An algorithm for finding entire solutions of systems of difference equations

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Abstract

We present an algorithm that computes the entire solutions of systems of two difference equations and of systems of one differential equation and one difference equation, all with complex polynomials coefficients. The problem of the determination of such solutions arose in the field of diophantine approximation. Our algorithm, which uses previous works by Abramov and Petkovšek, allows also to determine, for each of the systems considered, all the solutions of the form $R_1(z)e^{\delta_1 z} + \cdots + R_S(z)e^{\delta_S z}$, with $S \in \mathbb{N}$, $\delta_1, \ldots, \delta_S \in \mathbb{C}$ and $R_1(z), \ldots, R_S(z) \in \mathbb{C}(z)$.

Keywords: difference equations, exponential polynomials, exponential fractions, linear differential equations, P-recursive functions, quasirational functions.

Résumé

On présente un algorithme qui calcule les solutions entières, i.e. holomorphes sur tout le plan complexe, de systèmes de deux équations aux différences à coefficients polynomiaux et de systèmes formés d'une équation aux différences et d'une équation différentielle linéaire à coefficients polynomiaux. Cet algorithme, qui utilise certains travaux de S.A. Abramov et M. Petkovšek, permet aussi de déterminer pour chacun des systèmes considérés toutes les solutions de la forme $R_1(z)e^{\delta_1 z} + \cdots + R_S(z)e^{\delta_S z}$, avec $S \in \mathbb{N}$, $\delta_1, \ldots, \delta_S \in \mathbb{C}$ et $R_1(z), \ldots, R_S(z) \in \mathbb{C}(z)$.

Mots-clés: Équations aux différences, polynômes exponentiels, fractions exponentielles, fonctions P-récursives, fonctions quasi-rationnelles

1 Introduction

An entire function is a function from $\mathbb{C}$ to $\mathbb{C}$ holomorphic over all the complex plane. An exponential polynomial is an entire function of the form $z \mapsto \sum_{s=1}^{S} H_s(z)e^{\delta_s z}$ with $S \in \mathbb{N}$, H_s in $\mathbb{C}[z]$ and δ_s in $\mathbb{C}$ for all s. By analogy, we call exponential fraction a function, not necessarily entire, of the form $z \mapsto \sum_{s=1}^{S} R_s(z)e^{\delta_s z}$ with R_s in $\mathbb{C}(z)$ and δ_s in $\mathbb{C}$ for all s. Let f denote an entire solution of the system of difference equations

$$\begin{cases} \displaystyle\sum_{0\le m\le M} P_m(z)f(z+m\alpha) = \Phi(z), \\ \displaystyle\sum_{0\le n\le N} Q_n(z)f(z+n\beta) = \Psi(z), \end{cases} \tag{1}$$

where $(P_m)_{0\le m\le M}$ and $(Q_n)_{0\le n\le N}$ are finite sequences over $\mathbb{C}[z]$ such that $P_M Q_N P_0 Q_0 \neq 0$, α and β two complex numbers linearly independent over $\mathbb{R}$, and Φ and Ψ two exponential fractions. Such a system, with $\Phi = \Psi = 0$, arose in the proof by Gramain (1981) that an entire function which maps the ring of Gauss integers $\mathbb{Z}[i] = \{a + ib;\, a, b \in \mathbb{Z}\}$ into itself and which satisfies some growth condition is necessarily a polynomial with complex coefficients.

In answer to a question of Masser (1982), Bézivin and Gramain (1993, 1996) proved that f is necessarily an exponential fraction. They also proved that this result remains true when the second difference equation is replaced by a differential equation with complex polynomial coefficients, i.e. when f is an entire function solution of the system

$$\begin{cases} \displaystyle\sum_{0\le m\le M} P_m(z)f(z+m\alpha) = \Phi(z), \\ \displaystyle\sum_{0\le n\le N} Q_n(z)f^{(n)}(z) = \Psi(z), \end{cases} \tag{2}$$

where $(P_m)_{0\le m\le M}$ and $(Q_n)_{0\le n\le N}$ are finite sequences over $\mathbb{C}[z]$ with $P_M Q_N P_0 Q_0 \neq 0$, α is any nonzero complex number, Φ and Ψ being two exponential fractions.

Brisebarre and Habsieger (1999) generalized Bézivin and Gramain's result about (1) to the case $\alpha/\beta \in \mathbb{R}\backslash\mathbb{Q}$ (which was also done by Loeb (1997)). They proposed also a new approach to this problem that allowed to find again some results of Bézivin and Gramain (1993, 1996).

Once the structure of the entire solutions of (1) and (2) known, the problem of the computation of these solutions arose. Brisebarre and Habsieger (1999) gave the sketch of an algorithm, inspired by Chapter 8 of Petkovšek et al. (1996), that computes the entire solutions of system (1) where it was assumed that $\Phi = \Psi = 0$. Moreover, a slight adaptation of this algorithm yielded a second algorithm that computes the entire solutions of (2), still under the assumption that $\Phi = \Psi = 0$.

In this article, we give an improved version of our two algorithms: we deal with a more general situation, since Φ and Ψ are any exponential fractions, and a more careful analysis makes our algorithms more efficient. Moreover, we give all the details of these algorithms and of their proofs (the previous versions of the algorithms and their proofs were only sketched in Brisebarre and Habsieger (1999)). Let us mention that our two algorithms can give all the exponential fractions solutions of (1) and (2). The paper is organized as follows. In Section 2, we introduce parametrized versions of two algorithms of Abramov and Petkovšek that are used in our algorithms. In Section 3, we give and prove the algorithm Pasrec that provides us with the entire solutions of system (1). We briefly present in Section 4 the algorithm Pasrecd that constructs the entire solutions of (2). Eventually, we compare in Section 5 our algorithms to another algorithm of Abramov (1991) for constructing the exponential fractions solutions of one linear differential or one finite-difference equation with polynomial coefficients. A brief analysis shows that:

- when we know beforehand that (1) or (2) has an exponential fraction solution, it is worth replacing some parts of our algorithms by Abramov's algorithm;
- otherwise, it is useless to take into account Abramov's algorithm.

In the first case, we give the modifications that we have to do in our algorithms.

2 Parametrized versions of some algorithms of Abramov and Petkovšek

Given a linear difference equation

$$p_M(z)R(z+M)+\cdots+p_1(z)R(z+1)+p_0(z)R(z)=b(z) \tag{3}$$

where $M\in\mathbb{N}$ and $p_0,\ldots,p_M,b\in K[z]$ with K a field of characteristic zero, the combination of these algorithms allows to find all rational solutions R of (3). The algorithm of Abramov (1995) constructs a polynomial D divisible by the denominator of any rational solution of (3). On the other hand, the algorithm Poly of Petkovšek (see Petkovšek et al. (1996)) gives us the polynomial solutions of (3). Thus, every rational solution R of (3) can be written as H/D where $H\in K[z]$ and D is the polynomial given by Abramov's algorithm. Then we use algorithm Poly to find all the solutions H in $K[z]$ of the equation

$$\frac{p_M(z)}{D(z+M)}H(z+M)+\cdots+\frac{p_1(z)}{D(z+1)}H(z+1)+\frac{p_0(z)}{D(z)}H(z)=b(z).$$

One can find the algorithm Poly and its theoretical background in Chapter 8 of Petkovšek et al. (1996). In the present work, we need slightly modified versions of these two algorithms since we deal with more general difference equations.

Let $\alpha\in K^*$, we consider the linear difference equation

$$p_M(z)R(z+M\alpha)+\cdots+p_1(z)R(z+\alpha)+p_0(z)R(z)=b(z) \tag{4}$$

where $M\in\mathbb{N}$ and $p_0,\ldots,p_M,b\in K[z]$. When looking for the polynomial solutions of (4), a careful reading of the proof of correctness of Poly allows to give the following adaptation (if P is a polynomial, $\mathrm{lc}(P)$ denotes its leading coefficient and we agree that $\deg P = -\infty$ if $P = 0$):

Algorithm Poly with parameter α

Input: The polynomials $p_0, \dots, p_M, b$ from equation (4), $\alpha \in K^*$.

Output: The general polynomial solution of (4) over K.

1. **For** $j = 0, \dots, M$, $G_j \leftarrow \sum_{m=j}^{M} \binom{m}{j} p_m$.

2. $t_1 \leftarrow \max_{0 \le j \le M}(\deg G_j - j)$,

$$T(z) \leftarrow \sum_{\substack{0 \le j \le M \\ \deg G_j - j = t_1}} \mathrm{lc}(G_j)\alpha^j z(z-1)\cdots(z-j+1),$$

$t_2 \leftarrow \max\{z \in \mathbb{N},\ T(z) = 0\}$,
$d \leftarrow \max\{\deg b - t_1, -t_1 - 1, t_2\}$.

3. Plug $R(z) = \sum_{i=0}^{d} c_i z^i$ into (4).
   Solve the linear system whose unknowns are the c_i.

In the sequel of the paper, we only need the following shortened algorithm.

Algorithm ShortPoly with parameter α

Input: The polynomials $p_0, \dots, p_M, b$ from equation (4), $\alpha \in K^*$.

Output: d, an upper bound for the degree of polynomial solutions of (4).

1. Step 1 of Poly with parameter α.

2. Step 2 of Poly with parameter α.

In the previous two algorithms, we can assume that $p_0, \dots, p_M, b \in K(z)$: it suffices, for instance, to multiply the members of (4) by a nonzero common denominator of these fractions to get an input of polynomials.

Now, we give the adaptation (easy to obtain) of the algorithm from Abramov (1995). If P and Q are polynomials, $\mathrm{Res}_z(P(z), Q(z))$ denotes their resultant.

Abramov's algorithm with parameter α

Input: The polynomials p_0 and p_M from equation (4), $\alpha \in K^*$.

Output: D, a denominator of the rational solutions of equation (4).

1. [Initialization] $A_1(z) \leftarrow p_M(z - M\alpha), A_2(z) \leftarrow p_0(z)$, $D \leftarrow 1$.

2. $R(m) \leftarrow \mathrm{Res}_z(A_1(z), A_2(z + m\alpha))$.

3. **If** $R(m)$ has some nonnegative integer root **then**
   $L \leftarrow$ the largest nonnegative integer root of $R(m)$,
   for $i = L, L-1, \dots, 0$,
   $d(z) := \gcd(A_1(z), A_2(z + i\alpha))$,
   $A_1(z) := A_1(z)/d(z)$,
   $A_2(z) := A_2(z)/d(z - i\alpha)$,
   $D(z) := D(z)d(z)d(z-\alpha)\cdots d(z - i\alpha)$,

3 The case of two difference equations: algorithm Pasrec

Let f be an entire function solution of (1) where α and β are two complex numbers linearly independent over $\mathbb{Q}$. We know from Bézivin and Gramain (1996) and Brisebarre and Habsieger (1999) that f is necessarily a finite sum of the form $\sum_{s=1}^{S} R_{\delta_s}(z)e^{\delta_s z}$ with R_{δ_s} in $\mathbb{C}(z)$ and δ_s in $\mathbb{C}$ for all s. The complex numbers δ_s and the rational fractions $R_{\delta_s}(z)$ are respectively called the *frequencies* and the *coefficients* of the exponential fraction f. Let g be an exponential fraction. We denote $\mathrm{Spec}(g)$ its *spectrum*, i.e. the set of the frequencies of g, and $R_{g,\delta}$ denotes the coefficient of g associated to $e^{\delta z}$ (it may be 0). If E is a finite set, $\#E$ denotes the number of elements of E. If z is a nonzero complex number, $\mathrm{Log}\,(z)$ is the complex number $\log|z| + i\mathrm{Arg}\, z$ with $\mathrm{Arg}\, z \in \,]-\pi,\pi]$, $\mathrm{Re}(z)$ is the real part of z and $\mathrm{Im}(z)$ is the imaginary part of z.

First, we state the algorithm and explain the improvements it contains with respect to the algorithm of Brisebarre and Habsieger (1999). In a second subsection, we give the refinement we can get in the case $\alpha/\beta \in \mathbb{C}\backslash\mathbb{R}$. Then, we give a worked example. In a fourth subsection, we provide a proof of correctness of the algorithm. Eventually, we discuss the particular situation that we can deal with in the case where $\alpha/\beta \in \mathbb{R}\setminus\mathbb{Q}$.

3.1 Algorithm Pasrec

Input: $\alpha, \beta \in \mathbb{C}^*$ such that $\alpha/\beta \in \mathbb{C}\setminus\mathbb{Q}$,
$P_0, \ldots, P_M, Q_0, \ldots, Q_N \in \mathbb{C}[z]$,
Φ, Ψ two exponential fractions.
Output: The general entire solution of (1).

1. [Determination of a set containing the frequencies δ_s]

   (a) $Spec \leftarrow \mathrm{Spec}(\Phi) \cup \mathrm{Spec}(\Psi)$,
   $d_P \leftarrow \max_{0\le m\le M} \deg P_m$, $d_Q \leftarrow \max_{0\le n\le N} \deg Q_n$,
   $a_m \leftarrow$ coefficient of z^{d_P} in $P_m(z)$, $b_n \leftarrow$ coefficient of z^{d_Q} in $Q_n(z)$.

   (b) Solve $\sum_{0\le m\le M} a_m X^m = 0 \Rightarrow J$ distinct nonzero roots $x_1, \ldots, x_J$.
   Solve $\sum_{0\le n\le N} b_n Y^n = 0 \Rightarrow L$ distinct nonzero roots $y_1, \ldots, y_L$.

   (c) **For** $\delta \in \mathrm{Spec}(\Phi)\backslash\mathrm{Spec}(\Psi)$,
   if $e^{\beta\delta} \notin \{y_1, \ldots, y_L\}$, **then**
   return: no entire solution.
   For $\delta \in \mathrm{Spec}(\Psi)\backslash\mathrm{Spec}(\Phi)$,
   if $e^{\alpha\delta} \notin \{x_1, \ldots, x_J\}$, **then**
   return: no entire solution.

   (d) **For** $j = 1, \ldots, J$, **for** $l = 1, \ldots, L$,
   If there exists a solution $(k, k') \in \mathbb{Z}^2$ of
   $k/\alpha - k'/\beta = (\mathrm{Log}\,(y_l)/\beta - \mathrm{Log}\,(x_j)/\alpha)/(2i\pi)$
   then
   $Spec \leftarrow Spec \cup \{(\mathrm{Log}\,(x_j) + 2ik\pi)/\alpha\}$.

(e) **If** $\#Spec = 0$ **then**
return: no entire solution but 0
else
$Spec$ is a set containing $\mathrm{Spec}(f)$.

2. [Determination of a common denominator of the coefficients of the exponential fractions solutions of (1)]
$D_\Phi \leftarrow$ g.c.d. of the coefficients of Φ, $D_\Psi \leftarrow$ g.c.d. of the coefficients of Ψ,
$D_\alpha \leftarrow$ Abramov's algorithm with parameter α applied to $(D_\Phi P_M, D_\Phi P_0)$,
$D_\beta \leftarrow$ Abramov's algorithm with parameter β applied to $(D_\Psi Q_N, D_\Psi Q_0)$,
$D \leftarrow \gcd(D_\alpha, D_\beta)$.

3. [Determination of the coefficients $R_{f,\delta}(z)$]
For $\delta \in Spec$,

(a) $deg_\alpha \leftarrow$ ShortPoly with parameter α applied to the equation

$$\sum_{m=0}^{M} \frac{P_m(z)e^{\delta m\alpha}}{D(z+m\alpha)} H_\delta(z+m\alpha) = R_{\Phi,\delta}(z). \tag{5}$$

$deg_\beta \leftarrow$ ShortPoly with parameter β applied to the equation

$$\sum_{n=0}^{N} \frac{Q_n(z)e^{\delta n\beta}}{D(z+n\beta)} H_\delta(z+n\beta) = R_{\Psi,\delta}(z). \tag{6}$$

$d \leftarrow \min(deg_\alpha, deg_\beta)$.

(b) Plug a generic polynomial $\sum_{j=0}^{d} a_j z^j$ into (5) and (6).
Solve the resulting linear system in the unknowns $a_0, \ldots, a_d$.
If there is no solution **then**
return: no entire solution
else
$H_{\delta,0} \leftarrow$ a particular solution of (5) and (6),
$F_\delta \leftarrow$ a basis of the direction of the $\mathbb{C}$-affine solution space of (5) and (6).

4. [Determination of the linear combination that makes the function entire]
Solve the linear system in the unknowns $\lambda_{\delta,k} \in \mathbb{C}$ of equations

$$\left(z \mapsto \sum_{\delta \in Spec} \left[H_{\delta,0}(z) + \sum_{k=1}^{\#F_\delta} \lambda_{\delta,k} F_{\delta,k}(z) \right] e^{\delta z} \right)^{(l_j)} (z_j) = 0, \tag{7}$$

$1 \le j \le r$, $0 \le l_j \le \nu_j - 1$, where $z_1, z_2, \ldots, z_r$ are the complex roots of D with respective multiplicity orders $\nu_1, \nu_2, \ldots, \nu_r$.

In step 1b, as we do not care about the order of multiplicity of the roots, we can replace the polynomial $P(X) = \sum_{0 \le m \le M} a_m X^m$ by a squarefree divisor having the same irreducible factors like $P(X)/\gcd(P(X), P'(X))$. The same applies to $Q(Y) = \sum_{0 \le n \le N} b_n Y^n$.

This algorithm allows to determine all the exponential fractions solutions of (1): we only have to skip step 4 which is the only step where we require that the solution is entire.

Let us now see how this new version improves on Brisebarre and Habsieger (1999). First, the systems of difference equations we can deal with are more general: the systems in Brisebarre and Habsieger (1999) were homogeneous whereas our algorithm solves systems whose right members can be any exponential fractions. Then, in the previous version of the algorithm, steps 2 and 3 were done using only the first equation of (1). We used the second equation to check the solutions found. In the present version, we use both equations at each step. This may decrease the degrees of the common denominator given by step 2 and of the numerator of each coefficient given by step 3a, which has the following consequences:

- if we get a smaller degree for the common denominator, this speeds up the applications of ShortPoly (remember that, according to Remark 2, equations (5) and (6) must be changed into equations with polynomial coefficients). It also speeds up step 4 since there are fewer equations to create when forming the system solved at step 4. Moreover, as we need to know the roots of this common denominator, it is of course important to get it with a degree as small as possible;
- if, in step 3a, we get a smaller degree for the numerator of the coefficient, this speeds up step 3b since the linear system is smaller.

3.2 The case $\alpha/\beta \in \mathbb{C} \setminus \mathbb{R}$

When α/β belongs to $\mathbb{C} \setminus \mathbb{R}$, we can replace step 1d with
For $j = 1, \ldots, J$, **for** $l = 1, \ldots, L$,
$k \leftarrow \mathrm{Re}(\beta \mathrm{Log}\,(x_j) - \alpha \mathrm{Log}\,(y_l))/\big(2\pi \mathrm{Im}(\beta/\alpha)\big)$,
$k' \leftarrow \mathrm{Re}(\mathrm{Log}\,(y_l)/\beta - \mathrm{Log}\,(x_j)/\alpha)/\big(2\pi \mathrm{Im}(\alpha/\beta)\big)$,
if $(k, k') \in \mathbb{Z}^2$, then
$Spec \leftarrow Spec \cup \{(\mathrm{Log}\,(x_j) + 2ik_{j,l}\pi)/\alpha\}$.

3.3 Worked example

We determine the entire functions f solutions of the system

$$\left\{ \begin{array}{l} (z+3)f(z+3) + (z^2 + (1-e)z - 2(1+e))f(z+2) \\ \qquad\qquad -((1+e)z^2 + z - e)f(z+1) + ez^2 f(z) = \Phi(z), \\ 3(z+3i)f(z+3i) - \left(2+3e^i\right)(z+2i)f(z+2i) \\ \qquad\qquad + \left(2e^i - 1\right)(z+i)f(z+i) + e^i z f(z) = \Psi(z), \end{array} \right. \tag{8}$$

with

$$\Phi(z) = 3((e-1)z + e^2 - e)e^{z+1} \text{ and } \Psi(z) = -3i(-3e^{2i} + 2e^i + 1)e^{z+i}.$$

Step 1. Here we have $\alpha = 1$ and $\beta = i$ and the ratio α/β belongs to $\mathbb{C} \setminus \mathbb{R}$. We also have $\mathrm{Spec}(\Phi) = \mathrm{Spec}(\Psi) = \{1\}$. We start by determining the potential frequencies of the solutions. We use the notations of Subsections 3.1 and 3.2. We have

$$d_P = \max_{0 \le m \le 2} \deg P_m = 2 \quad \text{and} \quad d_Q = \max_{0 \le n \le 3} \deg Q_n = 1.$$

We have to solve the equations

$$X^2 - (1+e)X + e = 0 \tag{9}$$

and

$$3Y^3 - (2 + 3e^i)Y^2 + (2e^i - 1)Y + e^i = 0. \tag{10}$$

The roots of the equation (9) are $x_1 = 1$ and $x_2 = e$, those of (10) are $y_1 = 1, y_2 = -1/3$ et $y_3 = e^i$. We put in Table 1 the values of the couples $(k_{j,l}, k'_{l,j})_{\substack{1 \le j \le 2 \\ 1 \le l \le 3}}$ defined by

$$k_{j,l} = \mathrm{Re}(i\mathrm{Log}\,(x_j) - \mathrm{Log}\,(y_l))/2\pi,$$

and

$$k'_{l,j} = \mathrm{Re}(i\mathrm{Log}\,(y_l) + \mathrm{Log}\,(x_j))/2\pi.$$

The only solutions in $\mathbb{Z}^2$ are $(k_{1,1}, k'_{1,1}) = (0,0)$ which is associated with fre-

	$l=1$	$l=2$	$l=3$
$l=1$	$(0,0)$	$\left(\frac{\mathrm{Log}\,(3)}{2\pi}, -\frac{1}{2}\right)$	$\left(0, -\frac{1}{2\pi}\right)$
$l=2$	$\left(0, \frac{1}{2\pi}\right)$	$\left(\frac{\mathrm{Log}\,(3)}{2\pi}, -\frac{1}{2} + \frac{1}{2\pi}\right)$	$(0,0)$

Table 1: Pairs $(k_{j,l}, k'_{l,j})$

quency $\delta_1 = \mathrm{Log}\,(x_1) + 2ik_{1,1}\pi = 0$ and $(k_{2,3}, k'_{3,2}) = (0,0)$ associated with frequency $\delta_2 = \mathrm{Log}\,(x_2) + 2ik_{2,3}\pi = 1$, which was already in *Spec*. Therefore, *Spec* is the set $\{0, 1\}$.

Step 2. Then, applying Abramov's algorithm with parameter 1 yields a common denominator to the rational fractions solutions of the first equation of the system (8). We have here $A_1(z) = P_3(z-3) = z$ and $A_2(z) = P_0(z) = ez^2$. Let $R_1(m) = \mathrm{Res}_z(A_1(z), A_2(z+m)) = em^2$ whose 0 is the only one integer root. Therefore, as the largest nonnegative integer root is 0, we have $D_1(z) = \gcd(A_1(z), A_2(z)) = z$. Now, applying Abramov's algorithm with parameter i to the couple $A_1(z) = Q_3(z-3i) = 3z$ and $A_2(z) = Q_0(z) = e^i z$, we get $D_i(z) = z$. Finally, the common denominator searched is $D(z) = \gcd(D_1(z), D_i(z)) = z$.

Step 3. In order to determine the rational fractions $R_{f,\delta}$ coefficients of the functions $e^{\delta z}$ (with $\delta \in \{0,1\}$), we have to solve the equations

$$\frac{z+3}{z+3}H_\delta(z+3) + \frac{z^2+(1-e)z-2(1+e)}{z+2}e^{2\delta}H_\delta(z+2) \\ - \frac{(1+e)z^2+z-e}{z+1}e^\delta H_\delta(z+1) + \frac{ez^2}{z}H_\delta(z) = R_{\Phi,\delta}(z),$$

and

$$\frac{3(z+3i)e^{3i\delta}H_\delta(z+3i)}{z+3i} - \frac{\left(2+3e^i\right)(z+2i)e^{2i\delta}H_\delta(z+2i)}{z+2i} \\ + \frac{\left(2e^i-1\right)(z+i)e^{i\delta}H_\delta(z+i)}{z+i} + \frac{e^i z H_\delta(z)}{z} = R_{\Psi,\delta}(z).$$

After simplifications, we obtain the equivalent equations

$$e^{3\delta}H_\delta(z+3) + (z-(1+e))e^{2\delta}H_\delta(z+2) \\ - ((1+e)z-e)e^\delta H_\delta(z+1) + ezH_\delta(z) = R_{\Phi,\delta}(z), \quad (11)$$

and

$$3e^{3i\delta}H_\delta(z+3i) - \left(2+3e^i\right)e^{2i\delta}H_\delta(z+2i) \\ + \left(2e^i-1\right)e^{i\delta}H_\delta(z+i) + e^iH_\delta(z) = R_{\Psi,\delta}(z). \quad (12)$$

We assume $\delta = 0$. We have $R_{\Phi,0} = R_{\Psi,0} = 0$. We use algorithm ShortPoly with parameter 1 to determine an upper bound deg_1 for the degree of the polynomial solutions H_0 of (11). We compute the polynomials $G_j = \sum_{m=j}^3 \binom{m}{j} P_m$ where P_m denotes the coefficient of $H_0(z+m)$ in (11). We find

$$G_0 = 0, \quad G_1 = (1-e)(z+1), \quad G_2 = z+2-e \text{ and } G_3 = 1.$$

Hence, $t_1 = \max_{0\le j\le 3}(\deg G_j - j) = \deg G_1 - 1 = 0$. Moreover, we notice that

$$T(z) = \sum_{\substack{0\le j\le 3 \\ \deg G_j - j = t_1}} \mathrm{lc}(G_j)z(z-1)\cdots(z-j+1) = (1-e)z,$$

where $\mathrm{lc}(G_j)$ denotes the leading coefficient of G_j. The degree of the polynomial solution is thus bounded by deg_1 with

$$deg_1 = \max\big(-t_1-1, \max\{z\in\mathbb{N},\, T(z)=0\}\big) = \max(-1,0) = 0.$$

The same computations give an upper bound deg_i for the degree of the polynomial solutions H_0 of (12) equal to zero. We plug a generic polynomial of degree zero into (11) and (12). Hence, we obtain that 0 is a particular solution of (11) and (12) and $\{1\}$ is a basis of the direction of the $\mathbb{C}$-affine space of polynomial solutions of (11) and (12). In the same way, if we assume $\delta = 1$, we get a particular solution $H_{1,0} = 3z$ and a basis $F_1 = \{1\}$.

Step 4. Finally, we have to determine the complex numbers λ_0 and λ_1 that make the linear combination $z \mapsto \dfrac{\lambda_0 + (\lambda_1 + 3z)e^z}{z}$ an entire function. Therefore we solve the equation

$$\lambda_0 + \lambda_1 = 0.$$

Hence, the necessary condition is $\lambda_0 = -\lambda_1$. We finally showed that the entire solutions of the system (8) are the functions

$$z \mapsto \lambda \frac{e^z - 1}{z} + 3e^z,\ \lambda \in \mathbb{C}.$$

3.4 Proof of correctness of algorithm Pasrec

We know from Bézivin and Gramain (1993, 1996) that f is necessarily an exponential fraction. So, we write f as $z \mapsto \sum_{\delta\in\mathbb{C}} R_{f,\delta}(z)e^{\delta z}$, where the $R_{f,\delta}(z)$ are almost all zero, into the first equation of (1). Then, we get

$$\sum_{m=0}^{M} P_m(z) \left(\sum_{\delta\in\mathbb{C}} R_{f,\delta}(z+m\alpha)e^{\delta m\alpha}e^{\delta z} \right) = \sum_{\delta'\in\mathrm{Spec}(\Phi)} R_{\Phi,\delta'}(z)e^{\delta' z}.$$

The family of functions $\left(e^{\delta\cdot}\right)_{\delta\in\mathbb{C}}$ is free over $\mathbb{C}(z)$ (see for example Waldschmidt (1974)). Therefore

$$\sum_{m=0}^{M} P_m(z)R_{f,\delta}(z+m\alpha)e^{\delta m\alpha} = R_{\Phi,\delta}(z),\ \text{for all } \delta \in \mathbb{C}. \tag{13}$$

Likewise, we get for every $\delta \in \mathbb{C}$

$$\sum_{n=0}^{N} Q_n(z)R_{f,\delta}(z+n\beta)e^{\delta n\beta} = R_{\Psi,\delta}(z). \tag{14}$$

Step 1. We first determine a set, called *Spec*, containing the spectrum of f. From (13) and (14), we see that, necessarily, $\mathrm{Spec}(\Phi) \cup \mathrm{Spec}(\Psi) \subset \mathrm{Spec}(f)$, hence the first line of step 1a. Then, we search for the other elements of $\mathrm{Spec}(f)$. Let d_P and d_Q be as in step 1a, let $\delta \in \mathrm{Spec}(f) \setminus \big(\mathrm{Spec}(\Phi) \cup \mathrm{Spec}(\Psi)\big)$. Relation (13) is equivalent to the equation

$$\sum_{m=0}^{M} \frac{P_m(z)}{z^{d_P}} \frac{R_{f,\delta}(z+m\alpha)}{R_{f,\delta}(z)} e^{\delta m\alpha} = 0.$$

Let a_m, for $m = 0, \ldots, M$, be as in step 1a. We let z tend to infinity: we get $\sum_{m=0}^{M} a_m e^{\delta m\alpha} = 0$, which is a polynomial equation in $e^{\delta\alpha}$. Likewise, let b_n, for $n = 0, \ldots, N$, be as in step 1a, we have $\sum_{n=0}^{N} b_n e^{\delta n\beta} = 0$. Therefore, in step 1b, we solve the equations

$$\sum_{m=0}^{M} a_m X^m = 0 \ \text{ and } \ \sum_{n=0}^{N} b_n Y^n = 0,\ \text{for } X = e^{\delta\alpha} \text{ and } Y = e^{\delta\beta}. \tag{15}$$

We can notice that, for $\delta \in \mathrm{Spec}(\Phi) \setminus \mathrm{Spec}(\Psi)$ (resp. $\mathrm{Spec}(\Psi) \setminus \mathrm{Spec}(\Phi)$), if $e^{\beta\delta}$ (resp. $e^{\alpha\delta}$) is not a root of $Q(Y)$ (resp. $P(X)$), then there is no entire solution of (1). This remark corresponds to step 1c.

The solving of (15) yields J ($\leq M$) distinct classes of solutions in $\mathbb{C}/2i\pi\alpha^{-1}\mathbb{Z}$ for the first equation and L ($\leq N$) distinct classes of solutions in $\mathbb{C}/2i\pi\beta^{-1}\mathbb{Z}$ for the second equation that we have to intersect to find the possible missing frequencies. This is done as follows. Let x be a nonzero root of $P = \sum_{m=0}^{M} a_m X^m$ and y a nonzero root of $Q = \sum_{n=0}^{N} b_n Y^n$. We associate with x a frequency of the form $(\mathrm{Log}\,(x) + 2i\pi k)/\alpha$ and with y a frequency of the form $(\mathrm{Log}\,(y) + 2i\pi k^{'})/\beta$ where $k, k^{'}$ are in $\mathbb{Z}$. So, we have to solve for k and $k' \in \mathbb{Z}$ the equations

$$\frac{\mathrm{Log}\,(x) + 2i\pi k}{\alpha} = \frac{\mathrm{Log}\,(y) + 2i\pi k'}{\beta},$$

with x in the set of the nonzero roots of P and y in the set of the nonzero roots of Q. These equations are equivalent to

$$\frac{k}{\alpha} - \frac{k'}{\beta} = \frac{1}{2i\pi}\left(\frac{\mathrm{Log}\,(y)}{\beta} - \frac{\mathrm{Log}\,(x)}{\alpha}\right). \tag{16}$$

From the irrationality of α/β, we are sure to find a finite number S of solutions: it is easy to see that for each pair (x, y) there is at most one solution $(k, k') \in \mathbb{Z}^2$ of equation (16). The solving of the equations (16) allows us to exhibit a finite set (that can be empty) of possible frequencies $\{\delta_1, \ldots, \delta_S\}$.

When $\alpha/\beta \in \mathbb{C}\setminus\mathbb{R}$, solving (16) is trivial: considering (16) and its conjugate yields

$$k = \frac{\mathrm{Re}(\beta \mathrm{Log}\,(x) - \alpha \mathrm{Log}\,(y))}{2\pi \mathrm{Im}(\beta/\alpha)} \quad \text{and} \quad k' = \frac{\mathrm{Re}(\mathrm{Log}\,(y)/\beta - \mathrm{Log}\,(x)/\alpha)}{2\pi \mathrm{Im}(\alpha/\beta)}.$$

We will discuss in Subsection 3.5 the case where $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$.

The union of the sets $\{\delta_1, \ldots, \delta_S\}$ and $\mathrm{Spec}(\Phi) \cup \mathrm{Spec}(\Psi)$ gives a set, that we call *Spec*, of possible frequencies. Note that $\mathrm{Spec}(f) \subset$ *Spec* (this inclusion can be strict). That implies in particular that if *Spec* is empty then $\mathrm{Spec}(f)$ is also empty: 0 is then the only entire solution.

Step 2. We turn now to the determination of a common denominator D of the coefficients of the exponential fractions solutions of (1), i.e. of the rational fractions solutions of equations (13) and (14). Abramov's algorithm with parameter α applied to (13) gives us D_α, a common denominator of the rational solutions of this equation. Since this algorithm requires that the right member of the difference equation is a polynomial, we first multiply (13) by D_Φ, a common denominator of the coefficients of Φ. We recall that only the polynomials $D_\Phi(z - M\alpha)P_M(z - M\alpha)$ and $D_\Phi(z)P_0(z)$ are involved in Abramov's algorithm. Therefore, the computation of a common denominator is independent of frequency δ.

Similarly, applying Abramov's algorithm with parameter β to (14) (multiplied by D_Ψ, a common denominator of the coefficients of Ψ), we obtain D_β, a common denominator of the rational solutions of this equation. As the denominator of every rational solution of (13) and (14) should divide both D_α and D_β, the g.c.d. of these two polynomials is a suitable common denominator.

Step 3. The next step is the determination of the coefficients $R_{f,\delta}$. Since we know, from step 2, a common denominator D of these coefficients, we replace $R_{f,\delta}$ in (13) and (14) with H_δ/D. Hence, for $\delta \in Spec$, we are looking for the polynomial solutions H_δ of (5) and (6). Algorithm ShortPoly with parameter α applied to (5) gives an upper bound deg_α for the degree of polynomial solutions H_δ of this equation. Likewise, we get an upper bound deg_β for the degree of polynomial solutions H_δ of equation (6). So, the maximal degree of the polynomials H_δ searched is bounded by $d = \min(deg_\alpha, deg_\beta)$. Then, as in step 3 of Algorithm Poly, we plug a generic polynomial $\sum_{j=0}^{d} a_j z^j$ into (5) and (6). We get a linear system of equations in the unknowns $a_0, \ldots, a_d \in \mathbb{C}$. If we get no particular solution of it, that means that there is no entire solution of (1). Otherwise, we have obtained a particular solution $H_{\delta,0}$ of (5) and (6) and a basis (which can be empty) $F_\delta = \{F_{\delta,k}\}_{k=1,\ldots,\#F_\delta}$ of the direction of the $\mathbb{C}$-affine space of solutions of (5) and (6).

Step 4. We have to find the complex coefficients $\lambda_{\delta,k}$ such that the function

$$z \mapsto \frac{\sum_{\delta\in Spec}\left[H_{\delta,0}(z) + \sum_{k=1}^{\#F_\delta}\lambda_{\delta,k}F_{\delta,k}(z)\right]e^{\delta z}}{D(z)}$$

is entire. If $D \in \mathbb{C}$ (therefore equal to 1, cf. Section 2), then the algorithm ends and the entire solutions are the sum of $f_0 = \sum_{\delta\in Spec} H_{\delta,0}(z)e^{\delta z}$ and of every linear combination over $\mathbb{C}$ of the elements of the families $\left\{F_{\delta,k}(z)e^{\delta z}\right\}_{\substack{\delta\in Spec,\\ k=1,\ldots,\#F_\delta}}$.
Otherwise, we have to find the $\lambda_{\delta,k} \in \mathbb{C}$ for which the roots of D are also roots (counted with multiplicity) of the exponential polynomial

$$\sum_{\delta\in Spec}\left[H_{\delta,0}(z) + \sum_{k=1}^{\#F_\delta}\lambda_{\delta,k}F_{\delta,k}(z)\right]e^{\delta z}.$$

If $z_1, z_2, \ldots, z_r$ denote the roots of D with respective order $\nu_1, \nu_2, \ldots, \nu_r$, the coefficients $\lambda_{\delta,k}$ are the solutions of the linear system whose equations are (7).

3.5 The case $\alpha/\beta \in \mathbb{R}\setminus\mathbb{Q}$

This is the solving of equations (16) that makes difficult the computer implementation of our algorithm in the case $\alpha/\beta \in \mathbb{R}\setminus\mathbb{Q}$. Indeed, as the subgroup $\mathbb{Z}+\mathbb{Z}\alpha/\beta$ is dense in $\mathbb{R}$, we can have several couples of rational integers apparently solutions of these equations, whereas there should be at most one solution in $\mathbb{Z}^2$ because of the irrationality of α/β. Though a solution to the general case is theoretically impossible to get, our algorithm works if we assume that the size of the frequencies is bounded by some constant. This would imply that the k and k' solutions of (16) are bounded by another constant K. The algorithm LLL (1982), that looks for short vectors in lattices, is able to give us all couples $(k,k') \in \mathbb{Z}^2$ solutions of (16) such that $\max(|k|,|k'|) \le K$. Another way to do it is to use the continued fraction expansion of α/β, as it was originally done in Baker and Davenport (1969). In that situation, the only problem comes from the fourth step of the algorithm, in which we want to combine the functions

$R_{f,\delta}(z)e^{\delta z}$ solutions of the system in order to get an entire solution. Indeed, if we missed some frequencies belonging to $\mathrm{Spec}(f)$ but whose modulus is larger than the bound we chose, we can have no entire solution at the end, because of the loss of some components, though there were really entire solutions of the initial system (1). Nevertheless, we can skip this problem in two cases where step 4 becomes useless:

- if the common denominator of the coefficients $R_{f,\delta}(z)$ determined in step 2 is a constant;
- if we only search for the exponential fractions solutions.

Another reason for being able to handle the case $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$ is the fact, proved by Loeb (1997), that the continuous functions $f : \mathbb{R} \to \mathbb{R}$ solutions of (1) are also exponential fractions.

4 The mixed case differential and difference equations, the algorithm Pasrecd

Now, we are dealing with the entire functions f solutions of (2). We know from Bézivin and Gramain (1993, 1996) that f is necessarily an exponential fraction. Here again, we have algorithms to find rational solutions of linear differential equations (see Abramov and Kvashenko (1991) for example). The main change with regard to Pasrec are the steps 1c and 1d which become

1. (c) **For** $\delta \in \mathrm{Spec}(\Phi)\setminus \mathrm{Spec}(\Psi)$,
   if $\delta \notin \{y_1, \ldots, y_L\}$, **then**
   return: no entire solution.
   For $\delta \in \mathrm{Spec}(\Psi)\setminus \mathrm{Spec}(\Phi)$,
   if $e^{\alpha\delta} \notin \{x_1, \ldots, x_J\}$, **then**
   return: no entire solution.

   (d) **For** $j = 1, \ldots, J$, for $l = 1, \ldots, L$,
   $k \leftarrow (\alpha y_l - \mathrm{Log}\,(x_j))/(2i\pi)$
   if $k \in \mathbb{Z}$, **then**
   $Spec \leftarrow Spec \cup \{y_l\}$

Proof of correctness of the algorithm. We know from Subsection 3.4 that $\mathrm{Spec}(\Phi) \subset \mathrm{Spec}(f)$. We plug $f = \sum_{\delta\in\mathbb{C}} R_{f,\delta}(z)e^{\delta z}$, where the $R_{f,\delta}(z)$ are almost all zero, into the second equation of system (2) and we get

$$\sum_{\delta\in\mathbb{C}}\sum_{n=0}^{N} Q_n(z)\left(R_{f,\delta}(z)e^{\delta z}\right)^{(n)} = \sum_{\delta'\in\mathrm{Spec}(\Psi)} R_{\Psi,\delta'}(z)e^{\delta' z}. \tag{17}$$

Leibniz's formula gives, for every $n \in \mathbb{N}$ and $\delta \in \mathbb{C}$,

$$\left(R_{f,\delta}(z)e^{\delta z}\right)^{(n)} = e^{\delta z}\sum_{k=0}^{n}\binom{n}{k}\delta^{n-k}R_{f,\delta}^{(k)}(z).$$

It follows from (17) and from the linear independence over $\mathbb{C}(z)$ of the family $\left(e^{\delta \cdot}\right)_{\delta\in\mathbb{C}}$ that

$$\sum_{n=0}^{N} Q_n(z)\left(\sum_{k=0}^{n}\binom{n}{k}\delta^{n-k}R_{f,\delta}^{(k)}(z)\right) = R_{\Psi,\delta}(z),$$

for every $\delta \in \mathbb{C}$. Hence $\mathrm{Spec}(\Psi) \subset \mathrm{Spec}(f)$. Let $\delta \in \mathrm{Spec}(f) \setminus (\mathrm{Spec}(\Phi) \cup \mathrm{Spec}(\Psi))$, we then have

$$\sum_{n=0}^{N} \frac{Q_n(z)}{z^{d_Q}}\left(\sum_{k=0}^{n}\binom{n}{k}\delta^{n-k}\frac{R_{f,\delta}^{(k)}(z)}{R_{f,\delta}(z)}\right) = 0.$$

Let b_n denote the coefficient of z^{d_Q} in $Q_n(z)$ and let z tend to infinity. We get the equation $\sum_{n=0}^{N} b_n\delta^n = 0$. Hence, we have to solve the following polynomial equations in $X = e^{\alpha\delta}$ and $Y = \delta$:

$$\sum_{m=0}^{M} a_m X^m = 0 \ \text{ and } \ \sum_{n=0}^{N} b_n Y^n = 0.$$

So, we find J $(\leq M)$ distinct classes of solutions in $\mathbb{C}/(2i\pi\alpha^{-1}\mathbb{Z})$ for the first equation that we easily intersect with the L $(\leq N)$ distinct solutions in $\mathbb{C}$ of the second equation. It gives a finite set that we add to $\mathrm{Spec}(\Phi) \cup \mathrm{Spec}(\Psi)$ to get *Spec*, a set that contains $\mathrm{Spec}(f)$. Here again, we can notice that, for $\delta \in \mathrm{Spec}(\Phi) \setminus \mathrm{Spec}(\Psi)$ (resp. $\mathrm{Spec}(\Psi) \setminus \mathrm{Spec}(\Phi)$), if δ (resp. $e^{\alpha\delta}$) is not a root of $\sum_{n=0}^{N} b_n Y^n$ (resp. $\sum_{m=0}^{M} a_m X^m$), then there is no entire solution of (2).

5 Comparison with an algorithm of Abramov

Abramov (1991) gives an algorithm that finds all the exponential fractions solutions of either one difference equation or one differential equation with polynomial coefficients. These solutions have the form $\sum_{i=1}^{m} \lambda_i^x R_i(x)$ in the first case and $\sum_{i=1}^{m} e^{\lambda_i x} R_i(x)$ in the second case with $\lambda_1, \ldots, \lambda_m \in K$ and $R_1, \ldots, R_m \in K(x)$, where K is what Abramov calls a feasible field, namely a field of characteristic 0 with an algorithm for finding integer-valued roots of polynomials with coefficients in K. In the sequel, the term "Abramov's algorithm" refers to the algorithm from Abramov (1991) and not to the one from Section 2. We make our comparison only in the finite difference case, since the comparison in the differential case is analogous.

The situations in Abramov (1991) and in our article are different: we have a second equation and we want the function f to be entire. Our context, with two equations, gives us more precise information about the exponential fractions solutions since we can determine explicitely the frequencies δ whereas Abramov (1991), with only one equation, only provides e^{δ}.

Abramov's algorithm is able, after the computation of a common denominator, to get both the exponentials of the frequencies – which corresponds to step 1b of our algorithm – and a maximal degree for the polynomial solutions of resulting equations – which corresponds to step 3a of our algorithm –. Moreover, the procedure that leads to this simultaneous determination allows to skip some frequencies not involved in the solution that our algorithm may get rid of only at the end of the third step of our algorithm.

The use of Abramov's algorithm to improve our algorithm depends on what is known beforehand about the system of equations. Generically, a system (1) or (2) has no exponential fraction solution. To give an idea of it, we can note that the first step alone of Algorithm Pasrec requires that the intersection of two countable subsets of $\mathbb{C}$ is non empty. Hence, if we deal with a system for which existence of exponential fractions solutions is not ensured, we use Algorithm Pasrec, without taking into account Abramov's algorithm, since Pasrec starts by a quite restrictive filter of determination of the possible frequencies, without computing previously a common denominator, which is done in Abramov (1991) on the other hand. The only case where it is worth using Abramov's algorithm is when we know beforehand that the system has an exponential fraction solution. Then, we use it in the following way. We first execute step 2 of Pasrec, then step 1 in which 1b is replaced by Abramov's algorithm (which does step 3a in the same time), step 3b and, if we want the exponential fraction to be entire, we execute step 4.

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