cells where transitions $\mathsf{A}, \ldots, \mathsf{H}$ are applicable, i.e.:

$$a(x) = |x|_{\mathtt{000}}, \quad b(x) = |x|_{\mathtt{001}}, \quad c(x) = |x|_{\mathtt{100}}, \quad d(x) = |x|_{\mathtt{101}},$$
$$e(x) = |x|_{\mathtt{010}}, \quad f(x) = |x|_{\mathtt{011}}, \quad g(x) = |x|_{\mathtt{110}}, \quad h(x) = |x|_{\mathtt{111}}.$$

For instance, consider rule BCG. For convenience, we denote by $p = 1/n$ the probability that a given cell is updated under fully asynchronous dynamics. Applying the transitions $\mathsf{A}, \ldots, \mathsf{D}$ increases the number of 1s by one and applying $\mathsf{E}, \ldots, \mathsf{H}$ decreases it by one. The expected variation of the number of 1s for configuration x in one step is then immediately $p \cdot (b(x) + c(x) - g(x))$. When the context is clear, the argument x will be omitted.

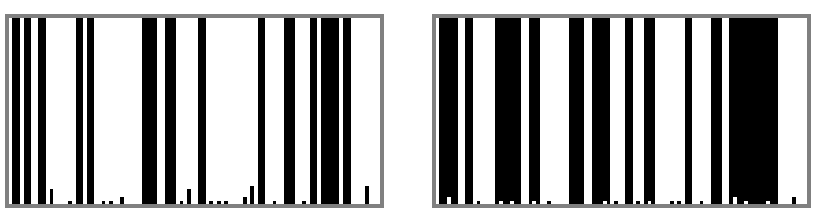
Clearly, parsing properly configuration x gives the following useful relationships.

Fact 8 *For all configurations $x \in Q^U$, the following equalities hold:*

$$\begin{aligned} |x|_{\mathtt{01}} &= b + d = e + f = c + d = e + g = |x|_{\mathtt{10}}, \\ |x|_{\mathtt{001}} &= b = c = |x|_{\mathtt{100}}, \\ |x|_{\mathtt{011}} &= f = g = |x|_{\mathtt{110}}. \end{aligned}$$

Let us now analyze the worst expected convergence time for DQECAs.

5.1 "Coupon collector" DQECAs



(a) E = **200** (b) DE = **232** (majority rule)

Figure 2: Sequential space-time diagrams for rules E et DE and $n = 100$. Time goes from bottom to top ; we go from one line to another after n iterations (i.e., lines represent states of the automaton for times $i \cdot n, i \in \mathbb{N}$). This convention is kept for the rest of the paper.

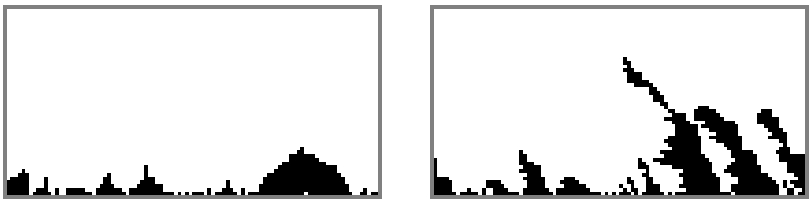
The behavior of the DQECAs in this class (see Fig. 2) is similar to the classic Coupon Collector random process (e.g., [12]).

Theorem 9 *Under fully asynchronous dynamics, DQECAs* E *and* DE *converge almost surely to a fixed point on any initial configuration. Their worst expected convergence times are $\Theta(n \ln n)$. The fixed points for* E *and* DE *respectively are the configurations without isolated* 1 *and the configurations without isolated* 0 *and* 1.

Proof. These rules simply erase either isolated 0s, isolated 1s or both. They never create any of them (by Fact 2), and reach a fixed point as soon as no more 0 or 1 are isolated (by Fact 3). These processes are then similar to a coupon collector process that has to collect all the isolated 0s or 1s, by drawing at each time step a random location uniformly in $\{1, \ldots, n\}$ (see e.g., [12]). If the number of remaining isolated 0s and 1s is i, the probability to draw one of them is i/n, and then, one of them is drawn on expectation after n/i steps. The expected convergence time is then bounded by $n(1 + \frac{n}{2} + \cdots + \frac{1}{n}) = O(n \ln n)$.

Finally, configuration $(010)^{\lfloor n/3 \rfloor} 0^{n \bmod 3}$, which is a proper coupon collector process, provides a lower bound of $\Omega(n \ln n)$ for both rules. □

5.2 Quadratic DQECAs



(a) DEFG = **160** (b) BEFG = **130**

Figure 3: Space-time diagrams of "quadratic" rules. Rule DEFG is the conjuguate rule of BCDE (it thus converges to 0^n).

Figure 3 illustrates the typical space-time diagram in this class. All the results of this section are obtained by finding a proper *variant* whose convergence implies the convergence of the DQECA, and which decreases by a constant on expectation.

Lemma 10 *Given an initial configuration x, for each DQECA* B, BC, BDE, BCDE, BCDEG, BE, EF, BCE, EFG, BCEFG, *and* BEFG, *there exists a sequence (X_t) of random variables with values in $\{0, \ldots, n\}$ (the* variant*), such that:*

(a) if $X_t = 0$, then x^t is a fixed point.

(b) for all t such that x^t is not a fixed point, $\mathbb{E}[\Delta X_{t+1} | X_t] \leqslant -p$.

Proof. **Rules B and BC.** Set $X_t = |x^t|_0$ the number of 0s in x^t. (a) is clear since $X_t = 0$ implies that $x^t = 1^n$. We obtain (b) by noticing that each application of transitions B or C decreases X_t by one, and that for any non fixed-point configuration, an active transition is performed with probability greater or equal to p.

Similarly, $X_t = |x^t|_1$ is suitable for rules EF and EFG.

Remaining rules. We need to take into account the presence of isolated 0s and 1s. We set $X_t = |x^t|_0 + Z(x^t)$ for rules BDE, BCDE, BE, BCE, and BCDEG; and $X_t = |x^t|_1 + Z(x^t)$ for rule BEFG. Consider automaton BEFG. Clearly, $X_t \in \{0, \ldots, n\}$, and we have (a) $X_t = 0$ implies that $x^t = 0^n$. First, for this rule,

$$\mathbb{E}[\Delta X_{t+1} | x^t] = p \cdot (b - e - f - g)(x^t) - p \cdot e(x^t),$$

since only transition E acts on $Z(x^t)$. By Fact 8, one can rewrite

$$\mathbb{E}[\Delta X_{t+1}|x^t] = -p \cdot (d+e+g)(x^t).$$

Second, if x is not a fixed point, then $(b+e+f+g)(x) > 0$. But by Fact 8, if $d+e=0$, then $b=f=g$. Thus, $b+e+f+g>0$ implies $d+e+g>0$. We conclude that if x^t is not a fixed point, we have (b). The proof is similar for all the remaining automata. □

We can now state the theorem.

Theorem 11 *Under fully asynchronous dynamics, DQECAs* B, BC, BDE, BCDE, BCDEG, BE, EF, BCE, EFG, BCEFG, *and* BEFG *converge almost surely to a fixed point on any initial configuration. Their worst expected convergence times are* $\Theta(n^2)$. *Only the DQECAs* B, BC, BE, *and* BCE *have fixed points that are distinct from* 0^n *and* 1^n, *which are all the configurations where all the* 0*s are isolated.*

Proof. The property on the fixed points is a direct application of Fact 3. Consider now one of the rules. Let X_t be the variant given by Lemma 10. X_t does not exactly verify the hypotheses of Lemma 4: X_t needs to be extended beyond a fixed point if it is reached before $X_t = 0$. We consider the random sequence X'_t defined as follow: $X'_t = X_t$ if x^t is not a fixed point, and $X'_t = 0$ otherwise. Thus, $X'_t = 0$ if and only if x^t is a fixed point, and we can now apply Lemma 4 with $m=0$, $m'=n$ and $\epsilon = p$ and we obtain $\mathbb{E}[T] \leqslant X_0/p = O(n^2)$.

The lower bound $\Omega(n^2)$ on the convergence time is simply given by considering the following initial configuration $x = 0^{\lceil n/2 \rceil}1^{\lfloor n/2 \rfloor}$. Note that $X_t = |x^t|_1$ works for all the rules on initial configuration x and its exact expected convergence time is straightforward to compute by first step analysis (see [13]). □

Observe that we can divide this class in two subcategories: the automata that are monotonic, for which the variant is a non-increasing function of time, and the non-monotonic, for which the variant follows a biased random walk (see Tab. 1). Interestingly enough, this distinction is observed on the space-time diagrams (see Fig. 3).

5.3 Cubic DQECAs

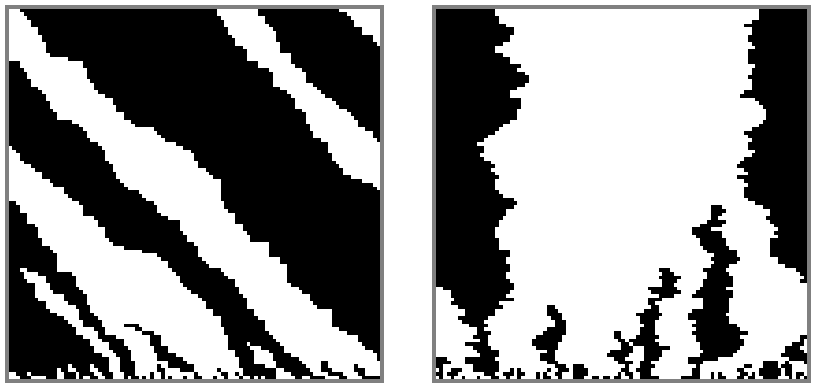
Figure 4 illustrates the typical behavior of this class: one can observe that the dynamics of the sizes of the regions in the space-time diagram are similar to unbiased random walks. Furthermore, one can observe that the process of the frontiers between regions is similar to annihilating random walks (e.g.,[10]): each frontier follow a random walk and two frontiers vanish when they meet.

All the results of this section are obtained by coupling the process with a suitable unbiased bounded random walk, such that the DQECA is guaranteed to reach a fixed point before the walk reaches a (or one distinguished) boundary.

Lemma 12 *Given an initial configuration x, for each DQECA* BDEF, BDEG, *and* BCDEFG, *there exists an integer m and a random integer sequence (X_t) of type I (see section 4.2) with values in $\{0,\dots,m\}$, such that: for all t, if $X_t = 0$ or $X_t = m$, then x^t is a fixed point.*

Proof. Rules BDEG **(Shift 170) and** BDEF. Set $X_t = |x^t|_1$. X_t takes its values in $\{0,\dots,n\}$, and $X_t \in \{0,n\}$ implies that x^t is a (trivial) fixed point. According to Fact 8, reading the code of the rule gives for all t,

$$\mathbb{E}[\Delta X_{t+1}|x^t] = p \cdot (b+d-e-g)(x^t) = 0.$$



(a) BDEG = **170** (shift) (b) BCEFG = **146**

Figure 4: Space-time diagrams of "cubic" rules.

X_t is thus a martingale.

For every time t such that $0 < X_t < n$,

$$\begin{aligned}\Pr\{\Delta X_{t+1} \geqslant 1|x^t\} &= p \cdot (b+d)(x^t)\\ &= p \cdot (e+g)(x^t)\\ &= \Pr\{\Delta X_{t+1} \geqslant 1|x^t\}\\ &= p|x^t|_{\mathsf{01}}\\ &\geqslant p.\end{aligned}$$

X_t is then of type I. The proof is similar for rule BDEG.

Rule BCDEFG. Because of special side effects due to transitions D and E, we need to use a more intricate sequence (X_t). We introduce two random sequences (D_t) and (E_t) that respectively count the number of applications of transitions D and E during time interval $[0, t)$.

For $t \geqslant 0$ such that x^t is not a fixed point, we define $X_t = Z(x^0) + |x^t|_{\mathsf{1}} + D_t - E_t$. Since for all t, $Z(x^0) - E_t \geqslant 0$, and $D_t \leqslant Z(x^0) \leqslant n/2$, X_t takes its values in $\{0, \ldots, 2n\}$. Furthermore, if $X_t = 0$ or $X_t = 2n$, then x^t is $\mathsf{0}^n$ or $\mathsf{1}^n$ respectively, and the process has converged.

Using Fact 8,

$$\mathbb{E}[\Delta X_{t+1}|x^t, D_t, E_t] = p \cdot (-b-c-d+e+f+g)(x^t) + p \cdot (e-d)(x^t) = 0.$$

Furthermore, assume that x^t is not a fixed point, we have $(b+c+d+e+f+g)(x^t) \geqslant 1$, i.e., $(2b+d+e+2g)(x^t) \geqslant 1$. Thus, at least one of b, d, e or g is greater or equal to 1 on x^t. We conclude :

$$\begin{aligned}\Pr\{\Delta X_{t+1} \geqslant 1|x^t, D_t, E_t\} &= \Pr\{\Delta X_{t+1} \leqslant -1|x^t, D_t, E_t\}\\ &= p \cdot (2e+f+g)(x^t)\\ &= p \cdot (b+d+e+g)(x^t)\\ &\geqslant p.\end{aligned}$$

In order to get a process of type I, we need to extend X_t beyond the fixed point, until it reaches either 0 or $2n$. We proceed as follows: for $t > 0$ such that x^t is a fixed point and $X_t \notin \{0, 2n\}$, X_{t+1} is $X_t + 1$ or $X_t - 1$ with equal probability $\frac{1}{2}$. X_t is then a suitable process of type I for rule BCDEFG. □

Lemma 13 *Given an initial configuration x, for each DQECA* BEF, BEG, *and* BCEFG, *there exists an integer m and a random integer sequence (X_t) of type II (see section 4.2) with values in $\{0, \ldots, m\}$, such that for all t, if $X_t = 0$, then x^t is a fixed point.*

Proof. **Rule** BEF. We define the process X_t as follows. First, $X_0 = |x^0|_1$. Then, as long as x^t is not a fixed point, X_{t+1} is computed according to the neighborhood of the cell updated at time t as follows :

- if the transition applied is E or F, then $X_{t+1} = X_t - 1$;
- if the transition applied is B or the neighborhood of the selected cell is 101 (i.e., the site of a fictitious transition D — this trick makes the process symmetric), then $X_{t+1} = \min(n-1, X_t + 1)$;
- otherwise, $X_{t+1} = X_t$.

Clearly, for all t, $X_t \in \{0, \ldots, n-1\}$ and $X_t \geqslant |x^t|_1$, i.e., X_t bounds from above the number of 1s in the configuration at any time t. As a consequence, the fixed point 0^n has been reached at or before time t if $X_t = 0$.

Assume again that x^t is not a fixed point. If $X_t < n-1$, then by Fact 8 :

$$\mathbb{E}[\Delta X_{t+1} | x^t, X_t] = p \cdot (b + d - e - f)(x^t) = 0$$

and

$$\begin{aligned} \Pr\{\Delta X_{t+1} \geqslant 1 | x^t, X_t\} &= \Pr\{\Delta X_{t+1} \leqslant -1 | x^t, X_t\} \\ &= p \cdot (b + d)(x^t) \geqslant p. \end{aligned}$$

Otherwise, $X_t = n - 1$ and $\Pr\{\Delta X_{t+1} | x^t, X_t\} = p \cdot (e + f)(x^t) \geqslant p$.

In order to obtain a proper process of type II, we need to extend (X_t) beyond the fixed point with two extra last steps: if x^t is a fixed point, then $X_{t+1} = 0$ or $X_{t+1} = n - 1$ with respective probabilities $\frac{X_t}{n-1}$ and $1 - \frac{X_t}{n-1}$; and if $X_{t+1} = n - 1$, then $X_{t+2} = 0$.

The designed (X_t) is then a suitable process of type II for rule BEF. By symmetry, exchanging f and g, in the definition of X_t gives a suitable process of type II for BEG.

Rule BCEFG. The definition of the process (X_t) is more subtle. First, set $X_0 = |x^0|_1 + Z(x^0)$. Assume that x^t is not a fixed point. The value of X_{t+1} is computed again according to the neighborhood of the cell selected at time t. We denote by ℓ the transition corresponding to the neighborhood of the cell updated in x^t at time t. X_{t+1} is given by:

- if $X_t \leqslant n - 2$, then: $X_{t+1} = \begin{cases} X_t + 2, & \text{if } \ell = \mathsf{D} \\ X_t + 1, & \text{if } \ell \in \{\mathsf{B}, \mathsf{C}\} \\ X_t - 1, & \text{if } \ell \in \{\mathsf{F}, \mathsf{G}\} \\ X_t - 2, & \text{if } \ell = \mathsf{E} \\ X_t, & \text{otherwise} \end{cases}$

- if $X_t = n-1$, then: $X_{t+1} = \begin{cases} X_t + 1, & \text{if } \ell \in \{\mathsf{B}, \mathsf{D}\} \\ X_t - 1, & \text{if } \ell \in \{\mathsf{F}, \mathsf{E}\} \\ X_t, & \text{otherwise} \end{cases}$

- if $X_t = n$, then: $X_{t+1} = \begin{cases} X_t - 1, & \text{if } \ell \in \{\mathsf{F}, \mathsf{G}, \mathsf{E}\} \\ X_t, & \text{otherwise} \end{cases}$

By induction, for any t, we have $0 \leqslant |x^t|_1 + Z(x^t) \leqslant X_t \leqslant n$, and then if $X_t = 0$, the process has reached the fixed point $\mathsf{0}^n$. Assume again that x^t is not a fixed point, then $b + c + e + f + g \geqslant 1$, which implies that $b + e + f \geqslant 1$. Now,

- if $X_t \leqslant n-2$, then:

$$\mathbb{E}[X_{t+1}|x^t, X_t] = p \cdot (b + c + 2d - 2e - f - g)(x^t) = 0,$$

  and

$$\begin{aligned} \Pr\{\Delta X_{t+1} \geqslant 1|x^t, X_t\} &= \Pr\{\Delta X_{t+1} \leqslant -1|x^t, X_t\} \\ &= p \cdot (2e + f + g)(x^t) \\ &= p \cdot (e + f + b + d)(x^t) \\ &\geqslant p, \end{aligned}$$

  since $b + e + f \geqslant 1$;

- If $X_t = n-1$, then:

$$\mathbb{E}[X_{t+1}|x^t, X_t] = p \cdot (b + d - e - f)(x^t) = 0,$$

  and

$$\begin{aligned} \Pr\{\Delta X_{t+1} \geqslant 1|x^t, X_t\} &= \Pr\{\Delta X_{t+1} \leqslant -1|x^t, X_t\} \\ &= p \cdot (b + d)(x^t) \\ &\geqslant p, \end{aligned}$$

  since $b + d = e + f$ and $b + e + f \geqslant 1$;

- if $X_t = n$, then:

$$\begin{aligned} \Pr\{\Delta X_{t+1} \leqslant -1|x^t, X_t\} &= p \cdot (f + g + e)(x^t) \\ &\geqslant p, \end{aligned}$$

  since $b + d = e + f$ and $b + e + f \geqslant 1$.

We then use the same technics as before to extend (X_t) to a process of type II beyond the fixed point. The resulting sequence is then a suitable process of type II for BECFG. $\square$

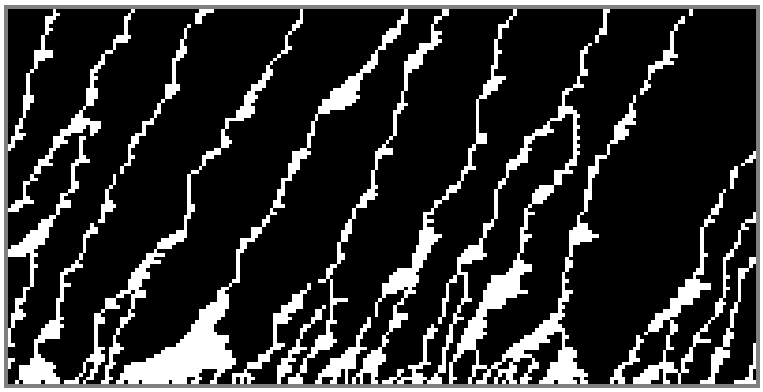
We can now conclude the theorem.

Theorem 14 *Under fully asynchronous dynamics, DQECAs* BDEF*,* BDEG*,* BCDEFG*,* BEF*,* BEG*, and* BCEFG *converge almost surely to a fixed point on any initial configuration. Their worst expected convergence times are* $\Theta(n^3)$*. All of them admit only* 0^n *and* 1^n *as fixed point. For DQECAs* BDEF*,* BDEG*, and* BCDEFG*, the fixed points* 0^n *and* 1^n *can be reached from any configuration (respectively distinct from* 1^n *and* 0^n*). For DQECAs* BEF*,* BEG*, and* BCEFG*, any configuration distinct from* 1^n *converges to* 0^n*.*

Proof. The upperbounds are straightforward applications of Lemmas 12 and 13 in combination with probabilistic lemmas 6 and 7.

The lower bounds are obtained by computing the exact expected convergence time of the automata on initial configuration $x = 0^{\lceil n/2 \rceil}1^{\lfloor n/2 \rfloor}$. Again, for this configuration, $X_t = |x^t|_1$ is a valid variant for all the rules, and its expected convergence time can be exactly computed by first step analysis (see [13]). □

5.4 Exponential DQECA



(a) BCEF = **210**

Figure 5: Space-time diagram of the "exponential" rule BCEF. The non-connexity of patterns of 1's are an artifact of the conventions of representations (see figure 2).

Figure 1(e) illustrates the typical behavior of this class. The illustrated process will eventually converge to 0^n. The trajectory of the 0-regions is similar to a coalescing random walk : the 0-regions follow a kind of coalescing random walk and merge when they meet, until only one 0-region remains. The size of the remaining 0-region then follows a random walk, biased towards 1, that will eventually converge to n after an exponential time (note that a 0-region cannot disappear for rule BCEF). This result is obtained by coupling the process with a process applying the same rule on a suitable single 0-region configuration. The following lemma analyzes the latter process first. Note that the expected convergence time is independent of the initial (non-fixed point) configuration, up to a multiplicative constant.

Lemma 15 *From any initial configuration* x *with exactly one* 0*-region and one* 1*-region,* BCEF *converges almost surely to the fixed point* 0^n*, after* $\Theta(n2^n)$ *iterations on expectation.*

Proof. The assertion about fixed points is given by Fact 3. The reachable configurations from x are $\mathtt{0}^{n-i}\mathtt{1}^i$, $0 \leqslant i \leqslant n-1$ (up to circular permutations). The process restricted to this set of configurations is fully described by the evolution of $X_t = |x^t|_{\mathtt{1}}$, which behaves as a kind of biased random walk on $\{0, \ldots, n-1\}$.

More precisely, we have $X_0 = |x|_{\mathtt{1}}$. For all t, if $0 \leqslant X_t \leqslant n-2$, then $X_{t+1} = X_t + 1$ with probability $2p$, $X_{t+1} = X_t$ with probability $(1-3p)$ and $X_{t+1} = X_t - 1$ with probability p; otherwise, if $X_t = n-1$, then $X_{t+1} = X_t$ with probability $(1-p)$ and $X_{t+1} = X_t - 1$ with probability p. The state $\mathtt{0}^n$ of the random walk is a fixed point and the expected convergence time T for the DQECA is defined by $T = \min\{t : X_t = 0\}$.

Let T_i denote the expected convergence time starting from configuration $\mathtt{0}^{n-i}\mathtt{1}^i$. First-step analysis (see [13]) gives the equations:

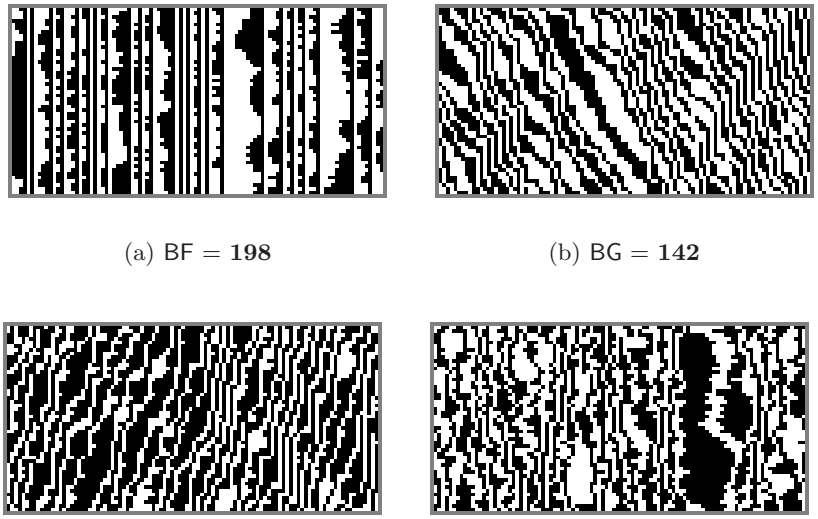
$$\begin{aligned} T_i &= 1 + pT_{i-1} + (1-3p)T_i + 2pT_{i+1}, \quad \text{pour } 1 \leqslant i \leqslant n-2 \\ T_{n-1} &= 1 + pT_{n-2} + (1-p)T_{n-1}, \\ T_0 &= 0. \end{aligned}$$

It can be checked that the solution of these equations is $T_i = \frac{2^n}{p}(1-2^{-i}) - \frac{i}{p} = \Theta(n2^n)$ for all $i \in \{1, \ldots, n-1\}$. □

Theorem 16 *The fixed points of DQECA* BCEF *are* $\mathtt{0}^n$ *and* $\mathtt{1}^n$*. From any non-fixed point initial configuration, DQECA* BCEF *converges almost surely to* $\mathtt{0}^n$ *and its expected convergence time is exactly* $\Theta(n2^n)$*.*

Proof. We couple any asynchronous execution of rule BCEF on initial configuration x with the asynchronous execution of the same rule from a single 0-region initial configuration y as follows. We first mark one arbitrary 0-region in x. y is the configuration obtained by complementing the marked 0-region with one single 1-region. At every time t, the same position is updated in x^t and y^t according to rule BCEF. After each transition, configuration y^t is realigned with configuration x^t, such that the right borders of the 0-region in y^t and of the marked 0-region in x^t coincide. This realignment does not affect the behavior of the two processes and ensures that, at every time t, the 0-region of y^t is included into the marked 0-region of x^t. Proceed by induction. Assume that the 0-region of y^t is included into the marked 0-region of x^t. This is clearly still true at time $t+1$ if transition B or C is applied on y, since this shrinks its 0-region. Now, if transition F is applied to y^t, the neighborhood of the updated cell in y^t is 011, and $01q$ in x^t. But the update is the same in x^t and y^t, whenever $q = \mathtt{1}$ (clearly) or $q = \mathtt{0}$ (transition E); and we get the again result for time $t+1$. Finally, if transition E is applied on y^t, then $y^t = \mathtt{1}^{n-1}\mathtt{0}$ (up to shifting), and by induction hypothesis, $x^t = y^t$ or $x^t = \mathtt{0}^n$, which validates the result for time $t+1$. Since the only reachable fixed point is $\mathtt{0}^n$, this guarantees that the convergence of the coupled process (y^t) implies the convergence of the original process (x^t). By Lemma 15, the expected convergence times of y^t and x^t are then $O(n2^n)$.

The lower bound on the expected convergence time relies simply on the fact that every process eventually reaches a configuration with a single 0-region and a single 1-region, and then takes $\Omega(n2^n)$ extra steps on expectation to converge (Lemma 15). □



(a) BF = **198** (b) BG = **142**

(c) BCF = **214** (d) BCFG = **150**

Figure 6: Space-time diagrams of "non-converging" rules.

5.5 Diverging DQECAs

Figure 6 illustrates the typical behavior of a divergent DQECA: the number of regions is conserved, and all reachable configurations from a given initial configuration are accessed an infinite number of times almost surely. The proof of the following result relies essentially on applying Fact 3.

Theorem 17 *Under fully asynchronous dynamics, the DQECAs* BF*,* BG*,* BCF*, and* BCFG *diverge almost surely on any initial configuration that is not one of the three following fixed points* 0^n*,* 1^n *and, if* n *is even,* $(01)^{n/2}$*. Furthermore, given an initial configuration, all reachable configurations are accessed an infinite number of times almost surely.*

Proof. According to Fact 3, the only possible non-trivial fixed points for these automata are configurations where all 0s and all 1s are isolated. Thus, only when n is even, these automata admit an extra fixed point, $(01)^{n/2}$, in addition to 0^n and 1^n. Furthermore, according to Fact 2, the number $Z(x^t)$ of alternations from 0 to 1 is constant, because none of these automata contains transitions D nor E. Thus, none of the fixed points can ever be reached from non-fixed point configurations, since $Z(0^n) = Z(1^n) = 0 < Z(x) < n/2 = Z((01)^{n/2})$ for every other configurations x.

The second part of the theorem consists in proving that there exists a finite length sequence of transitions between any pairs of configurations reachable from the same initial condition. Since there are finitely many pairs of configurations, each of these paths is followed with uniformly bounded positive probability, which yields the result. The existence of finite length paths between any pairs of reachable configurations follows simply from the reversibility of these automata: it can be easily verified that for these rules, if a configuration x^t is updated into a configuration x^{t+1}, then there exists an update such that $x^{t+2} = x^t$. As a consequence, there exists a finite length path from every reachable configuration to the initial configuration and then to any other reachable configuration, which concludes the proof. □

Note that for rules BG, BCF, and BCFG the set reachable points is simply the set of configurations with the same number of alternations from 0 to 1 as the initial configuration.

6 Conclusion

In this paper, we have characterized the convergence of the sixty-four double-quiescent elementary automata under fully asynchronous dynamics. Our results use the essential property that the number of regions is a non-increasing function of time. The coding introduced allowed us to easily determine the behavior of these regions and then to couple the evolution of each automaton with an appropriate stochastic process.

One may wonder what happens if more than one cell can be updated at each time step. Although the number of regions is no longer a non-increasing function of time, some results presented here are still valid [14].

Natural extensions of our work on the sensibility to asynchronism of cellular automata includes:

- What can be said for neighborhoods with radii larger than 1? Note that in this case, double-quiescence is not sufficient to ensure that the number of regions is non-increasing.
- What can be said when the number of states is greater than 2? Does new types of convergence arise?
- What can be said of the two-dimensional case?
- Can the study of asynchronous behavior of a cellular automaton help us to understand its synchronous behavior?

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